Brief Communication: Modified KdV equation for Rossby-Khantadze waves in a sheared zonal flow of the ionospheric E-layer

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Abstract

The system of nonlinear equations for Rossby-Khantadze waves in a weakly ionized conductive ionospheric plasma with sheared zonal flow is given. Use of multiple-scale analysis allows reduction of obtained set of equations to one-dimensional so called nonlinear modified KdV (MKdV) equation describing the propagation of solitary Rossby-Khantadze waves.

Keywords: Rossby-Khantadze waves; nonlinearity, sheared zonal flow

1. Introduction

Different satellite and ground-based investigations indicate presence of zonal flows in various atmospheric regions around the Earth (Pedlosky, 1987). The presence of sheared flow along the meridians with inhomogeneous velocity, is connected with the ultra-low-frequency perturbations in E and F regions of the ionosphere (Satoh, 2004; Champeaux et al. 2008; Shukla et al., 2003; Onishchenko et al. 2004; Kaladze T.D., 2007; Kaladze et al., 2008). Effect of sheared flow can be seen in the linear and nonlinear properties of the waves, and conditions suitable for that are available in Earth's ionosphere. This gives rise to a variety of linear and nonlinear phenomena like zonal flows. Rossby waves may exist and this may occur in the upper atmosphere and in the oceans; and these play a significant role in the global atmospheric circulation.

The reason for the existence of zonal flows is the non-uniform warming of the Earth's atmospheric regions by the sun. In recent decades, several nonlinear phenomena related to the excitation of sheared (zonal) flows were investigated. For instance, Benkadda et al., (2011) investigated the excitation of zonal flows in fusion plasmas by taking drift waves into account. Using Hasegawa Wakatani model, Champeaux et al. (2008) investigated the excitation of zonal flows in drift wave turbulence. Lately, zonal flow generations for short wavelength electromagnetic Rossby waves have also been studied by taking Reynolds stresses into account (Shukla et al., 2003; Onishchenko et al. 2004).

The production of zonal flow in E-ionospheric layer by Rossby waves was investigated by Kaladze et al. (2007). The authors considered the effects of the zonal flows on nonlinear structures in Rossby waves and it was shown that such structures split into various segments based on the collection of zonal flow energy (Kaladze et al., 2008). A different concept was considered by Benkadda et al. (2011) where they emphasized on the interaction of high frequency drift waves with those having lower frequencies.

More recently, it was seen, that propagation of coupled Rossby-Khantadze (RK) waves could be self-organized into dipolar (solitary) vortices (Kaladze, 2014). The spatial inhomogeneity of Coriolis parameter and ambient magnetic field along the meridians causes the propagation of such coupled RK waves. The numerical simulation of the same problem was carried out by Kaladze, et al. (1999) where the possibility of modulation mechanism in ionospheric plasmas was discussed, in turn leading to the formation and transport of dust flows and particles respectively. Numerical solutions of RK waves with sheared zonal flow in weakly ionized ionospheric E-plasma region was investigated as well (Futatani et al., 2015). In his work he has pointed out the splitting of vortices, where the energy is transported by sheared flow into multiple pieces. The equatorial Rossby waves were discussed under similar sheared flows in the beginning of the millennia (Qiang et al., 2001) and the existence of solitons was confirmed by the observations of *Freja and Viking satellites* (Qiang et al., 2001; Bostrom, 1992; Lindqvist et al., 1994; Dovner et al. 1994, YunLong Shi et al. 2018). The generation of shear flow by Rossby-Khantadze waves in E-ionospheric region has also been discussed (Kaladze, et al. 2014). Earlier, Kaladze et al. (2009) had earlier investigated the properties of magnetized solitary Rossby waves with the interaction of zonal flows (sheared) and developed the mKdV equation. By considering the β-plane approximation, similar work was done by Jian et al., (2009).

In the present paper, we have considered the effect of magnetic field for the MKdV equation for RK waves. Such case has not been reported so far and thus provides novelty to this work. Here we have investigated solitary Rossby-Khantadze (RK) waves by incorporating sheared zonal flows in a partially ionized conducting plasma, found in the ionospheric E-region. In Sec. 2, by using the multiple scale analysis and perturbation approach from a system of nonlinear two-dimensional equations we derive one-dimensional MKdV equation describing solitary Rossby-Khantadze waves' dynamics along with zonal (shear) flows. In Sec. 3, includes the discussion of the results.

