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Brief Communication: Modified KdV equation for Rossby-Khantadze waves in a sheared zonal flow of the ionospheric E-layer

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Abstract

The nonlinear system of equations for Rossby-Khantadze waves in a weakly ionospheric plasma by incooperating sheared zonal flow is given. It is shown that in the presence of multiple-scale analysis, our obtained set of equations can be decomposed into one-dimensional equation which we call nonlinear modified KdV (MKdV) equation illustrating the propagation of solitary Rossby-Khantadze waves.

Keywords: Rossby-Khantadze waves; nonlinearity, sheared zonal flow

1. Introduction

Different satellite and ground-based investigations indicates presence of zonal flows in various atmospheric regions around Earth (Pedlosky, 1987). In E and F regions of ionosphere the sheared flows present along the meridians including inhomogeneous velocity is linked with ultra-low-frequency perturbations (Satoh, 2004; Champeaux et al. 2008; Shukla et al., 2003; Onishchenko et al. 2004; Kaladze T.D., 2007; Kaladze et al., 2008). The additive effects are given by sheared property in linear and nonlinear effect of perturbed waves. Hence, suitable conditions are provided by ionospheric medium to give diverse nonlinear wave structures. The propagation of Rossby waves which exists both in oceans and in atmosphere attract significant attention because of its effects on the global atmospheric circulation.

The reason for the existence of zonal flows is the non-uniform warming of the Earth's atmospheric regions by the sun. In the past decade, several nonlinear phenomena were discussed for the excitation of sheared (zonal) flows. Recent work of Benkadda et al., (2011) for fusion plasmas discusses excitation of zonal flow by taking into account drift waves. Furthermore, streamers (dust) generation in the ionospheric plasma is also studied. The excitation of such zonal flows in drift-turbulence were investigated (Champeaux et al. 2008), where the Hasegawa-Wakatani model was considered. Few years ago, the generation of zonal flows for short-scale electromagnetic Rossby waves was considered by taking Reynolds stresses into account (Shukla et al., 2003; Onishchenko et al. 2004). The production of zonal flow in E-ionospheric layer by Rossby waves was investigated by Kaladze et al. (2007). The authors considered the effects of the zonal (sheared) flows on Rossby nonlinear structures and it was shown that the nonlinear structures split into various segments based on the collection of zonal flow energy into such nonlinear structures (Kaladze et al., 2008). A different concept

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was considered by Benkadda et al. (2011) who emphasized on the interaction of high frequency drift waves and those with lower frequencies.

Recently, it is implied that such coupled Rossby-Khantadze (RK) waves could be propagated and could be self-organized into dipolar (solitary) vortices (Kaladze, 2014). In the present work, the spatially inhomogeneous coriolis parameter and ambient magnetic field along the parallels is considered illustrating propagation of such coupled RK waves. The numerical simulation of the same problem was done in (Kaladze, et al. 1999) where the possibility of modulation mechanism in ionospheric plasmas was discussed, leading to the formation and transport of such dust flows and particles respectively. Numerical solutions of such shear RK waves with sheared zonal flow in such weakly ionized ionospheric E-plasma region were investigated (Futatani et al., 2015). In this work the splitting of the vortices which carried the energy by sheared flow into multiple pieces is pointed out. The equatorial Rossby waves were discussed under similar sheared flows (Qiang et al., 2001). Existence of solitons was observed by *Freja and Viking satellites* (Qiang et al., 2001; Bostrom, 1992; Lindqvist et al., 1994; Dovner et al. 1994, YunLong Shi et al. 2018). The production of shear flow by Rossby-Khantadze waves in E-ionospheric region were discussed (Kaladze, et al. 2014). Kaladze et al. (2009) studied the solitary properties of Rossby (magnetized) waves with the interaction of zonal flows (shear) and developed the 1D mKdV equation for Rossby waves. By considering β-plane approximation, similar work was studied (Jian et al., 2009).
The novelty of present paper, is by consideration of magnetic field we studied the MKdV equation for Rossby-Khantadze waves which is not reported so far.

The present problem is not reported before and the novelty of the present work is to study the solitary Rossby-Khantadze (RK) waves by incorporating the sheared zonal flows in partially ionized ionospheric E-region plasma. In Sec. 2, by using multiple scale analysis and perturbation approach; from a system of nonlinear two-dimensional equations we derive one-dimensional MKdV equation describing solitary Rossby-Khantadze waves' dynamics along with zonal (shear) flows is obtained. In next Sec. 3, includes the discussion of the results.

