S1 On the $\Gamma_{n,\lambda}$ distributions from Gaussians with $\sigma \neq 1$

We want to fit the $\sigma^2 = \frac{1}{\beta}$ distributions as a convolution of exponential distributions. Let's first consider which is the general expression of a $\Gamma_{n,\lambda}$ coming from gaussian distributions with $\sigma \neq 1$ in general.
Let x_i be independent gaussian distributed random variables with zero mean ($\mu_i = 0$) and same variance ($\sigma_i = \sigma$):

$$f(x_i) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x_i^2}{2\sigma^2}} \tag{S1}$$

Let's define:

$$Q = \sum_{i=1}^{2n} x_i^2 \tag{S2}$$

So, calling V the elemental shell volume at $Q(x)$, which is proportional to the $(2n-1)$-dimensional surface in $2n$-space for which $Q = \sum_{i=1}^{2n} x_i^2$,

$$\begin{aligned} P(Q)dQ &= \int_V \frac{e^{-\frac{1}{2\sigma^2}\left(x_1^2+x_2^2+\ldots+x_{2n}^2\right)}}{(2\pi)^{2n/2}\sigma^{2n}} dx_1 dx_2 \cdots dx_{2n} \\ &= \frac{e^{-\frac{Q}{2\sigma^2}}}{(2\pi)^n \sigma^{2n}} \int_V dx_1 dx_2 \cdots dx_{2n} \\ &= \frac{e^{-\frac{Q}{2\sigma^2}}}{(2\pi)^n \sigma^{2n}} \cdot \frac{2Q^{\frac{2n-1}{2}}\pi^n}{\Gamma(n)} \cdot \frac{dQ}{2Q^{1/2}} \\ &= \frac{Q^{n-1}e^{-\frac{Q}{2\sigma^2}}}{2^n \Gamma(n) \sigma^{2n}} \cdot dQ \end{aligned} \tag{S3}$$

The resulting normalized $\Gamma_{n,\lambda}$ distribution (where $\lambda = 1/2\sigma^2$) with $2n$ degrees of freedom (the number of the added gaussian variables) is:

$$\Gamma_{n,\lambda}(Q) = \frac{Q^{n-1}e^{-\frac{Q}{2\sigma^2}}}{(2\sigma^2)^n \Gamma(n)} = \frac{\lambda^n Q^{n-1} e^{-\lambda Q}}{\Gamma(n)} \tag{S4}$$

Imponing $2n = 2$ we obtain the exponential distribution:

$$\Gamma_{1,\lambda}(Q) = \frac{1}{2\sigma^2} \cdot e^{-\frac{Q}{2\sigma^2}} = \lambda e^{-\lambda Q} \tag{S5}$$

The mean value of a gamma distributed variable is:

$$\mu_{\Gamma,n} = \int_0^\infty Q \Gamma_{n,\lambda}(Q) dQ = \frac{n}{\lambda} \tag{S6}$$

The variance of a gamma distributed variable is:

$$\sigma^2_{\Gamma,n} = \int_0^\infty Q^2 \Gamma_{n,\lambda}(Q) dQ - \mu^2_{\Gamma,n} = \frac{n}{\lambda^2} \tag{S7}$$

Imponing $n = 1$ (the case of the exponential distribution) we obtain:

$$\sigma^2_{\exp} := \sigma^2_{\Gamma,1} = \frac{1}{\lambda^2} \tag{S8}$$

S2 On the convolution of $\Gamma_{1,\lambda}$ distributions

Let X and Y be two independent random variables distributed as $\Gamma_{1,\lambda}$ and $\Gamma_{1,k}$ respectively, see Equation (S5), where $\lambda = \frac{1}{2\sigma_X^2} > 0$ and $k = \frac{1}{2\sigma_Y^2} > 0$:

$$\begin{aligned} f_X(x) &= \lambda e^{-\lambda x}\Theta_{(0,\infty)}(x) \\ f_Y(y) &= k e^{-ky}\Theta_{(0,\infty)}(y) \end{aligned} \tag{S9}$$

where

$$\Theta_{(a,b)}(t) = \begin{cases} 1 & \text{if } a \leq t \leq b \\ 0 & \text{otherwise} \end{cases} \tag{S10}$$

