A review of "Lévy-noise versus Gaussian-noise-induced Transitions in the Ghil-Sellers Energy Balance Model " by Lucarini et al., NPG-2021-34

General comment

This manuscript studies the impact on the Ghil-Sellers energy balance model (EBM) of strongly non-Gaussian α-stable Lévy fluctuations ($0 < \alpha < 2$) of solar irradiance, whereas mainly the case of Gaussian fluctuations (α =2) has been considered so far, including on a wide range of metastable systems. By means of numerical simulations for $\alpha = \{0.5; 1.0; 1.5\}$, the authors show the existence of an $\varepsilon^{-\alpha}$ scaling law of the mean residence time (in each metastable state) under a vanishing α-stable Lévy multiplicative forcing intensity ($\varepsilon \rightarrow 0$, $\varepsilon|_{\alpha=0.5} \in [0.0001: 0.002]$; $\varepsilon|_{\alpha=1.0} \in [0.004: 0.04]$ and $\varepsilon|_{\alpha=1.5} \in [0.01: 0.1]$ numerically). Thus, the main and important result obtained is an extension to the Ghil-Sellers EBM of the scaling law found on simpler models (e.g., Imkeller and Pavlyukevich, 2006a,b; Debussche et al., 2013). This is certainly of great interest to NPG readership who have some familiarity with climate models and stochastic processes of various types.

However, in its very construction, the content of the manuscript itself seems excessively broad to allow readers to assess the generality/robustness and thus the impact of the results obtained. Several improvements seem to be needed before this article can be published. I hope the attached comments will be helpful for the revision.

Detailed comments

Introduction of the Ghil-Sellers energy balance model

The introduction of this model, with the help of Eqs 1-6 and its 9 parameters and 6 empirical functions (of location x), is quite abrupt whereas it is later summarised in three *physical* terms ($D_I - D_{III}$), in particular in Eq. 10. It would be useful to proceed in the opposite direction and address rather usual questions such as the sensitivity of the model to the details of these terms, and in particular to the values of their parameters, as well as the uncertainty on the empirical estimates. These questions are particularly important regarding the scaling law obtained, e.g., its robustness.

Introduction of the α-stable Lévy noises

The introduction of the α-stable Lévy noises is a bit surprising and disappointing. Whereas the manuscript includes a considerable list of various applications of α-stable Lévy noises, which can be understood as a vague argument of their potential interest for climate models, two main geophysical applications of the α-stable Lévy noises were forgotten:

- The multiplicative cascades generated by α-stable Lévy noises, often called "universal multifractals", have not only been widely used in geophysics but have been inspired by them (Schertzer and Lovejoy, 1987), specifically to analyse and simulate their ubiquitous intermittency and heavy tailed statistics, including at climate scales. This origin has been recognised by mathematicians who have used the term "Lévy multiplicative chaos" to emphasise their generality (e.g. Fan, 1987).
- The fractional Fokker-Planck equations for nonlinear SDE forced by non-Gaussian stable Lévy noises (e.g. Schertzer et al (2001) and references therein) used to analyse and simulate diffusion.

The authors argue for an α-stable Lévy forcing by referring to the paleoclimatic records exhibiting strong non-Gaussian behaviour. It may be worth mentioning though that these observed heavy-tailed distributions generally do not support a power law exponent $\alpha < 2$, but a larger one that can be deduced from (i), see e.g. Schmitt et al (1995).

It is also surprising that the fundamental and common property of α-stable Lévy ($0 < \alpha < 2$) and Gaussian ($\alpha = 2$) noises to be both stable (with a precise stability meaning of the index α) and attractive (often presented like the generalised central limit theorem) is not presented. The fundamental statistical difference between α-stable Lévy and Gaussian extremes is nor clearly presented, although indirectly evoked: namely α-stable Lévy have heavy tails and α (<2) is then both the critical exponent of the divergence of statistical moments and the exponent of these power-law tails, whereas Gaussian noises do not have these extreme behaviours. Similarly, the scaling law of the increments (nonetheless used in Eq.16) and that of the Lévy jump measure are not mentioned (only indirectly in Eq. A7), whereas they play a key role. Moreover, only symmetric α-stable Lévy noises are introduced, whereas unlike Gaussian noises, α-stable Lévy noises are easily skewed and only extremely skewed α-stable Lévy noises can generate multiplicative cascades (i) with finite (theoretical) statistics and avoid spurious (empirical) estimates. The latter may impact the rationale (L 309-312) to limit the range of intensity ε.

Overall, it seems that Appendix A is unbalanced mainly recalling properties of Lévy processes instead of the specific properties of the α-stable Lévy processes. On the contrary, the physical properties of the latter recapitulated above may help to answer the question: is there a physical reason to escape the "Gaussian rigidity" and to consider α-stable Lévy fluctuations?

Robustness of the result and the multiplicative nature of the forcing term

As already mentioned, it is necessary to discuss the robustness of the result, especially because of the many parameters and empirical functions involved in the climate model. Since the numerically observed scaling corresponds to that of an additive stochastic forcing, it is suggested to first assess the multiplicative character, i.e., whether the stochastic forcing in such a regime or only very weakly multiplicative. This depends exclusively on the effective temperature dependence of the albedo term (Eq. 5). It could be then important to estimate the latter.

Negative values of the solar irradiance

Despite the initially strong physical orientation of the manuscript, the authors do not hesitate to finally abandon some physical relevance, accepting negative values of the solar irradiance, "to be able to stick to the desired mathematical framework" (L 314-316), which is in turn substituted by a numerical framework with limited ranges of noise intensity ε. These can contribute to overlooking the fact that symmetric α-stable Lévy noises generate much more frequent negative fluctuations than Gaussian noises and thus a significant gap between mathematical convenience and physics. The authors partly acknowledge this problem in their conclusions in terms of general heuristics (e.g., L 449-452), but they could be more precise in their critical analysis.

Minor comments

- L 124: the expression "discontinuous càdlàg paths" seems a bit convoluted for what the authors have in mind
- L 129 and 216: Lokka et al (2004) do not consider multiplicative Lévy noise laws but a linear SPDE (Poisson equation).
- L160: m is not defined (nor is σ … but one can guess for the latter)
- L 170: T_m is not defined
- L 231: the Lévy-Itô decomposition is not fully introduced in the Appendix A and it does not help to understand Eq. 12
- L 232 $\Psi(t)$ is a Green's function for physicists