2. Mathematical Preliminaries

We have considered weakly conductive E-ionospheric region comprising of electrons, ions and neutrals. The ionospheric plasma is enclosed in a geomagnetic field. The local geomagnetic field's components are $\boldsymbol{B}_0 = \left(0, B_{0y}, B_{0z}\right)$ alongwith the angular velocity of the earth, as $\boldsymbol{\Omega} = \left(0, \Omega_{0y}, \Omega_{0z}\right)$. In this layer, the wave motion is considered two-dimensional and defined in terms of stream function $\psi(x, y, t)$, $\mathbf{v} = (u, \mathrm{v}, 0)$, with $u = -\frac{\partial \psi}{\partial y}$ and $\mathrm{v} = \frac{\partial \psi}{\partial x}$.

The nonlinear behavior of considered sheared Rossby-Khantadze waves is pointed out by a system having two-dimensional nonlinear equations. The first system of equation is obtained from the z-component of the curl of vorticity with $\zeta_z = \Delta\psi$ and the second one is obtained from the z-component of magnetic field of the Faraday's law by considering the spatial inhomogeneous background magnetic field and Coriolis parameter, $f = 2\Omega_{0z}$ (Kaladze, et al., 2014):

$$\begin{cases} \frac{\partial \Delta\psi}{\partial t} + \beta \frac{\partial \psi}{\partial x} + \mathrm{J}(\psi, \Delta\psi) - \frac{1}{\mu_0 \rho} \beta_B \frac{\partial h}{\partial x} = 0 \,, \\ \frac{\partial h}{\partial t} + \mathrm{J}(\psi, h) + \beta_B \frac{\partial \psi}{\partial x} + c_B \frac{\partial h}{\partial x} = 0 \,, \end{cases} \quad (1)$$

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where h represents the z-component of the perturbed field (magnetic); $\beta = \frac{\partial f}{\partial y} = \frac{2\partial \Omega_{0z}}{\partial y}$ describes the latitudinal inhomogeneity in vertical component of angular velocity; also $c_B = \beta_B / en\mu_0$ with $\beta_B = \frac{\partial \mathrm{B}_{0z}}{\partial y}$ is the latitudinal inhomogeneity in the background magnetic field ; $J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$ is the Jacobian (vector nonlinearity) and $\Delta = \partial_x^2 + \partial_y^2$.

The boundary conditions that are fulfilled in this system are given by

$$\psi(0) = \psi(1) = 0\,, \qquad (2)$$

which represents the flow's edges, specifically along the south and north direction [1, 2].

2.1 Perturbation and weakly nonlinear approach

The background stream function is considered as:

$$\Psi(\mathrm{y}) = -\int [\mathrm{U}(\mathrm{y}) - \mathrm{c}_0]\mathrm{dy}\,. \qquad (3)$$

Here *U(y)* describes the basic background flow with c_0 as a constant eigenvalue. We consider the whole stream function ψ is the sum of background (zonal flow) stream function *Ψ(y)* and a disturbed stream ψ' function, along with a normalized small parameter $\varepsilon \ll 1$, forms a weak nonlinear problem, that is the subject of this study. While the perturbed magnetic field is also characterized by a small a parameter ε. Therefore the stream function and the magnetic perturbations takes the form,

$$\begin{aligned} &\psi = \Psi(y) + \varepsilon\psi' = -\int [U(y) - c_0]dy + \varepsilon\psi', \\ &h = \varepsilon h' \end{aligned} \qquad (4)$$

Using Eq (4) into (1) gives

$$\begin{cases} \frac{\partial \Delta\psi'}{\partial t} + (U(y) - c_0)\frac{\partial \Delta\psi'}{\partial x} + (\beta - U'')\frac{\partial \psi'}{\partial x} + \frac{\beta_B}{\mu_0\,\rho}\frac{\partial h'}{\partial x} + \varepsilon \mathrm{J}(\psi', \Delta\psi') = 0\,, \\ \frac{\partial h'}{\partial t} + \varepsilon \mathrm{J}(\psi', h') + (U(y) - c_0)\frac{\partial h'}{\partial x} + \beta_B \frac{\partial \psi'}{\partial x} + c_B \frac{\partial h'}{\partial x} = 0. \end{cases} \qquad (5)$$

where $U'' = \frac{d^2 U}{dy^2}$. It is also noted that parameter ε involves magnitude of nonlinear products.

