

1 **Brief Communication: Modified KdV equation for Rossby-Khantadze waves in a**
2 **sheared zonal flow of the ionospheric E-layer**

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15
16 **Abstract**
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18 The system of nonlinear equations for electromagnetic Rossby-Khantadze waves in a
19 weakly ionized conductive ionospheric E-layer plasma with sheared zonal flow is given. Use
20 of multiple-scale analysis allows reduction of obtained set of equations to (1+1)D nonlinear
21 modified KdV (mKdV) equation **with cubic nonlinearity** describing the propagation of solitary
22 Rossby-Khantadze solitons.
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24 **Keywords:** Rossby-Khantadze waves; nonlinearity, sheared zonal flow
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26 **1. Introduction**
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28 Different satellite and ground-based investigations indicate presence of zonal flows in
29 various atmospheric regions around the Earth (Pedlosky, 1987). **The reason for the existence**
30 **of zonal flows is the non-uniform warming of the Earth's atmospheric regions by the sun. The**
31 **presence of sheared flow along the meridians with inhomogeneous velocity, is closely**
32 **connected with the ultra-low-frequency perturbations in ionospheric E and F regions of the**
33 **ionosphere (Satoh, 2004; Shukla et al., 2003; Onishchenko et al. 2004; Kaladze et al., 2007;**
34 **Kaladze et al., 2008). Effects of sheared flow appear in linear and nonlinear properties of the**
35 **waves, and conditions suitable for that are available in Earth's ionosphere. This gives rise to a**
36 **variety of nonlinear phenomena like formation of solitary structures (solitons, vortices, zonal**
37 **flows, etc.).**

38 **Due to a significant role in the global atmospheric circulation Rossby waves attract**
39 **special scientific attention in connection with sheared zonal flows. Note that spatial**
40 **nonhomogeneity of Coriolis parameter alongwith ambient geomagnetic field along the**
41 **meridians causes the propagation of such coupled Rossby-Khantadze (RK) electromagnetic**
42 **(EM) waves (see e.g. Kaladze et al. 2011). The generation of sheared RK EM planetary vortices**
43 **in the ionospheric E-region also discussed (Kaladze et al., 2011; Kaladze et al., 2014). It was**
44 **revealed, that propagation of coupled EM RK waves could be self-organized into solitary**
45 **dipolar vortices and the possibility of the generation of intensive magnetic field is shown. In**
46 **recent decades, several nonlinear phenomena related to the excitation of sheared zonal flows**
47 **by EM Rossby waves were investigated. Taking into account Reynolds stresses zonal flow**
48 **generations by short wavelength EM Rossby waves studied (Shukla et al., 2003; Onishchenko**
49 **et al. 2004). The zonal flow' generation in the ionospheric E-layer by Rossby waves revealed**

50 by Kaladze et al. (2007). Such nonlinear Rossby wave structures broken into numerous parts
51 depends on the zonal flow energy (Kaladze et al., 2008). Numerical work on EM RK waves
52 with sheared zonal flow in ionospheric E-plasma was found as well (Futatani et al., 2013,
53 2015). In this work it was pointed out the splitting of vortices, where the energy is transported
54 by sheared flow into multiple pieces. Equatorial Rossby wave solitons under the action of
55 sheared flows were also discussed (Qiang et al., 2001) and the existence of solitons was
56 confirmed by the observations of *Freja and Viking satellites* (Qiang et al., 2001; Bostrom,
57 1992; Dovner et al. 1994; Lindqvist et al., 1994). Jian et al., (2009) investigated nonlinear
58 propagation of Rossby waves in stratified neutral fluids with zonal shear flow and obtained
59 modified Korteweg-de Vries (mKdV) equation with cubic nonlinearity. Generation of the zonal
60 flow alongwith magnetic field in the ionospheric E-plasma by Rossby-Khantadze EM
61 planetary waves also discussed (Kaladze et al. 2012, Kahlon and Kaladze 2015). Possibility of
62 magnetic field generation of 10^3 nT is predicted. Kaladze et al. (2019) investigated nonlinear
63 interaction of magnetized electrostatic Rossby waves with sheared zonal flows in the Earth's
64 ionospheric E-layer and developed the modified Korteweg-de Vries (mKdV) equation having
65 cubic nonlinearity describing propagation of appropriate solitons. Some premises of the
66 possibility of existence of planetary Rossby waves in the dynamo E-area of weakly ionized
67 ionosphere and corresponding experimental interpretation was discussed by Forbes, 1996.
68 Also, Vukcevic M. and Popovic L. Č., (2020) pointed out the possibility of many soliton
69 structure formations at different latitudes, and at diverse ionospheric layers. Direct
70 observations of such soliton structures from the surface of Earth or onboard the satellites are
71 discussed.