2. Mathematical Preliminaries

We consider weakly ionized E-ionospheric region comprising of electrons, ions and neutrals particles, where the ionospheric plasma is enclosed in a geomagnetic field $\boldsymbol{B}_0 = (0, B_{0y}, B_{0z})$. Similarly, the angular velocity of the earth contains no x-component, $\boldsymbol{\Omega} = (0, \Omega_{0y}, \Omega_{0z})$. In this layer therefore, two-dimensional consideration of the wave motion provides complete information about its propagation in terms of stream function $\psi(x, y, t)$, $\mathbf{v} = (u, \mathrm{v}, 0)$, with $u = -\frac{\partial\psi}{\partial y}$ and $\mathrm{v} = \frac{\partial\psi}{\partial x}$.

The nonlinear behavior of the sheared Rossby-Khantadze waves can be described by the following system of 2D equations, where the first equation is obtained from the z-component of the curl of vorticity with $\zeta_z = \Delta\psi$ and the second one is obtained from the z-component of magnetic field, through Faraday's law. Here we obtained the following system of Eqs. (1) under the assumption that electron and ion flows due to the small concentration number (compared to the neutral particles) gives the contribution only in the inductive current (Kaladze, et al. 2013). The quantity $\zeta_z = \boldsymbol{e}_z \cdot \nabla \times \mathbf{v}$ is the z-component of the vorticity. It should also be noted that the ambient magnetic field and Coriolis parameter are spatially inhomogeneous, $f = 2\Omega_{0z}$ (Kaladze, et al., 2014):

$$\begin{cases} \frac{\partial\Delta\psi}{\partial t} + \beta\frac{\partial\psi}{\partial x} + \mathrm{J}(\psi, \Delta\psi) - \frac{1}{\mu_0\rho}\beta_B\frac{\partial h}{\partial x} = 0\,, \\ \frac{\partial h}{\partial t} + \mathrm{J}(\psi, h) + \beta_B\frac{\partial\psi}{\partial x} + c_B\frac{\partial h}{\partial x} = 0\,, \end{cases} \tag{1}$$

In Eq. (1), h represents the z-component of perturbed magnetic field and $\beta = \frac{\partial f}{\partial y} = \frac{2\partial \Omega_{0z}}{\partial y}$ describes the latitudinal inhomogeneity present in the vertical component of angular velocity. Also the parameter $c_B = \beta_B / en\mu_0$ with $\beta_B = \frac{\partial B_{0z}}{\partial y}$, describes the latitudinal inhomogeneity in the background magnetic field, n is the number density of the charged particles, μ_0 is the magnetic permeability and $J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$ is the Jacobian (responsible for the vector nonlinearity) and $\Delta = \partial_x^2 + \partial_y^2$.

The boundary conditions that are fulfilled in this system are given as,

$$\psi(0) = \psi(1) = 0\,, \qquad (2)$$

which represents the flow's edges, specifically along the south and north direction [1, 2].

2.1 Perturbation and weakly nonlinear approach

The background stream function is considered as:

$$\Psi(\mathrm{y}) = -\int [\mathrm{U}(\mathrm{y}) - \mathrm{c}_0]\mathrm{dy}\,. \qquad (3)$$

Here *U*(y) describes the basic background flow with c_0 as a constant eigenvalue. The whole stream function ψ is considered as the sum of background (zonal flow) stream function *Ψ*(y) and a disturbed stream ψ' function, along with a normalized small parameter $\varepsilon \ll 1$. Which forms a weakly nonlinear system, that is the subject of this study. While the perturbed magnetic field is also characterized by a small a parameter ε. Therefore the stream function and the magnetic perturbations takes the form,

$$\begin{aligned} &\psi = \Psi(y) + \varepsilon\psi' = -\int [U(y) - c_0]dy + \varepsilon\psi', \\ &h = \varepsilon h' \end{aligned} \qquad (4)$$

Using Eq (4) into (1) gives

$$\begin{cases} \frac{\partial \Delta\psi'}{\partial t} + (U(y) - c_0)\frac{\partial \Delta\psi'}{\partial x} + (\beta - U'')\frac{\partial \psi'}{\partial x} + \frac{\beta_B}{\mu_0\,\rho}\frac{\partial h'}{\partial x} + \varepsilon \mathrm{J}(\psi', \Delta\psi') = 0\,, \\ \frac{\partial h'}{\partial t} + \varepsilon \mathrm{J}(\psi', h') + (U(y) - c_0)\frac{\partial h'}{\partial x} + \beta_B\frac{\partial \psi'}{\partial x} + c_B\frac{\partial h'}{\partial x} = 0. \end{cases} \qquad (5)$$

where $U'' = \frac{d^2U}{dy^2}$.