By using the multiple scale analysis, we obtain the asymptotic solution where we can take the spatial and temporal parameters as $X = \varepsilon\, x$ and time $T = \varepsilon^3 t$ respectively of such nonlinear investigations. Also, by eliminating h' from 5(b) into 5(a) we get the single equation for ψ'

$$\mathcal{L}_0(\psi) + \varepsilon^2 \mathcal{L}_1(\psi) + \varepsilon\, \mathrm{J}\left(\psi, \frac{\partial^2 \psi}{\partial y^2}\right) + \varepsilon^3\, \mathrm{J}\left(\psi, \frac{\partial^2 \psi}{\partial X^2}\right) + \varepsilon^4\, \frac{\partial^3 \psi}{\partial T \partial X^2} = 0\,. \qquad (6)$$

Throughout the apostrophe of perturbed stream function is dropped.

Here we introduce the following linear differential operators:

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$$\mathcal{L}_0 = \left[(U - c_0)\frac{\partial^2}{\partial y^2} + p(y) + \frac{\alpha(y)}{U - c_0 + c_B}\right]\frac{\partial}{\partial x}\,, \mathcal{L}_1 = \frac{\partial}{\partial T}\frac{\partial^2}{\partial y^2} + (U - c_0)\frac{\partial^3}{\partial X^3}\,, \quad (7)$$

where $\alpha(y) = \frac{\beta_B^2}{\mu_0\,\rho}$ and $p(y) = \beta - U''$. Here the parameter α involves the spatial inhomogeneous background magnetic field which is not considered before in Kaladze et al. [16].

Furthermore, we denote the disturbed stream function ψ (the asymptotic expansion) as:

$$\psi = \psi_0 + \varepsilon\,\psi_1 + \varepsilon^2\psi_2 + \cdots. \quad (8)$$

By using Eq. (8) into Eq. (6), from the lowest order $O(\varepsilon^0)$, we get the following equation with conditions of boundary:

$$\mathcal{L}_0[\psi_0] = 0, \quad \text{with} \quad \psi_0 = 0 \; for \; y = 0,1\,. \quad (9)$$

The above equation (9) is a linear differential equation. By performing a separation of variables method for $\psi_0 = A(X,T)\,\Phi_0(y)$ into this form and substitute it into Eq. (7) we get the following equation with conditions of boundary:

$$\left(\frac{d^2}{dy^2} + \frac{p(y)}{U - c_0} + \frac{\alpha(y)}{(U - c_0)(U - c_0 + c_B)}\right)\Phi_0 = 0\,, \quad \text{with} \quad \Phi_0(0) = \Phi_0(1) = 0. \quad (10)$$

Here we consider $U - c_0 \neq 0$ *and* $U - c_0 + c_B \neq 0$. This is an eigenvalue problem for eigen value c_0 . By specifying $p(y)$ and $\alpha(y)$, $\Phi_0(y)$ can be found. Since $p(y)$ and $\alpha(y)$ have dependent on the variable y, it is not easy to solve this eigen problem analytically. From the lowest order $O\,(\varepsilon^0)$, we have used a couple of approximations: the first is the wave space structure is clear; the other one is about it is not time-dependent equation. Thus, it didn't help to find information about amplitude of such waves. From the next order $O\,(\varepsilon^1)$

$$\mathcal{L}_0[\psi_1] = -\mathrm{J}\left(\psi_0, \frac{\partial^2\psi_0}{\partial y^2}\right) \equiv F_1 = A\frac{\partial A}{\partial X}\left(\frac{p(y)}{U - c_0} + \frac{\alpha}{(U - c_0)(U - c_0 + c_B)}\right)_y \Phi_0^2\,, \quad (11)$$

