Supplementary Material 1 : Ornstein-Uhlenbeck processes example in detail
Correcting for Model Changes in Statistical Post-Processing

Jonathan Demaeyer and Stéphane Vannitsem
Institut Royal Météorologique de Belgique, Avenue Circulaire, 3, 1180 Brussels, Belgium and
European Meteorological Network, Avenue Circulaire, 3, 1180 Brussels, Belgium

In this supplementary note, we detail the computations that lead to the main Section 2 results.

I. ORNSTEIN-UHLENBECK PROCESSES AS A FORECAST MODEL CHANGE

We consider two Ornstein-Uhlenbeck processes supposedly representing the reality $x(\tau)$ and its forecast model $y(\tau)$ obeying the following equations:

$$\dot{x} = -\lambda_x \; x + K_x + Q_x \, \xi_x(\tau) \tag{S1}$$

$$\dot{y} = -\lambda_y \; y + K_y + Q_y \, \xi_y(\tau) \tag{S2}$$

where ξ_x and ξ_y are white noise processes such that

$$\begin{aligned} \langle \xi_x(\tau) \rangle = \langle \xi_y(\tau) \rangle &= 0 \\ \langle \xi_x(\tau) \; \xi_x(\tau') \rangle &= \delta(\tau - \tau') \\ \langle \xi_y(\tau) \; \xi_y(\tau') \rangle &= \delta(\tau - \tau') \\ \langle \xi_x(\tau) \; \xi_y(\tau') \rangle &= 0 \end{aligned}$$

These dynamics are thus uncorrelated Ornstein-Uhlenbeck processes with noise amplitudes Q_x and Q_y.

A change of the model $y(\tau)$ toward a model $\hat{y}(\tau)$ is then considered, possibly improving or degrading the forecast performances:

$$\dot{\hat{y}} = -\lambda_y \; \hat{y} + K_y + Q_y \, \xi_y(\tau) + \Psi_y(\tau) \tag{S3}$$

where

$$\Psi_y(\tau) = -\kappa \left(\delta K + \delta Q \; \xi_y(\tau) \right) \tag{S4}$$

with $\delta K = K_y - K_x$ and $\delta Q = Q_y - Q_x$. It can represent for example a better parameterization of subgrid-scale processes. Note that if $\kappa = 1$, the correction is perfect.

A post-processing scheme constructed before the model change, and then applied after it, would give a deteriorated correction. Therefore, a post-processing scheme is in general recomputed after such a change. The object of the present note is to show that response theory can correct the prior post-processing scheme and gives thus an approximation to the one computed after the change. Moreover, in the present case involving Ornstein-Uhlenbeck processes, this correction is exact. First, this post-processing scheme is described.

A. The post-processing of x

We consider an EVMOS [4] post-processing of the variable x :

$$x_C(\tau) = \alpha(\tau) + \beta(\tau) \cdot y(\tau) \tag{S5}$$

The coefficient α and β are given by the equations:

$$\alpha(\tau) = \langle x(\tau) \rangle - \beta(\tau) \cdot \langle y(\tau) \rangle \tag{S6}$$

$$\beta(\tau) = \sqrt{\frac{\sigma_x^2(\tau)}{\sigma_y^2(\tau)}} \tag{S7}$$

with

$$\sigma_x^2(\tau) = \left\langle \big(x(\tau) - \langle x(\tau)\rangle\big)^2 \right\rangle \tag{S8}$$

$$\sigma_y^2(\tau) = \left\langle \big(y(\tau) - \langle y(\tau)\rangle\big)^2 \right\rangle \tag{S9}$$

where the average is taken over initial conditions of the x process. Indeed, both the forecast models y and $\hat{y}$ are initialized with the initial conditions of x. The corrected quantity x_C depends thus on the lead time τ.

B. Post-processing before and after the model change

Before the model change, $\Psi_y(\tau) = 0$, and a direct computation gives:

$$\langle x(\tau)\rangle = \langle x(0)\rangle\, e^{-\lambda_x \tau} + \frac{K_x}{\lambda_x}\, \left(1 - e^{-\lambda_x \tau}\right) \tag{S10}$$

$$\langle y(\tau)\rangle = \langle x(0)\rangle\, e^{-\lambda_y \tau} + \frac{K_y}{\lambda_y}\, \left(1 - e^{-\lambda_y \tau}\right) \tag{S11}$$

and

$$\sigma_x^2(\tau) = \sigma_x^2(0)\, e^{-2\lambda_x \tau} + \frac{Q_x^2}{2\,\lambda_x}\, \left(1 - e^{-2\,\lambda_x \tau}\right) \tag{S12}$$

$$\sigma_y^2(\tau) = \sigma_x^2(0)\, e^{-2\lambda_y \tau} + \frac{Q_y^2}{2\,\lambda_y}\, \left(1 - e^{-2\,\lambda_y \tau}\right) \tag{S13}$$

since the model is initialized with the same initial conditions as the reality:

$$\langle y(0)\rangle = \langle x(0)\rangle \qquad , \qquad \sigma_y^2(0) = \sigma_x^2(0) \quad . \tag{S14}$$