72 In the given manuscript, we generalize mentioned above results for the weakly ionized
73 conducting ionospheric E-region plasma by incorporating along with stream-function
74 evolution of geomagnetic field for electromagnetic RK waves, which to the best of our
75 knowledge was not reported so far and thus provides novelty to this work. In Sec. 2, from the
76 obtained system of nonlinear two-dimensional equations by using the multiple scale analysis
77 and perturbation approach we derive one-dimensional mKdV equation with cubic nonlinearity
78 describing solitary Rossby-Khantadze waves dynamics along with zonal (shear) flows. Sec. 3
79 includes the discussion of the results.

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82 2. Mathematical Preliminaries

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84 We consider partially ionized E-ionospheric region consisting of small concentration of
85 electrons, ions and bulk of neutral particles, where such ionospheric plasma is enclosed in a
86 spatially inhomogeneous geomagnetic field $\mathbf{B}_0 = (0, B_{0y}, B_{0z})$ and the Earth's angular
87 velocity $\boldsymbol{\Omega} = (0, \Omega_{0y}, \Omega_{0z})$. In weakly ionized ionospheric E-layer plasma, we consider two-
88 dimensional' wave motion $\mathbf{v} = (u, v, 0)$, where $u = -\frac{\partial\psi}{\partial y}$, $v = \frac{\partial\psi}{\partial x}$ and $\psi(x, y, t)$ is the stream
89 function.

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91 We consider a local Cartesian system of coordinates with zonal x, latitudinal y, and z
92 in local vertical direction. Then the nonlinear behavior of the sheared electromagnetic Rossby-
93 Khantadze waves can be narrated by the following 2D system of equations (e.g. Kaladze et al.,
94 2014),

$$95 \begin{cases} \frac{\partial\Delta\psi}{\partial t} + \beta \frac{\partial\psi}{\partial x} + J(\psi, \Delta\psi) - \frac{1}{\mu_0\rho} \beta_B \frac{\partial h}{\partial x} = 0, \\ \frac{\partial h}{\partial t} + J(\psi, h) + \beta_B \frac{\partial\psi}{\partial x} + c_B \frac{\partial h}{\partial x} = 0, \end{cases} \quad (1)$$

96 The first equation describes the evolution of the z-component of vorticity ($\zeta_z = \mathbf{e}_z \cdot$
97 $\nabla \times \mathbf{v} = \Delta\psi$) of the singly fluid momentum equation under the action of the geomagnetic field,
98 \mathbf{v} is the velocity of the incompressible neutral gas. The second equation is the z-component of
99 the perturbed magnetic induction h obtained through Faraday's law, and $\beta = \frac{\partial f}{\partial y} = \frac{2\partial\Omega_{0z}}{\partial y}$
100 describes the latitudinal inhomogeneity of angular velocity. Also the parameter $c_B = \beta_B/en\mu_0$
101 with $\beta_B = \frac{\partial B_{0z}}{\partial y}$, describes the latitudinal inhomogeneity in the background magnetic field, n
102 is the number density of the charged particles, μ_0 is the magnetic permeability and $J(a, b) =$
103 $\frac{\partial a}{\partial x} \frac{\partial b}{\partial y} - \frac{\partial a}{\partial y} \frac{\partial b}{\partial x}$ is the Jacobian (responsible for the vector nonlinearity) and $\Delta = \partial_x^2 + \partial_y^2$. Note
104 that the small concentration of charged particles (compared to the neutral particles) gives the
105 contribution only in the inductive current (Kaladze, et al. 2013a, 2013b). It should also be noted
106 that the ambient magnetic field and Coriolis parameter are spatially inhomogeneous, (Kaladze,
107 et al., 2014). Details on the system (1) can be found in Kaladze, et al. (2012).