Furthermore, we use $\psi_1 = \frac{1}{2}\,A^2(X,T)\,\Phi_1(y)$ for non-singular neutral solutions into (11)

$$\left(\frac{d^2}{dy^2} + \frac{p(y)}{U - c_0} + \frac{\alpha}{(U - c_0)(U - c_0 + c_B)}\right)\Phi_1 = \left(\frac{p(y)}{U - c_0} + \frac{\alpha}{(U - c_0)(U - c_0 + c_B)}\right)_y \frac{\Phi_0^2}{(U - c_0)}, \quad (12)$$

from the given boundary conditions $\Phi_1(0) = \Phi_1(1) = 0$. To get amplitude we will solve next such as $O(\varepsilon^2)$ which gives

$$\mathcal{L}_0[\psi_2] = -\mathcal{L}_1[\psi_0] - \mathrm{J}\left(\psi_0, \frac{\partial^2\psi_1}{\partial y^2}\right) - \mathrm{J}\left(\psi_1, \frac{\partial^2\psi_0}{\partial y^2}\right) \equiv F_2, \quad (13)$$

with $\psi_2(0) = \psi_2(1) = 0$.

Here it is pointed out that the dispersion effect competes with weakly nonlinear effect.

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Furthermore, we take $\psi_2 = B(X,T)\Phi_2(y)$ and multiply Eq. (13) by ψ_0 and integration over y give

$$\int_0^1 dy\, \frac{F_2}{U-c_0}\Phi_0 = 0\,. \tag{14}$$

By substituting F_2 and use $\psi_1 = \frac{1}{2}\, A^2(X,T)\, \Phi_1(y)$ into above equation (14) give the modified KdV (MKdV) equation

$$\frac{\partial A}{\partial T} + N\, A^2\, \frac{\partial A}{\partial X} + D\, \frac{\partial^3 A}{\partial X^3} = 0\,. \tag{15}$$

Here

$$N = \frac{I_2}{I_0}\,, \qquad D = -\frac{I_1}{I_0}, \tag{16}$$

$$\begin{cases} I_0 = \int_0^1 dy\, \Phi_0^2(y)\, [\frac{p(y)}{(U-c_0)^2} + \frac{\alpha}{(U-c_0)^2(U-c_0+c_B)}], \\ I_1 = \int_0^1 dy\, \Phi_0^2(y) \\ I_2 = \int_0^1 dy\, \frac{\Phi_0^2(y)}{U-c_0} \begin{Bmatrix} \frac{3}{2}\left(\frac{p(y)}{U-c_0} + \frac{\alpha}{(U-c_0)(U-c_0+c_B)}\right)_y \Phi_1(y) \\ -\frac{1}{2}\Phi_0^2(y)\left[\left(\frac{p(y)}{U-c_0} + \frac{\alpha}{(U-c_0)(U-c_0+c_B)}\right)_y \frac{1}{U-c_0}\right]_y \end{Bmatrix}, \end{cases} \tag{17}$$

Kaladze et al. [16] and Jian et al. [17] also obtained the same MKdV equation for Rossby waves as equation (15) and pointed out that the background flow shear is a necessary condition for the existence of solitary waves, whereas in this work, we get the MKdV for the Rossby-Khantadze waves where the coefficients have been modified by inclusion of inhomogeneity in geomagnetic field. Moreover, the effect of shear basic flow on the spatial structure, propagation velocity and wave width of solitary Rossby waves have been studied.

Here the function $\beta(y)$, $\alpha(y)$ and $U(y)$ are related to the coefficients N and D.

The solution of modified KdV equation (15) is,

$$A(x,t) = \pm\sqrt{\frac{6c}{N}}\, sech\left(\sqrt{\frac{c}{D}}\,(x-ct)\right). \tag{18}$$

3. Discussion

In the given paper, we studied the nonlinear dynamics of large-scale electromagnetic Rossby-Khantadze waves with zonal flows in E-ionospheric plasma. Both the latitudinal inhomogeneities in angular velocity of the earth's rotation and the geomagnetic field are taken into account. The inhomogeneity of the magnetic field with latitude is rtesponsible for coupled and Rossby–Khantadze waves. Such coupling results in an appearance of dispersion of Khantadze waves. To derive the nonlinear modified KdV we particularly, used the weakly nonlinear perturbation and multiple scale approaches. From the lowest order of O (ε^0), we get an eigen value problem with constant eigen value c_0 alongwith boundary conditions. The parameters $p(y)$ and $\alpha(y)$ have dependent on the variable y, it is not easy to solve this eigen

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problem analytically. From the next order O (ε^1), by using separation of variables techniques and after doing some mathematical manipulations we get modified KdV equation of one dimension. The derived quantity $\sqrt{\frac{6c}{N}}$ describes the amplitude of solitary waves. The obtained coefficients N and D depend on spatial inhomogeneous Coriolis force $\alpha(y)$ and background magnetic field $\beta(y)$, respectively. The parameter $\sqrt{\frac{6c}{N}}$ obtained above describes the amplitude of obtained such Rossby-Khantadze waves.