By using the multiple scale analysis, we obtain the asymptotic solution where we can take the spatial and temporal parameters as $X = \varepsilon\, x$ and time $T = \varepsilon^3 t$ respectively. Further by eliminating h' from 5(b) into 5(a) we get the single equation for ψ'

$$\mathcal{L}_0(\psi) + \varepsilon^2\mathcal{L}_1(\psi) + \varepsilon\,\mathrm{J}\left(\psi, \frac{\partial^2\psi}{\partial y^2}\right) + \varepsilon^3\,\mathrm{J}\left(\psi, \frac{\partial^2\psi}{\partial X^2}\right) + \varepsilon^4\,\frac{\partial^3\psi}{\partial T\partial X^2} = 0\,. \qquad (6)$$

Throughout Eq. (6) the prime on the perturbed stream function is dropped.

Here we introduce the following linear differential operators:

$$\mathcal{L}_0 = \left[(U - c_0)\frac{\partial^2}{\partial y^2} + p(y) + \frac{\alpha(y)}{U - c_0 + c_B}\right]\frac{\partial}{\partial x}\,,\; \mathcal{L}_1 = \frac{\partial}{\partial T}\frac{\partial^2}{\partial y^2} + (U - c_0)\frac{\partial^3}{\partial X^3}\,, \quad (7)$$

where $\alpha(y) = \frac{\beta_B^2}{\mu_0\,\rho}$ and $p(y) = \beta - U''$. Here the parameter α involves the spatial inhomogeneous background magnetic field which was not considered before in Kaladze et al. [16].

Furthermore, we denote the disturbed stream function ψ (the asymptotic expansion) as:

$$\psi = \psi_0 + \varepsilon\,\psi_1 + \varepsilon^2\psi_2 + \cdots . \quad (8)$$

By using Eq. (8) into Eq. (6), from the lowest order O(ε^0), we get the following equation with conditions of boundary:

$$\mathcal{L}_0[\psi_0] = 0, \qquad \text{with} \qquad \psi_0 = 0\; for\; y = 0{,}1\,. \quad (9)$$

The above equation (9) is a linear differential equation. By performing a separation of variables method for $\psi_0 = A(X,T)\,\Phi_0(y)$ into this form and substitute it into Eq. (7) we get the following equation with conditions of boundary:

$$\left(\frac{d^2}{dy^2} + \frac{p(y)}{U - c_0} + \frac{\alpha(y)}{(U - c_0)(U - c_0 + c_B)}\right)\Phi_0 = 0\,, \quad \text{with} \quad \Phi_0(0) = \Phi_0(1) = 0. \quad (10)$$

Here we consider $U - c_0 \neq 0$ *and* $U - c_0 + c_B \neq 0$. This is an eigenvalue problem for eigen value c_0 . By specifying $p(y)$ and $\alpha(y)$, $\Phi_0(y)$ can be found. Since $p(y)$ and $\alpha(y)$ have dependence on the variable y, it is not easy to solve this eigen value problem analytically. From the lowest order $O\,(\varepsilon^0)$, we see that the problem is time independent, but cannot be analytically solved as we have not substituted any definite dependence on y for the parameters $p(y)$ and $\alpha(y)$.Thus, in order to gain more information about the amplitude of these waves, we go to the next order i.e. O (ε^1) from Eqs. (7) and (8), we obtain

$$\mathcal{L}_0[\psi_1] = -\mathrm{J}\left(\psi_0, \frac{\partial^2\psi_0}{\partial y^2}\right) \equiv F_1 = A\frac{\partial A}{\partial X}\left(\frac{p(y)}{U - c_0} + \frac{\alpha}{(U - c_0)(U - c_0 + c_B)}\right)_y \Phi_0^2\,, \quad (11)$$

Furthermore, we carry out a separation of variables in the following manner $\psi_1 = \frac{1}{2}A^2(X,T)\,\Phi_1(y)$ for non-singular neutral solutions into (11)

$$\left(\frac{d^2}{dy^2} + \frac{p(y)}{U - c_0} + \frac{\alpha}{(U - c_0)(U - c_0 + c_B)}\right)\Phi_1 = \left(\frac{p(y)}{U - c_0} + \frac{\alpha}{(U - c_0)(U - c_0 + c_B)}\right)_y \frac{\Phi_0^2}{(U - c_0)}, \quad (12)$$