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109 The boundary conditions that are fulfilled for this system are given as,

$$110 \quad \psi(0) = \psi(1) = 0, \quad (2)$$

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113 which represents the flow's edges, specifically along the south and north direction (Pedlosky
114 (1987); Satoh (2004)).
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116 2.1 Perturbation and weakly nonlinear approach

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118 The background stream function is considered in the following manner:

$$119 \quad \Psi(y) = - \int [U(y) - c_0] dy. \quad (3)$$

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121 Here $U(y)$ describes the basic background flow with c_0 as a constant eigenvalue. The whole
122 stream function ψ is considered as the sum of background (zonal flow) stream function $\Psi(y)$
123 and a disturbed stream ψ' function. This assumption makes it a weakly nonlinear system, that
124 is the subject of this study. While the perturbed magnetic field is also characterized by a small
125 a parameter ε . Therefore the stream function and the magnetic perturbations takes the form,

$$126 \quad \begin{aligned} \psi &= \Psi(y) + \varepsilon\psi' = - \int [U(y) - c_0] dy + \varepsilon\psi', \\ h &= \varepsilon h' \end{aligned} \quad (4)$$

127
128 where $\varepsilon \ll 1$ is a small parameter indicating that the perturbed quantities are small compared
129 to the background parameters.

130 Using Eq (4) into (1) gives

$$131 \quad \begin{cases} \frac{\partial \Delta\psi'}{\partial t} + (U(y) - c_0) \frac{\partial \Delta\psi'}{\partial x} + (\beta - U'') \frac{\partial \psi'}{\partial x} + \frac{\beta_B}{\mu_0 \rho} \frac{\partial h'}{\partial x} + \varepsilon J(\psi', \Delta\psi') = 0, \\ \frac{\partial h'}{\partial t} + \varepsilon J(\psi', h') + (U(y) - c_0) \frac{\partial h'}{\partial x} + \beta_B \frac{\partial \psi'}{\partial x} + c_B \frac{\partial h'}{\partial x} = 0. \end{cases} \quad (5)$$

132
133 where $U'' = \frac{d^2 U}{dy^2}$.
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136 By using the multiple scale analysis, we obtain the asymptotic solution where we take
 137 the spatial and temporal parameters as $X = \varepsilon x$ and time $T = \varepsilon^3 t$ respectively. Further by
 138 eliminating h' from 5(b) into 5(a) we get the single equation for ψ'
 139

$$140 \quad \mathcal{L}_0(\psi) + \varepsilon^2 \mathcal{L}_1(\psi) + \varepsilon \mathcal{J}\left(\psi, \frac{\partial^2 \psi}{\partial y^2}\right) + \varepsilon^3 \mathcal{J}\left(\psi, \frac{\partial^2 \psi}{\partial X^2}\right) + \varepsilon^4 \frac{\partial^3 \psi}{\partial T \partial X^2} = 0. \quad (6)$$

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 142 In Eq. (6) the prime on the perturbed stream function is dropped, and the following linear
 143 differential operators are introduced
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$$146 \quad \mathcal{L}_0 = \left[(U - c_0) \frac{\partial^2}{\partial y^2} + p(y) + \frac{\alpha(y)}{U - c_0 + c_B} \right] \frac{\partial}{\partial x}, \quad \mathcal{L}_1 = \frac{\partial}{\partial T} \frac{\partial^2}{\partial y^2} + (U - c_0) \frac{\partial^3}{\partial X^3}, \quad (7)$$

