Reply to the comments by the referee #3

I appreciate the referee for helpful comments. I have revised the manuscript after carefully considering the comments. I have also slightly revised the figures of the convergence of the objective function J because I found that the iteration numbers in each plot (Figures 1–4 and 7–9) were wrongly shifted by one. (The initial guesses before the iterations were plotted as the estimates with the first iteration in the previous version.)

In the following, the comments by the referee are quoted in bold, and followed by my responses.

They have demonstrated the approach functions, and the sensitivity of convergence to certain parameters, with application to the toy Lorenz 96 (L96) problem with the largest tested state vector being 400 elements long. The presentation of the results, however, in section 7 is somewhat lacking. I would suggest they give the reader more of an impression of what the algorithm is attempting to do by presenting for a specific state times series of the truth, overlaid by the observed and estimated values. They may also wish to present a time series of the error between the true and estimated states. At present the convergence of the objective function J with iteration is somewhat too abstract.

I appreciate the referee for the helpful suggestion. I have added the figures to demonstrate the convergence of the estimated time series (Figures 5 and 10) for the two typical cases. I have also added the figures to show the time series of the errors between the truth and estimate (Figure 6 and 11).

Additionally, I don't think the author can state that they have shown that the approach is applicable to high dimensional geophysical systems. It may well be, but one cannot say as such given the suit of presented experiments. Whilst L96 has been applied many times in the literature for such problems it does not have many of the properties that are intrinsic and essential to high dimensional geophysical systems. Specifically: there is no notion of physical space and hence no spatial error correlation; it does not have long lived serial temporal error correlations; and perhaps most importantly it is not multi-scale. The author either needs to convince me that this 4DEnVar method will also perform well in multi-scale systems, or significantly tone down their claims. Note, I would not class the two level subgrid version of L96 multi-scale either, it is dual scale.

I believe that the theoretical discussion provided in this paper is applicable even to high-dimensional multi-scale problems. Indeed, our group has applied the ensemble transform version of the iterative method to a practical geodynamo model by Takahashi (Phys. Earth. Planet. Inter., v. 226, p. 83, 2014, https://doi.org/10.1016/j.pepi/2013.11.006), which contains spatio-temporal correlation with multiple scales, and obtained a reasonable result (Minami et al., 2020).

However, I agree that the performance of the random ensemble version of the iterative method in high-dimensional geophysical problems has not yet been demonstrated and that it should be assessed in the future. In the revised version, I have added a comment to say that this paper has not assess the performance of the random ensemble version with Eq. (36) in practical high-dimensional problems (L. 461–464).

List of changes made in the manuscript

- The figures to demonstrate the convergence of the estimated time series (Figures 5 and 10) have been added. The figures to show the time series of the errors between the truth and estimate (Figure 6 and 11) have also been added.
- I have added a comment to say that this paper has not assessed how well the method with a random ensemble generation with Eq. (36) works in practical high-dimensional problems (L. 461–464).
- Since the iteration numbers in Figures 1–4 and 7–9 were wrongly shifted by one, these figures have been revised.
- Some other errors have been fixed (Eq. (69) and Algorithm 2).