For the given boundary conditions $\Phi_1(0) = \Phi_1(1) = 0$. To get amplitude we solve Eqs.(7) and (8) in the next order i.e. O(ε^2) which gives

$$\mathcal{L}_0[\psi_2] = -\mathcal{L}_1[\psi_0] - \mathrm{J}\left(\psi_0, \frac{\partial^2\psi_1}{\partial y^2}\right) - \mathrm{J}\left(\psi_1, \frac{\partial^2\psi_0}{\partial y^2}\right) \equiv F_2, \quad (13)$$

with $\psi_2(0) = \psi_2(1) = 0$.

Here it is pointed out that the dispersion effect competes with weakly nonlinear effect.

Furthermore, we take $\psi_2 = B(X,T)\Phi_2(y)$ and multiply Eq. (13) by ψ_0 and integrate over y, which yields

$$\int_0^1 dy\, \frac{F_2}{U-c_0}\Phi_0 = 0\,. \tag{14}$$

By substituting F_2 and using $\psi_1 = \frac{1}{2}A^2(X,T)\,\Phi_1(y)$ into above Eq. (14) results in the modified KdV (MKdV) equation

$$\frac{\partial A}{\partial T} + N\,A^2\,\frac{\partial A}{\partial X} + D\,\frac{\partial^3 A}{\partial X^3} = 0\,. \tag{15}$$

Here

$$N = \frac{I_2}{I_0}\,, \qquad D = -\frac{I_1}{I_0}, \tag{16}$$

$$\begin{cases} I_0 = \int_0^1 dy\,\Phi_0^2(y)\left[\frac{p(y)}{(U-c_0)^2} + \frac{\alpha}{(U-c_0)^2(U-c_0+c_B)}\right], \\ I_1 = \int_0^1 dy\,\Phi_0^2(y) \\ I_2 = \int_0^1 dy\,\frac{\Phi_0^2(y)}{U-c_0}\begin{Bmatrix} \frac{3}{2}\left(\frac{p(y)}{U-c_0} + \frac{\alpha}{(U-c_0)(U-c_0+c_B)}\right)_y \Phi_1(y) \\ -\frac{1}{2}\Phi_0^2(y)\left[\left(\frac{p(y)}{U-c_0} + \frac{\alpha}{(U-c_0)(U-c_0+c_B)}\right)_y \frac{1}{U-c_0}\right]_y \end{Bmatrix}, \end{cases} \tag{17}$$

Kaladze et al. [16] and Jian et al. [17] also obtained the same MKdV equation for Rossby waves as equation (15) and pointed out that the background flow shear is a necessary condition for the existence of solitary waves, whereas in this work, we get the MKdV for the Rossby-Khantadze waves where the coefficients have been modified by inclusion of inhomogeneity in geomagnetic field. Moreover, the effect of shear basic flow on the spatial structure, propagation velocity and wave width of solitary Rossby waves have been studied.

Here the function $\beta(y)$, $\alpha(y)$ and $U(y)$ are related to the coefficients N and D.

The solution of modified KdV equation (15) is,

$$A(x,t) = \pm\sqrt{\frac{6c}{N}}\,sech\left(\sqrt{\frac{c}{D}}(x-ct)\right). \tag{18}$$

Where the coefficients D, and N are defined above.

3. Discussion

In the given paper, we studied the nonlinear dynamics of large-scale electromagnetic Rossby-Khantadze waves with zonal flows in E-ionospheric plasma. Both the latitudinal inhomogeneities in angular velocity of the earth's rotation and the geomagnetic field are taken into account. The inhomogeneity of the magnetic field with latitude is responsible for coupled Rossby–Khantadze waves. Such coupling results in an appearance of dispersion of Khantadze waves. To derive the nonlinear modified KdV we used the multiple scale analysis technique.

From the lowest order of O (ε^0), we get an eigen value problem with constant eigen value c_0 along with the boundary conditions. The parameters $p(y)$ and $\alpha(y)$ have dependence on the variable y, making it not possible to solve this eigen value problem analytically. From the next order O (ε^1), by using separation of variables techniques and after doing some mathematical manipulations we arrive at the modified KdV equation of one dimension. The derived quantity $\sqrt{\frac{6c}{N}}$ describes the amplitude of solitary waves. The obtained coefficients N and D depend on spatial inhomogeneous Coriolis force $\alpha(y)$ and background magnetic field $\beta(y)$, respectively. The parameter $\sqrt{\frac{6c}{N}}$ obtained above describes the amplitude of obtained such Rossby-Khantadze waves.

