Review of *Stieltjes Functions and Spectral Analysis in the Physics of Sea Ice by Kenneth M. Golden et al.*

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1 Overview

This manuscript is a review paper on how various physical properties of sea ice, which is considered either as a microstructured composite or a uniaxial polycrystaline material. In either scenario, the analytic continuation method (ACM) has been applied successfully to 'parameterize' the various physical properties of the sea ice pertinent to the climate modeling in terms of the spectrum of the underlying self-adjoint operator defining the effective properties.

In this manuscript, some results obtained by applying this ACM approach for sea ice are shown by the authors to be counterparts in other fields such as the Anderson localization. These parallels are intellectually very interesting and beautiful. The preprint starts with the percolation models regarding the volume fraction and the electric conductivity of sea ice, followed by electromagnetic properties of sea ice as a two-phase composites of isotropic brine and ice, and then as uniaxial polycrystals (crystals with two different values among all possible crystal axes). In these sections, the focus is on how the microstructure is coded in the effective properties via the Stieltjes function in the ACM. In Section 5, the bounds derived from the ACM method are used to solve the inverse homogenization problems for using measurement of effective properties to bound the range of volume fractions. The last two sections, Sections 6 and 7, are devoted to the topic of ACM for diffusive processes and the Random matrix theory for sea ice physics, where the transition of the Eigenvalus Spacing Distribution (ESD) from Poissonian to universal Wigner-Dyson (WD) statistics of Gaussian Orthogonal Ensembles (GOE) for sea ice near the percolation threshold $\phi \approx 0.50$ is demonstrated by applying ACM to the CT image of sea ice. Also included are the project method describe in the appendix.

2 Specific comments

Topic-wise, this is a very good paper. All the results are from refereed papers. However, there are several points that need to be clarified.

1. In Figure 3, is the noise level known? Also, in the caption, it states "... a delta function singularity in the spectral function $\mu(\lambda)$ develops at $\lambda = 0$'. However, it is not clear to me how to see this in the graph of the spectral measure; indeed, it looks like the graph for $\phi = 0.12$ has the biggest jump near $\lambda = 0$ among all the three cases.

2. In several places in the manuscript, the word 'connectivity' appears several times. For example, in the paragraph between line 260 and line 264. It seems that as the volume fraction approaches 0.51, the gap vanishes and delta-measure starts to appear near the spectral end-point. But why is this a SIGNATURE of the composites being more connected? Again, in Line 395, it is stated that 'Connectivity information is also embedded in the spectral measure.' How do we measure the 'connectedness' from the spectral measure?

3. The entire Section 6 is very disconnected from the rest of the manuscript, partly because there are many un-defined symbols and missing essential details. For example, in Line 610, it states 'Non-dimensionalizing and homogenizing (26) shows...', but it is not clear what is the parameter over which the (26) is being homogenized; what is small? Or in Line 611, $\tilde{\phi}$ is mentioned but where to be found in the following equation, which is eqn (28). In Line 620, 'over the space-time period cell for periodic flows', why suddenly the average over space and TIME? Section 6 needs to be carefully rewritten to ensure the clarity and the consistency of the symbols and notations.

4. It would be great if the authors could add a paragraph or two elaborating on how these ACM results can directly contribute to the current climate model, and mention any known limitation, if any, of the ACM method in the context of sea ice modeling.

3 Technical comments

1. Line 148, please add references after 'its role in climate'.

2. Eqn. (21): $\frac{\partial}{\partial s}$ should be changed to $\frac{d}{ds}$.

3. Line 451: Please add references to the end of the sentence "These relations estimate permeability of a porous material characterizing ....'

4. Page 9, Figure 6: The numbering of these figures needs to be corrected.

5. Line 536: Is 'c-axis' crystal axis?