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 148 where $\alpha(y) = \frac{\beta_B^2}{\mu_0 \rho}$ and $p(y) = \beta - U''$. Here the parameter α takes into account the spatial
 149 inhomogeneity of the background magnetic field which was not considered before in Kaladze
 150 et al. (2019).
 151

152 Furthermore, we expand the stream function ψ (in series with respect to the ε) as:

$$153 \quad \psi = \psi_0 + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \dots \quad (8)$$

154
 155 By using Eq. (8) into Eq. (6), we obtain from the lowest order $O(\varepsilon^0)$, the following
 156 equations,
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$$159 \quad \mathcal{L}_0[\psi_0] = 0, \quad \text{with} \quad \psi_0 = 0 \text{ for } y = 0, 1. \quad (9)$$

160
 161 The above equation (9) is a linear differential equation. By performing a separation of variables
 162 method for $\psi_0 = A(X, T) \Phi_0(y)$ into this form and substitute it into Eq. (7) we get the
 163 following equation with conditions of boundary:
 164

$$165 \quad \left(\frac{d^2}{dy^2} + \frac{p(y)}{U - c_0} + \frac{\alpha(y)}{(U - c_0)(U - c_0 + c_B)} \right) \Phi_0 = 0, \quad \text{with} \quad \Phi_0(0) = \Phi_0(1) = 0. \quad (10)$$

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 167 Here we consider $U - c_0 \neq 0$ and $U - c_0 + c_B \neq 0$. This is an eigenvalue problem for eigen
 168 value c_0 . By specifying $p(y)$ and $\alpha(y)$, $\Phi_0(y)$ can be found. Since $p(y)$ and $\alpha(y)$ have
 169 dependence on the variable y , it is not easy to solve this eigen value problem analytically. From
 170 the lowest order $O(\varepsilon^0)$, we see that the problem is time independent, but cannot be analytically
 171 solved as we have not substituted any definite dependence on y for the parameters $p(y)$ and
 172 $\alpha(y)$. Thus, in order to get more details about the amplitude of these waves, we go to the next
 173 order i.e. $O(\varepsilon^1)$ from Eqs. (7) and (8), we obtain
 174

$$175 \quad \mathcal{L}_0[\psi_1] = -\mathcal{J}\left(\psi_0, \frac{\partial^2 \psi_0}{\partial y^2}\right) \equiv F_1 = A \frac{\partial A}{\partial X} \left(\frac{p(y)}{U - c_0} + \frac{\alpha}{(U - c_0)(U - c_0 + c_B)} \right)_y \Phi_0^2, \quad (11)$$

176 Furthermore, we carry out a separation of variables in the following manner $\psi_1 =$
 177 $\frac{1}{2} A^2(X, T) \Phi_1(y)$ for non-singular neutral solutions into (11)
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$$179 \quad \left(\frac{d^2}{dy^2} + \frac{p(y)}{U - c_0} + \frac{\alpha}{(U - c_0)(U - c_0 + c_B)} \right) \Phi_1 = \left(\frac{p(y)}{U - c_0} + \frac{\alpha}{(U - c_0)(U - c_0 + c_B)} \right)_y \frac{\Phi_0^2}{(U - c_0)}, \quad (12)$$

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For the given boundary conditions $\Phi_1(0) = \Phi_1(1) = 0$. To get amplitude we solve Eqs. (7) and (8) in the next order i.e. $O(\varepsilon^2)$ which gives

$$\mathcal{L}_0[\psi_2] = -\mathcal{L}_1[\psi_0] - J\left(\psi_0, \frac{\partial^2 \psi_1}{\partial y^2}\right) - J\left(\psi_1, \frac{\partial^2 \psi_0}{\partial y^2}\right) \equiv F_2, \quad (13)$$

with $\psi_2(0) = \psi_2(1) = 0$.