AUTHOR DECLARATIONS:

Conflict of Interest

The authors have no conflicts to disclose.

Data Availability

The data that support the findings of this study are available within the article.

References

[1] Pedlosky J., Geophysical Fluid Dynamics, Springer-Verlag, New York, 1987.

[2] Satoh M., Atmospheric Circulation Dynamics and General Circulation Models, Springer, New York, 2004

[3] Champeaux S. & Diamond P.H., Streamer and zonal flow generation from envelope modulations in drift wave turbulence, Phys. Lett. A **288**, 214–219, 2001.

[4] Shukla P.K., & Stenflo L., Generation of zonal flows by Rossby waves, Phys. Lett. A, **307**, 154–157, 2003.

[5] Onishchenko O.G., Pokhotelov O.A., Sagdeev R.Z., Shukla P.K., & Stenflo L., Generation of zonal flows by Rossby waves in the atmosphere, Nonlinear Process. Geophys. **11**, 241–244, 2004.

[6] Kaladze T.D., Wu D.J., Pokhotelov O.A., Sagdeev R.Z., Stenflo L., & Shukla P.K., Rossby-wave driven zonal flows in the ionospheric E-layer, J. Plasma Phys. **73**, 131–140, 2007.

[7] Kaladze T.D., Pokhotelov O.A., Stenflo L., Rogava J., Tsamalashvili L.V., & Tsik-Lauri M., Zonal flow interaction with Rossby waves in the Earth's atmosphere: a numerical simulation, Phys. Lett. A **372**, 5177–5180, 2008.

[8] Benkadda S., Klochkov D.N., Popel S.I., & Izvekova Yu.N., Nonlinear excitation of zonal flows and streamers in plasmas, Phys. Plasmas **18**, 052306, 2011.

[9] Kaladze T., Kahlon L., Horton W., Pokhotelov O., & Onishchenko O., Shear flow driven Rossby-Khantadze electromagnetic planetary vortices in the ionospheric E-layer, Europhys. Lett. **106**, 29001, 2014.

[10] Kaladze T., Magnetized Rossby waves in the Earth's ionosphere, Plasma Phys. Rep. **25** (4), 284–287, 1999.

[11] Futatani S., Horton W., Kahlon L.Z., & Kaladze T.D., Rossby-Khantadze electromagnetic planetary waves driven by sheared zonal winds in the E-layer ionosphere, Phys. Plasmas **22**, 012906, 2015.

[12] Qiang Z., Zuntao F., & Shikuo L., Equatorial envelope Rossby solitons in a

shear flow, Adv. Atmos. Sci. **18**, 418–428, 2001.

[13] Bostrom R., Observations of weak double layers on auroral field lines, IEEE Trans. Plasma Sci. **20**, 756–763, 1992.

[14] Lindqvist P.A., Marklud G.T., & Blomberg L.G., Plasma characteristics determined by the Freja electric field instrument, Space Sci. Rev. **70**, 593–602, 1994.

[15] Dovner P.O., Eriksson A.I., Bostrom R., Holback B., Freja multiprobe observations of electrostatic solitary structures, Geophys. Res. Lett. **21**, 1827–1830, 1994.

[16] Kaladze T., Tsamalashvili L., Kaladze D., Ozcan O., Yesil A., & Inc M., Modified KdV equation for magnetized Rossby waves in a zonal flow of the ionospheric E-layer, Physics Letter A, **383**, 125888, 2009.

[17] Jian S., & Lian-Gui Y., Modified KdV equation for solitary Rossby waves with β effect in barotropic fluids, Chin. Phys. B **18** (7) 2873–2877, 2009.

[18] Shi1 Y., Yang D., Feng X., Qi J., Yang H. and Yin B., One possible mechanism for eddy distribution in zonal current with meridional shear, Scientific Reports, **8**, 10106, 2018, DOI:10.1038/s41598-018-28465-z.

[19] Kaladze T.D., Horton W., Kahlon L.Z., Pokhotelov O., and Onishchenko O., Zonal flows and magnetic fields driven by large-amplitude Rossby-Alfvén-Khantadze waves in the E-layer ionosphere**,** J. Geophys. Res.: Space Physics **118**, 1-12 2013.