Revised Consolidated Response to Reviewers

We sincerely thank all reviewers for their insightful comments, which have greatly improved the manuscript. This document provides point-by-point responses, detailing the revisions made to address each comment while maintaining the original intent

Response to Reviewer #1

Reviewer's Comment 1:

I would recommend increasing the size of Figure 1. At its present size, it is difficult to read without zooming in very closely.

Response:

Thank you for the suggestion regarding Figure 1's readability. We have increased both the size and resolution of Figure 1. Specifically, we have:

Enlarged the figure

Increased the resolution

Enhanced the contrast of the curve lines and axis labels

Changes in Manuscript:

We have increased the size and resolution of Figure 1 to improve its readability and ensure all details are clearly visible.

Reviewer's Comment 2:

In line 88, is the parameter a just a numerical parameter, or does it have a name or definition?

Response:

Thank you for pointing out the need to clarify parameter a. In the revised manuscript, we have added a clear definition of this parameter. As shown in lines 84-85 of the revised manuscript, we now explicitly state that "a is a dimensionless control parameter that governs the system's nonlinear characteristics." This parameter was originally introduced by Agop et al. (2012) (referenced in line 83) to describe the system's nonlinear behavior.

Changes in Manuscript:

In line 84, we have added the following definition: "where a is a dimensionless control parameter that governs the system's nonlinear characteristics".

Reviewer's Comment 3:

In lines 91–93, the change in potential profile against current is discussed in Figure 1. Within these lines, there is mentioned a "certain state" and a "certain limit," as well as a "completely different state" before reaching steady state. Be more specific with what these thresholds and states really are, and if possible postulate on how they might come to be.

Response:

Thank you for requesting clarification about the state transitions. We have revised lines 91-98 in the manuscript to provide precise definitions and explanations of these states:

"Certain state": Now defined in line 91 as the initial stable state (Point A in Figure 1), characterized by monotonic current increase with voltage.

"Certain limit": Explained in lines 92-93 as critical threshold points (Points B and D) that mark transitions between stable states.

"Completely different state": Defined in lines 94-95 as the high-conductivity stable state (Point C) after transition.

Changes in Manuscript:

We have revised lines 91-98 as follows:

“As illustrated in Figure 1, for parameter a, the J-φ characteristic curve exhibits three distinct regions. There are two stable regions where $dJ/d\varphi > 0$: a low-conductivity state (segment AB) and a high-conductivity state (segment CD), both characterized by a monotonically increasing current with voltage. These stable regions are separated by an unstable region where $dJ/d\varphi < 0$, demonstrating negative differential resistance”

Response to Reviewer #2

Reviewer's Comment 1:

The main concern regards the connection between the formalism introduced in Section 2.2 and the formalisms and results presented in Sections 2.4 and 3. Indeed, what has been introduced in Section 2.2 is a classical theory of bifurcation for autonomous dynamical systems being written as a time-evolution mapping (continuous in this case) with a not implicit dependence on time in the forcing term. Conversely, what is introduced Eq. (10) is a non-autonomous dynamical system whose implicit variable is not time but one of the state variables (I). Thus, the connection among fixed points, instability, and other types of concepts cannot be simply ruled out. What the authors introduced in Eq. (10) is a mathematical description of the manifold or a dynamical bifurcation scenario for, at least, a 2-D dynamical system described by the state variables U and I. The authors need to carefully address these concepts and revise accordingly the manuscript by possibly considering a 2-D dynamical system of the form

$$dU/dt = f(U, I, u\_parameters)$$

$$dI/dt = g(I, U, i\_parameters)$$

where u_parameters and i_parameters refer to the bifurcation parameters leading eventually to critical transitions in the system.

Response:

Thank you for your insightful comment regarding the mathematical framework. We need to clarify that Equation (10), appearing in lines 160-165 of the revised manuscript, represents a calculation of differential resistance (dU/dI) rather than a dynamical system. The theoretical framework introduced in Section 2.2 establishes the connection between negative differential resistance and system instability, which we then apply to analyze lightning channel behavior.

Thank you for your valuable suggestion regarding the mathematical framework. We need to clarify several key points:

Framework Clarification (lines 160-165):

Equation (10) represents a calculation method rather than a dynamical system

The equation describes instantaneous channel properties rather than temporal evolution

Connection Between Sections:

Section 2.2 (lines 103-110): Establishes theoretical foundation for stability analysis

Section 2.4 (lines 160-165): Derives differential resistance calculation

Section 3 (lines 170-190): Applies framework to physical system

Relationship to Suggested 2-D System:

While we appreciate the suggestion of a 2-D system, our focus is on the instantaneous relationship between voltage and current rather than temporal dynamics.

Changes in Manuscript:

We have made the following revisions in Section 2.4 (lines 160-165):

"The differential resistance of a streamer channel is determined by the potential difference U across the streamer zone of the leader head and the channel current I. Equation (10) provides a mathematical expression for calculating this differential resistance, which serves as an indicator of channel stability rather than describing temporal evolution of the system"

Reviewer's Comment 2:

The second main concern is related to Figure 2. Indeed, what the authors reported is valid for bi-stable dynamical systems which are described by a double-well potential function. It is not straightforward the connection with Eq. (1) and the system introduced in Line 103 which seems to be more similar to a hysteresis cycle. Which are the stable and unstable fixed points in your system? If φ is treated as a parameter the system admits 3 fixed points provided that $J \neq 0$ and J is real. However, limit cycles could emerge when crossing the complex plane (Hopf bifurcation). Thus, more careful analysis of the bifurcations should be carried out.

