1. Theoretical foundation

Thank you for the comments. It made us realize that introducing too many equations and terms in the manuscript may confuse readers. Our original intention was to demonstrate the impact of nonlinear strain, but its presentation may have been too complicated. We have simplified the equations to avoid unnecessary repetition. At the same time, we will strengthen the explanation of nonlinear terms to clarify their physical meaning.

Based on the specific questions, we have the following responses:

(1): The rewriting of Eq. (2) as Eq. (5) in the manuscript is to show the approximate relationship between the nonlinear strain tensor E_{ij} and the linear strain tensor e_{ij} after neglecting the high-order nonlinear terms. We think this helps to understand the equivalence between E_{ij} and e_{ij} more clearly.

$$E_{ij} = \frac{1}{2}(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_k}{\partial x_i} \cdot \frac{\partial u_k}{\partial x_j}) \tag{2}$$

$$E_{ij} = e_{ij} + \frac{1}{2}e_{ij}^2 + \frac{1}{2}\left(e_{ij}r_{ij} - r_{ij}e_{ij}\right) - \frac{1}{2}r_{ij}^2 \tag{5}$$

(2): Eqs (6), (7), and (8) are introduced to describe the volume changes that occur in a medium under nonlinear conditions when subjected to external forces. The volume strain includes the fundamental terms of the linear theory and the nonlinear terms. These neglected nonlinear terms also affect the propagation characteristics of elastic waves. Therefore, we introduce the volume strain here to preserve the nonlinear terms resulting from the volume strain, which are of the same order as those in the nonlinear strain tensor, in order to better understand and simulate the propagation of elastic waves under nonlinear conditions. By retaining nonlinear terms of the same order in the volume strain as in the nonlinear strain tensor, not only the nonlinearities due to the strain tensor are taken into account, but also the nonlinearities due to the volume strain. We believe that the higher-order terms in the volume strain need to be extended and added to the nonlinear study. In future research, we will further discuss the difference in wave equations of motion considering only the nonlinear terms that the strain tensor brings to the propagation of elastic waves.

$$
\begin{aligned}
\theta &= \frac{(1+\theta_{xx})(1+\theta_{yy})(1+\theta_{zz})dxdydz - 1dxdydz}{dxdydz} \\
&= e_{xx} + e_{yy} + e_{zz} + e_{xx}e_{yy} + e_{xx}e_{zz} + e_{yy}e_{zz} + e_{xx}e_{yy}e_{zz}
\end{aligned} \tag{6}
$$

$$
\theta_e \approx e_{xx} + e_{yy} + e_{zz} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \tag{7}
$$

$$
\begin{aligned}
\theta_E &\approx E_{xx} + E_{yy} + E_{zz} + E_{xx}E_{yy} + E_{xx}E_{zz} + E_{zz}E_{yy} \approx \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} + \frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial y} + \frac{\partial u_x}{\partial x}\frac{\partial u_z}{\partial z} + \frac{\partial u_y}{\partial y}\frac{\partial u_z}{\partial z} \\
&+ \frac{1}{2}\left(\frac{\partial u_x}{\partial x}.\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial x}.\frac{\partial u_y}{\partial x} + \frac{\partial u_z}{\partial x}.\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial y}.\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial y}.\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial y}.\frac{\partial u_z}{\partial y} + \frac{\partial u_x}{\partial z}.\frac{\partial u_x}{\partial z} + \frac{\partial u_y}{\partial z}.\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial z}.\frac{\partial u_z}{\partial z} \right)
\end{aligned} \tag{8}
$$

(3): About Eq. (9) and Eq. (10): Eq. (9) is derived from the combination of linear strain tensor, linear volume strain, and related equations. This equation is the basic theoretical framework widely used in seismology in conventional linear theory. In contrast, Eq. (10) is obtained by combining the strain tensor, nonlinear volume strain, and related equations, and is designed to more comprehensively capture the nonlinear elastic effects that may occur during earthquakes, especially under strong or near-field earthquakes, which can significantly affect the characteristics and propagation patterns of seismic motion. By presenting the two equations in parallel, we want to visualize the essential differences between linear and nonlinear motion in mathematical expressions and their potential effects on seismic displacement field.

2. Mathematical errors and confusion in equations

Thank you for the comments. The mathematical errors and confusion in the equations were indeed unintentional on our part. We have reviewed and corrected the use of symbols in the equations to ensure consistency. In particular, regarding the confusing use of i, j, k and x, y, z, we have standardized the symbol system used and provided clear definitions and explanations. Thank you for your detailed advice and patience. We will improve the equation derivation process and present it better.

3. Benchmark required

Thank you for the comments. We will validate our simulation results under linear approximation conditions by selecting a benchmark test code based on your suggestions. The correctness and reliability of our code will be verified by comparing the results of our code with the benchmark test code.

4. English expression needs improvement

Thank you for the comments. We acknowledge that there are shortcomings in our English expression that make it difficult to read. We will seek professional language services to improve the English expression and ensure the readability and academic standard of the manuscript. At the same time, we will improve our English language skills to ensure we can express our research results more clearly and accurately in our future research.

In summary, we will address the theoretical foundation part by rewriting it according to the reviewers' comments, simplifying the equations, clarifying the derivation process, and removing unnecessary equations. For the numerical implementation, we will perform benchmark tests to ensure the consistency of the simulation results with known codes. We look forward to better presenting our research results in the revised version.