AUTHOR DECLARATIONS:

Conflict of Interest

The authors have no conflicts to disclose.

Data Availability

The data that support the findings of this study are available within the article.

References

[1] Pedlosky J., Geophysical Fluid Dynamics, Springer-Verlag, New York, 1987.
[2] Satoh M., Atmospheric Circulation Dynamics and General Circulation Models, Springer, New York, 2004
[3] Champeaux S. & Diamond P.H., Streamer and zonal flow generation from envelope modulations in drift wave turbulence, Phys. Lett. A **288**, 214–219, 2001.
[4] Shukla P.K., & Stenflo L., Generation of zonal flows by Rossby waves, Phys. Lett. A, **307**, 154–157, 2003.
[5] Onishchenko O.G., Pokhotelov O.A., Sagdeev R.Z., Shukla P.K., & Stenflo L., Generation of zonal flows by Rossby waves in the atmosphere, Nonlinear Process. Geophys. **11**, 241–244, 2004.
[6] Kaladze T.D., Wu D.J., Pokhotelov O.A., Sagdeev R.Z., Stenflo L., & Shukla P.K., Rossby-wave driven zonal flows in the ionospheric E-layer, J. Plasma Phys. **73**, 131–140, 2007.
[7] Kaladze T.D., Pokhotelov O.A., Stenflo L., Rogava J., Tsamalashvili L.V., & Tsik-Lauri M., Zonal flow interaction with Rossby waves in the Earth's atmosphere: a numerical simulation, Phys. Lett. A **372**, 5177–5180, 2008.
[8] Benkadda S., Klochkov D.N., Popel S.I., & Izvekova Yu.N., Nonlinear excitation of zonal flows and streamers in plasmas, Phys. Plasmas **18**, 052306, 2011.
[9] Kaladze T., Kahlon L., Horton W., Pokhotelov O., & Onishchenko O., Shear flow driven Rossby-Khantadze electromagnetic planetary vortices in the ionospheric E-layer, Europhys. Lett. **106**, 29001, 2014.
[10] Kaladze T., Magnetized Rossby waves in the Earth's ionosphere, Plasma Phys. Rep. **25** (4), 284–287, 1999.
[11] Futatani S., Horton W., Kahlon L.Z., & Kaladze T.D., Rossby-Khantadze electromagnetic planetary waves driven by sheared zonal winds in the E-layer ionosphere, Phys. Plasmas **22**, 012906, 2015.
[12] Qiang Z., Zuntao F., & Shikuo L., Equatorial envelope Rossby solitons in a shear flow, Adv. Atmos. Sci. **18**, 418–428, 2001.
[13] Bostrom R., Observations of weak double layers on auroral field lines, IEEE Trans. Plasma Sci. **20**, 756–763, 1992.
[14] Lindqvist P.A., Marklud G.T., & Blomberg L.G., Plasma characteristics determined

by the Freja electric field instrument, Space Sci. Rev. **70**, 593–602, 1994.
[15] Dovner P.O., Eriksson A.I., Bostrom R., Holback B., Freja multiprobe observations of electrostatic solitary structures, Geophys. Res. Lett. **21**, 1827–1830, 1994.
[16] Kaladze T., Tsamalashvili L., Kaladze D., Ozcan O., Yesil A., & Inc M., Modified KdV equation for magnetized Rossby waves in a zonal flow of the ionospheric E-layer, Physics Letter A, **383**, 125888, 2009.
[17] Jian S., & Lian-Gui Y., Modified KdV equation for solitary Rossby waves with β effect in barotropic fluids, Chin. Phys. B **18** (7) 2873–2877, 2009.
[18] Shi1 Y., Yang D., Feng X., Qi J., Yang H. and Yin B., One possible mechanism for eddy distribution in zonal current with meridional shear, Scientific Reports, **8**, 10106, 2018, DOI:10.1038/s41598-018-28465-z.