The two random variables X and Y are independent, so their joint probability distribution is:

$$f_{XY}(x, y) = f_X(x) \cdot f_Y(y) \tag{S11}$$

If we define a new random variable $Z = X + Y$, we will obtain:

$$\begin{aligned} f_Z(x, z) &= f_X(x) \cdot f_Y(z - x) \\ &= \lambda k e^{-\lambda x} e^{-k(z-x)}\Theta_{(0,\infty)}(x)\Theta_{(0,\infty)}(z - x) \\ &= \lambda k e^{-kz} e^{-(\lambda-k)x}\Theta_{(0,z)}(x) \end{aligned} \tag{S12}$$

So, we obtain the convolution:

$$\begin{aligned} f_Z(z) &= \int_0^\infty f_Z(x, z)dx \\ &= \lambda k e^{-kz} \int_0^z e^{-(\lambda-k)x}dx \\ &= \frac{\lambda k}{\lambda - k}(e^{-kz} - e^{-\lambda z}) \end{aligned} \tag{S13}$$

The normalization can be easily checked.

S2.1 The case where $\lambda = k$

In the case where $\lambda = k$, it means that X and Y are identically distributed, so Z is the sum of $n = 2$ exponential processes (positive variables with maximum entropy distribution). We can interpret Z as the sum of $2n = 4$ indipendent gaussian random variables with $\sigma_X^2 = \frac{1}{2\lambda}$:

$$Z \sim \Gamma_{2,\lambda}(z) = \lambda^2 z e^{-\lambda z} \tag{S14}$$

In this case the variance results:

$$\sigma_{\Gamma,2}^2 = \frac{2}{\lambda^2} \tag{S15}$$

In the following and in the paper, when we omit the pedix $n = 2$, we refer to the case $n = 2$:

$$\sigma_\Gamma^2 := \sigma_{\Gamma,2}^2 \tag{S16}$$

S3 The general case: the superstatistical combination of a gaussian distribution and a $\Gamma_{n,\lambda}$ distribution

The PDF of the total signal x is locally (in time) gaussian $N_{(0,z)}(x)$ with zero mean and the variance z varying (on longer time scales) following a $\Gamma_{n,\lambda}(z)$ distribution. The PDF $p(x)$ of the total signal can be obtained in the following way (look at integral 16 page 369 of [**1**]):

$$\begin{aligned}
p(x) &= \int_0^\infty \Gamma_{n,\lambda}(z) \cdot N_{(0,z)}(x) dz \\
&= \int_0^\infty \frac{\lambda^n}{\Gamma(n)} z^{n-1} e^{-\lambda z} \cdot \frac{1}{\sqrt{2\pi}\sqrt{z}} e^{-x^2/2z} dz \\
&= \frac{\lambda^n}{\Gamma(n)\sqrt{2\pi}} \int_0^\infty z^{(n-1)-\frac{1}{2}} \cdot e^{-\lambda z - \frac{x^2/2}{z}} dz \\
&= \frac{\lambda^n}{\Gamma(n)\sqrt{2\pi}} (-1)^{n-1} \sqrt{\pi} \frac{\partial^{n-1}}{\partial \lambda^{n-1}} \left(\frac{e^{-\sqrt{2\lambda x^2}}}{\sqrt{\lambda}} \right) \\
&= \frac{(-1)^{n-1}\lambda^n}{\Gamma(n)\sqrt{2}} \cdot \frac{\partial^{n-1}}{\partial \lambda^{n-1}} \left(\frac{e^{-\sqrt{2\lambda}\cdot|x|}}{\sqrt{\lambda}} \right)
\end{aligned} \tag{S17}$$

with zero mean and variance s^2:

$$\begin{aligned}
s^2 &:= var(x) = E[x^2] \\
&= 2 \int_0^\infty \frac{(-1)^{n-1}\lambda^n}{\Gamma(n)\sqrt{2}} \cdot x^2 \cdot \frac{\partial^{n-1}}{\partial \lambda^{n-1}} \left(\frac{e^{-\sqrt{2\lambda}\cdot x}}{\sqrt{\lambda}} \right) dx \\
&= \frac{(-1)^{n-1}\lambda^n\sqrt{2}}{\Gamma(n)} \cdot \frac{d^{n-1}}{d\lambda^{n-1}} \left(\frac{1}{\sqrt{\lambda}} \int_0^\infty x^2 e^{-\sqrt{2\lambda}\cdot x} dx \right) \\
&= \frac{(-1)^{n-1}\lambda^n\sqrt{2}}{\Gamma(n)} \cdot \frac{d^{n-1}}{d\lambda^{n-1}} \left(\frac{1}{\sqrt{\lambda}} \cdot \frac{1}{2\lambda} \cdot \frac{1}{\sqrt{2\lambda}} \underbrace{\int_0^\infty y^2 e^{-y} dy}_{=2} \right) \\
&= \frac{(-1)^{n-1}\lambda^n}{\Gamma(n)} \cdot \frac{d^{n-1}}{d\lambda^{n-1}} \left(\frac{1}{\lambda^2} \right) \\
&= \frac{(-1)^{n-1}\lambda^n}{\Gamma(n)} \cdot \frac{(-1)^{n-1} n!}{\lambda^{n+1}} \\
&= \frac{n}{\lambda}
\end{aligned} \tag{S18}$$

The normalization of $p(x)$ can be easily checked.
We note that:

$$s^2 = \frac{n}{\lambda} = \sqrt{n} \cdot \sigma_{\Gamma,n} = n \cdot \sigma_{\exp} \tag{S19}$$

S4 Our case

In our case:

$$\sigma_\star^2 \sim f(\sigma_\star^2) = \Gamma_{2,\lambda_\star}(\sigma_\star^2) = \lambda_\star^2 \sigma_\star^2 e^{-\lambda_\star \sigma_\star^2} \tag{S20}$$

Since our focus is on the variances (positive values), we can interpret $n = 2$ as the effective degree of freedom.
The velocity increment PDF is (see Eq. (S17) with $n = 2$):

$$\begin{aligned} p(\delta u) &= \int_0^\infty f(\sigma_{\delta u}^2) p(\delta u|\sigma_{\delta u}^2) d(\sigma_{\delta u}^2) \\ &= \int_0^\infty \lambda_{\delta u}^2 \sigma_{\delta u}^2 e^{-\lambda_{\delta u}\sigma_{\delta u}^2} \cdot \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sigma_{\delta u}^2}} e^{-\delta u^2/2\sigma_{\delta u}^2} \cdot d(\sigma_{\delta u}^2) \\ &= -\frac{\lambda_{\delta u}^2}{\sqrt{2}} \cdot \frac{\partial}{\partial \lambda_{\delta u}} \left(\frac{e^{-\sqrt{2\lambda_{\delta u}}\cdot|\delta u|}}{\sqrt{\lambda_{\delta u}}} \right) \\ &= \frac{\sqrt{2\lambda_{\delta u}} e^{-\sqrt{2\lambda_{\delta u}}|\delta u|} (\sqrt{2\lambda_{\delta u}}|\delta u| + 1)}{4} \end{aligned} \tag{S21}$$

and analogously for δv. The normalization can be easily checked. The PDF is symmetric.
The analytical velocity increment variance $s_\star^2$ is (see Eq. (S18) with $n = 2$):

$$s_{\delta u}^2 = \int_{-\infty}^{+\infty} \delta u^2 p(\delta u) d(\delta u) = \frac{2}{\lambda_{\delta u}} \tag{S22}$$

and analogously for δv.
In the case $n = 2$, Eq. (S19) reduces to:

$$\left.\begin{aligned} \sigma_{\text{exp}}^2 &= \frac{1}{\lambda_\star^2} \\ \sigma_\Gamma^2 &= \frac{2}{\lambda_\star^2} \end{aligned}\right\} s_\star^2 = \frac{2}{\lambda_\star} = \sqrt{2} \cdot \sigma_\Gamma = 2 \cdot \sigma_{\text{exp}}. \tag{S23}$$

References

[1] Gradshteyn, Izrail Solomonovich, and Iosif Moiseevich Ryzhik. Table of integrals, series, and products. Academic press, 2014.