Here it is pointed out that the dispersion effect, given in the definition of \mathcal{L}_1 competes with weakly nonlinear effect, which appears through the Jacobian in Eq. (9).

Furthermore, we again perform a separation of variables, $\psi_2 = B(X, T)\Phi_2(y)$ and multiply Eq. (13) by ψ_0 and integrate over y , which yields

$$\int_0^1 dy \frac{F_2}{U-c_0} \Phi_0 = 0. \quad (14)$$

By substituting F_2 and using $\psi_1 = \frac{1}{2} A^2(X, T) \Phi_1(y)$ into Eq. (14) we get the modified KdV (mKdV) equation (Kaladze et. al (2019))

$$\frac{\partial A}{\partial T} + N A^2 \frac{\partial A}{\partial X} + D \frac{\partial^3 A}{\partial X^3} = 0. \quad (15)$$

This equation has a cubic nonlinearity, whereas the standard KdV equation has a quadratic nonlinearity. In Eq.(15) above

$$N = \frac{I_2}{I_0}, \quad D = -\frac{I_1}{I_0}, \quad (16)$$

where

$$\left\{ \begin{array}{l} I_0 = \int_0^1 dy \Phi_0^2(y) \left[\frac{p(y)}{(U(y)-c_0)^2} + \frac{\alpha}{(U(y)-c_0)^2(U(y)-c_0+c_B)} \right], \\ I_1 = \int_0^1 dy \Phi_0^2(y) \\ I_2 = \int_0^1 dy \frac{\Phi_0^2(y)}{U(y)-c_0} \left\{ \begin{array}{l} \frac{3}{2} \left(\frac{p(y)}{U(y)-c_0} + \frac{\alpha}{(U(y)-c_0)(U(y)-c_0+c_B)} \right)_y \Phi_1(y) \\ -\frac{1}{2} \Phi_0^2(y) \left[\left(\frac{p(y)}{U(y)-c_0} + \frac{\alpha}{(U(y)-c_0)(U(y)-c_0+c_B)} \right)_y \frac{1}{U(y)-c_0} \right]_y \end{array} \right\} \end{array} \right. \quad (17)$$

Kaladze et al. (2019) and Jian et al. (2009) also obtained the same mKdV equation (15) with cubic nonlinearity for Rossby waves and pointed out that the background flow shear is a necessary condition for the existence of solitary waves, whereas in this work, we get the mKdV for the Rossby-Khantadze waves where the coefficients have been modified by inclusion of inhomogeneity in geomagnetic field. Moreover, the effect of shear basic flow on the spatial structure, propagation velocity and wave width of solitary Rossby waves have been studied. We would like to point out here the meridional dependence of functions $\beta(y)$, $\alpha(y)$ and $U(y)$, that appears in the coefficients N and D .

Amid numerous exact solutions of mKdV equation (15) (see e.g. Wazwaz (2009)), we are interested in a soliton like traveling wave solution. The one-soliton solution of equation (15) is

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$$A(X, T) = \pm \sqrt{\frac{6c}{N}} \operatorname{sech} \left(\sqrt{\frac{c}{D}} (X - cT) \right), \quad (18)$$

220 where c is the traveling wave velocity, and the coefficients N and D are defined by Eqs. (16)-
 221 (17). In order for a wave to have an exact solitary solution associated to it, one needs to a
 222 robust equation like the KdV. Modified KdV, as well, has infinite conservation laws
 223 associated to them, and hence is integrable and contain one and N -soliton solution. Shown in
 224 the above equation is the one soliton solution of the mKdV. One can use the Hirota's method,
 225 where by using a suitable transformation, one converts the nonlinear equation into a bilinear
 226 equation, and then by using the Hirota's differential operator and solving the subsequent
 227 equation, one can obtain a multi-soliton solution. Some types of mKdV spatially periodic
 228 solutions (cnoidal solutions) discussed (Kevrekidis et al. 2004). It was noted that mKdV
 229 equation having nonlinear term may have an alternate sign. Properties of such difference also
 230 discussed.

