Response to Reviewer #2's Comments

Dear Reviewer：

Thank you very much for your detailed comments and constructive suggestions. Below are our responses to your specific points, particularly regarding Figure 2 and its relationship to Equation (1), as well as clarifications on other parts of the manuscript:

1. Connection between the Formalism in Section 2.2 and the Results in Sections 2.4 and 3

Reviewer's Comment:

The main concern regards the connection between the formalism introduced in Section 2.2 and the formalisms and results presented in Sections 2.4 and 3. Indeed, what has been introduced in Section 2.2 is a classical theory of bifurcation for autonomous dynamical systems... Conversely, what is introduced in Eq. (10) is a non-autonomous dynamical system whose implicit variable is not time but one of the state variables (I). Thus, the connection among fixed points, instability, and other types of concepts cannot be simply ruled out. What the authors introduced in Eq. (10) is a mathematical description of the manifold or a dynamical bifurcation scenario for, at least, a 2-D dynamical system described by the state variables U and I. The authors need to carefully address these concepts and revise accordingly the manuscript by possibly considering a 2-D dynamical system of the form

dU/dt = f(U, I, u_parameters)

$dI/dt = g(I, U, i\_parameters)$

Our Response:

Thank you for your valuable feedback. We would like to clarify the following points:

1. Nature of Equation (10):

Equation (10) is not intended as a description of a dynamical system or non-autonomous system. Rather, it represents the calculation of the differential resistance dU/dI in the lightning channel. The purpose of this equation is to analyze the stability and critical transition points of the system by examining the sign of the differential resistance.

The misunderstanding may have arisen due to the text not clearly explaining that Equation (10) is a static tool used for stability analysis, not a model for the system's time evolution.

2. Improving the Theoretical Connection:

We realize that the connection between the theory in Section 2.2 and the practical analysis in Sections 2.4 and 3 could be better clarified. To address this:

In the revised manuscript, we will provide a clearer explanation of Equation (10), emphasizing that it is used to study the stability of fixed points in the system, and how the sign of the differential resistance indicates stability or instability.

We will also remove any references to "dynamical systems" or "time evolution" to avoid confusion and ensure that Equation (10) is interpreted as a stability analysis tool rather than a description of system dynamics.

2. Figure 2 and Its Relationship to Equation (1)

Reviewer's Comment:

The second main concern is related to Figure 2. Indeed, what the authors reported is valid for bi-stable dynamical systems which are described by a double-well potential function. It is not straightforward the connection with Eq. (1) and the system introduced in Line 103 which seems to be more similar to a hysteresis cycle. Which are the stable and unstable fixed points in your system? If φ is treated as a parameter the system admits 3 fixed points provided that J≠0 and J is real. However, limit cycles could emerge when crossing the complex plane (Hopf bifurcation). Thus, more careful analysis of the bifurcations should be carried out.

Our Response:

Thank you for your comments on Figure 2. We would like to clarify the following points regarding the figure and its relationship to Equation (1):

1. Core Concept of Equation (1):

Equation (1) defines the relationship between normalized voltage ϕ and normalized current J, where parameter a determines the strength of the system's non-linearity and the potential for bistable behavior. By varying a, Equation (1) can produce the following behaviors:

Monostable behavior: When a is small, the system exhibits a single stable state, with a monotonically increasing relationship between voltage and current.

Bistable behavior: As a increases, the system can exhibit bistability, with two stable points and one unstable region (negative differential resistance).

2. Clarification of Figure 2:

Figure 2 is a landscape diagram that visually represents the transition between stable and unstable states under different conditions, which we associate with the bistable behavior described by Equation (1). The figure shows how the system transitions between two stable states, with the critical points F_1 and F_2 indicating where the system undergoes a shift from one stable state to another. These critical points are not the stable states themselves, but rather the thresholds at which the transition occurs.

We will revise the manuscript to clarify that F_1 and F_2 are critical points, not stable states, and that they mark the boundary between different stable states in the system.

We will also explain how the hysteresis cycle observed in Figure 2 corresponds to the negative differential resistance region in Equation (1).

3. Universality and Literature Support:

Figure 2 highlights the universality of bistable behavior, which is commonly observed in various natural systems such as ecological systems (e.g., Scheffer's studies) and lightning channel dynamics. This figure visually demonstrates the critical transition phenomenon that can also be found in other complex systems.

We will include references to relevant literature (such as Scheffer et al.) to support the broader applicability of this model and its connection to critical transitions in complex systems.

3. Presentation and Structure of the Manuscript

Reviewer's Comment:

The third main concern is related to the presentation of the results and the overall structure of the manuscript. The authors need to carefully revise the manuscript to improve the quality of the figure as well as to check the consistency of the different type settings of the text, typos, references, etc. Please find a list below.

Our Response:

Thank you for pointing out areas for improvement in the manuscript presentation. We will address the following points in the revised version:

1. Figure Quality:

We will improve the resolution of all figures (Figure 1, Figure 2, Figure 3, Figure 4, and Figure 5) and increase the font size and clarity of the labels, particularly in Figure 2.

2. Text Consistency:

We will fix any mismatched symbols, such as the inconsistency between ϕ in Line 84 and its usage in Equation (1).

We will clarify the definitions of all symbols (e.g., ϕ in Line 104, J in Line 106).

We will also check the overall text formatting, correcting any typos, reference formatting issues, and other minor inconsistencies.

3. Clarification of Figure 2:

We will add more detailed descriptions to the figure caption to clarify how Figure 2 relates to Equation (1), as well as its role in illustrating bistable behavior and its physical significance.

Conclusion

We appreciate your constructive feedback, which has been instrumental in improving the manuscript. In the revised version, we will:

1. Provide clearer explanations of Equation (10) and its role in analyzing stability transitions.

2. Clarify the relationship between Figure 2 and Equation (1).

3. Improve the manuscript's presentation, including figure quality and text formatting, to ensure clarity and consistency.

We hope these revisions will address your concerns. Thank you again for your valuable input, and we look forward to your further suggestions.

Best regards,

On behalf of all authors