Response:

Thank you for raising these important points about the system's behavior and Figure 2. We have substantially revised our explanation in the manuscript to clarify:

The relationship between Equation (1) and system stability (lines 103-110):

For small a: monostable behavior with monotonic current-voltage relationship

For larger a: bistable behavior with negative differential resistance region

The physical interpretation of Figure 2 (following line 110):

Valleys represent stable states (low and high conductivity states)

Peaks correspond to unstable transition points

Points F1 and F2 mark critical transitions between states

Connection to Physical System:

Low-conductivity state: Initial channel condition

High-conductivity state: Fully developed discharge

Transitions: Observed as sudden channel brightening or extinction"

Changes in Manuscript:

We have enhanced the explanation in lines 103-110 to read:

"In nonlinear dynamics, negative differential resistance, bistability, and hysteresis are commonly observed. Considering the dynamic system $dJ/dt = f(J,\varphi)$, where J is the state variable and φ is a parameter. The equilibrium points are given by $f(J,\varphi) = 0$. At an equilibrium point, the system is unstable when $\partial f/\partial J > 0$ and stable when $\partial f/\partial J < 0$."

Reviewer's Comment 3:

The third main concern is related to the presentation of the results and the overall structure of the manuscript. The authors need to carefully revise the manuscript to improve the quality of the figure as well as to check the consistency of the different type settings of the text, typos, references, etc.

Please find a list below.

Check the font size for subsections

Check the font type for references through the text (sometimes italics, sometimes not)

All figures need to be improved for quality

Line 84: mismatching between φ and that used in Eq. (1)

Figure 1: increase font and labels

Line 104: missing definition of φ

Line 105: formally, the condition for fixed points should be met not for all J but for a specific solution J* or something similar

Line 106: missing definition of what the subscript J means (I assume derivative with respect to J)

Line 106: missing space and capital letter "if we let"

Line 110: the assumption is not straightforward and the connection between Eq. (1) and Line 103 is missing

Line 139: please delete double point.

Line 140: please delete the period before introducing the equation.

Line 145: which type of fit is used?

Figure 3: missing space Fig3

Figure 4: missing space Fig4

Figure 5: missing space Fig5.

Response:

Thank you for your thorough review of the technical details. We have made comprehensive revisions throughout the manuscript to address all formatting and consistency issues:

Changes in Manuscript:

1. Typography and Formatting:

Standardized subsection font sizes throughout

Unified reference formatting to non-italic style

Corrected figure spacing (e.g., "Fig. 3" instead of "Fig3")

2. Mathematical Notation:

Lines 84-85: Added consistent φ notation

Line 104: Added explicit definition of φ

Line 105: Clarified fixed point conditions

Line 106: Added definition of subscript J

3. Figure Quality:

Enhanced resolution of all figures

Increased font sizes in labels and annotations

Standardized figure formatting

4. Technical Content:

Line 145: Added explanation of fitting method: "Used nonlinear least squares fitting with double power-law model $E = aI^b + cI^d$"

Improved equation presentation and formatting throughout

Response to Reviewer #3

Reviewer's Comment

The manuscript explores stability and critical transitions in lightning discharge channels using concepts from nonlinear dynamics, particularly bi-stable systems. I recommend acceptance contingent upon revision. While Figures 1 and 2 effectively demonstrate the theoretical principles, they remain too abstract and do not correspond directly with the specific dynamics of lightning channels discussed. I suggest adding figures that depict the stability profiles derived from equations 9 and 10, as these directly describe the behavior of lightning systems. This enhancement will clarify the instability mechanisms within real lightning channels, making the application of theoretical models more comprehensible and scientifically rigorous.

Response:

Thank you for your constructive suggestion regarding the theoretical and practical aspects of our analysis. We need to clarify that Equations (9) and (10) represent calculations of potential difference and differential resistance rather than dynamical system equations. These equations directly relate to the physical behavior of lightning channels as follows:

Equation (9) (lines 170-175) calculates the total potential difference across the leader-streamer system.

Equation (10) (lines 175-180) determines the differential resistance, which indicates system stability.

Changes in Manuscript:

We have enhanced the explanation in Section 3 (lines 170-190) to clarify how these equations relate to physical observations: Section 3 (lines 170-190) now reads: "The theoretical framework established by Equations (9) and (10) directly corresponds to measurable lightning channel characteristics. Figure 4 demonstrates this connection by showing how differential resistance varies with channel current, identifying critical transition points that match observed behavior. The intersection points with zero differential resistance correspond to stability thresholds observed in lightning measurements."

Summary of Major Changes

We have made the following substantial improvements to the manuscript:

1. Enhanced theoretical framework clarity and connections

2. Improved mathematical consistency and notation

3. Upgraded figure quality and presentation

4. Strengthened links between theory and physical application

5. Standardized formatting throughout

We believe these revisions have substantially improved the manuscript while maintaining its scientific contribution. We again thank all reviewers for their valuable input.