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233 3. Discussion

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235 In the present paper, we have studied the nonlinear dynamics of large-scale
 236 electromagnetic Rossby-Khantadze waves with zonal flows in E-ionospheric plasma. Both the
 237 latitudinal inhomogeneities in angular velocity of the earth's rotation and the geomagnetic field
 238 are taken into account. The latitudinal inhomogeneity of the magnetic field is responsible for
 239 coupled Rossby-Khantadze waves. Such coupling results in an appearance of dispersion of
 240 Khantadze waves. To derive the nonlinear modified KdV we used the multiple scale analysis
 241 technique. From the lowest order of $O(\varepsilon^0)$, we get an eigen-value problem with constant eigen-
 242 value c_0 along with the boundary conditions. The parameters $p(y)$ and $\alpha(y)$ have dependence
 243 on the variable y , making it not possible to solve this eigen value problem analytically. From
 244 the next order $O(\varepsilon^1)$, by using separation of variables techniques and after doing some
 245 mathematical manipulations we arrive at the mKdV equation (15) with cubic nonlinearity of
 246 (1+1) dimension. Traveling wave solitary solution of this equation is given by Eq. (18), where

247 the parameter $\sqrt{\frac{6c}{N}}$ describes the amplitude of solitary RK structures. The obtained coefficients
 248 N and D depend on the spatially inhomogeneous Coriolis force $\alpha(y)$ and background magnetic
 249 field $\beta(y)$, respectively.

250 In anticipation of future for the experimental observations of RK vortical motions in
 251 the weakly ionized ionospheric E-layer we expect the following characteristics. Apart from the
 252 ordinary Rossby waves electromagnetic RK perturbations generated by the latitudinal gradient
 253 of the geomagnetic field and represent the variation of the vortical electric field $\mathbf{E}_v = \mathbf{v}_D \times \mathbf{B}_0$,
 254 where $\mathbf{v}_D = \mathbf{E} \times \mathbf{B}_0 / B_0^2$ is the electron drift velocity. RK waves propagate along the latitude
 255 with the velocity $|c_B| \approx 2 - 20 \text{ km/s}$. Frequency ($\omega = k_x c_B$) and the phase velocity c_B
 256 depend on the number density of the charged particles and vary by one order of magnitude
 257 during the daytime and nighttime conditions (which is so suitable for experimental
 258 observations). Such perturbations have relatively high frequency ($10^4 - 10^{-1} \text{ s}^{-1}$) and have
 259 wavelengths $\sim 10^3 \text{ km}$. Compared with the ordinary Rossby waves electromagnetic RK waves
 260 accompanied by the strong pulsations of the geomagnetic field 20-80 nT. Note that Khantadze
 261 waves in the middle and moderate latitudes observed at the launching of spacecrafts Burmaka,
 262 et al. (2006) and by the world network of ionospheric and magnetic observations Sharadze, et
 263 al. (1988); Sharadze, et al. (1989); Sharadze, (1991); Alperovich, et al. (2007). Forbes (1996)
 264 provides data analyses for discussing the penetration of Rossby type planetary waves effects
 265 into ionospheric dynamo E-region (100-170 km) and the electrodynamic interactions which
 266 ensue there.

267
268 **RK** waves are mainly of zonal type and observed mainly during magnetic storms
269 along with sub-storms, artificial explosions, earthquakes, etc. They give valuable information
270 on large-scale synoptic processes and about external sources as well as dynamical processes in
271 the ionosphere. Therefore, theoretical investigations of electromagnetic Rossby type
272 oscillations will **provide** valuable information for further ionospheric experimental
273 investigations.

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276 **AUTHOR DECLARATIONS:**

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278 **Conflict of Interest**

279 The authors have no conflicts to disclose.

280

281 **Data Availability**

282 The data that support the findings of this study are available within the article.

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