These equations allow us to compute the post-processing coefficients at every lead time τ:

$$\alpha(\tau) = \langle x(\tau)\rangle - \beta(\tau)\, \langle y(\tau)\rangle \tag{S15}$$

$$\beta(\tau) = \sqrt{\frac{\sigma_x^2(\tau)}{\sigma_y^2(\tau)}} \tag{S16}$$

It is easy to extend these results when model change Ψ is incorporated:

$$\langle \hat{y}(\tau)\rangle = \langle x(0)\rangle\, e^{-\lambda_y \tau} + \frac{K_y - \kappa\,\delta K}{\lambda_y}\, \left(1 - e^{-\lambda_y \tau}\right) \tag{S17}$$

and

$$\sigma_{\hat{y}}^2(\tau) = \sigma_x^2(0)\, e^{-2\lambda_y \tau} + \frac{(Q_y - \kappa\,\delta Q)^2}{2\,\lambda_y}\, \left(1 - e^{-2\,\lambda_y \tau}\right) \tag{S18}$$

with

$$\hat{\alpha}(\tau) = \langle x(\tau)\rangle - \hat{\beta}(\tau)\, \langle \hat{y}(\tau)\rangle \tag{S19}$$

$$\hat{\beta}(\tau) = \sqrt{\frac{\sigma_x^2(\tau)}{\sigma_{\hat{y}}^2(\tau)}} \tag{S20}$$

The variation of the moments of the dynamics are then

$$\langle \hat{y}(\tau)\rangle = \langle y(\tau)\rangle - \frac{\kappa\,\delta K}{\lambda_y}\left(1 - e^{-\lambda_y \tau}\right) \tag{S21}$$

$$\sigma_{\hat{y}}^2(\tau) = \sigma_y^2(\tau) + \frac{1}{2\,\lambda_y}\, \left(\kappa^2 \delta Q^2 - 2\,\kappa\, Q_y\, \delta Q\right)\, \left(1 - e^{-2\,\lambda_y \tau}\right) \tag{S22}$$

and it implies the following variation of the bias α:

$$\delta\alpha(\tau) = \hat{\alpha}(\tau) - \alpha(\tau) = \beta(\tau)\, \langle y(\tau)\rangle - \hat{\beta}(\tau)\, \langle \hat{y}(\tau)\rangle \tag{S23}$$

The ratio between the biases β is given by

$$\frac{\hat{\beta}(\tau)}{\beta(\tau)} = \sqrt{\frac{\sigma_y^2(\tau)}{\sigma_{\hat{y}}^2(\tau)}} \tag{S24}$$

For $\tau \gg \max(1/\lambda_x, 1/\lambda_y)$, we note that this ratio tend to

$$\frac{\hat{\beta}(\tau)}{\beta(\tau)} = \frac{1}{1 - \kappa\,\delta Q/Q_y} \tag{S25}$$

and the asymptotic limit for the variation of α is given by

$$\delta\alpha(\tau) = -\beta(\tau)\,\frac{K_y}{\lambda_y}\left[\frac{1-\kappa\,\delta K/K_y}{1-\kappa\,\delta Q/Q_y} - 1\right] \quad . \tag{S26}$$

C. (Non-Stationary) Response theory

The post-processing problem is typically a non-stationary initial value problem, since the initial conditions of the model equations (S2) and (S3) are typically chosen close to the reality (S1). As a consequence, the model actual averages relax toward the stationary response in the long-time limit, but the stationary response theory [3, 5] cannot provide us the short-time relaxation behaviors. We thus consider the short-time evolution of the averages. Therefore, the Ruelle time-dependent response theory should be used [2].

After the model change, the model is ruled by Eq. (S3):

$$\dot{\hat{y}} = -\lambda_y\,\hat{y} + K_y + Q_y\,\xi_y(\tau) + \Psi_y(\tau) \tag{S27}$$

with

$$\Psi_y(\tau) = -\kappa\left(\delta K + \delta Q\,\xi_y(\tau)\right) \tag{S28}$$

Given an observable A, its average after the model change can then be related to it's average before by

$$\langle A(\tau)\rangle_{\hat{y}} = \langle A(\tau)\rangle_y + \delta\langle A(\tau)\rangle_y + \delta^2\langle A(\tau)\rangle_y + \ldots \tag{S29}$$

If the perturbations (S28) is small, then the response theory states that the first order is given by:

$$\delta\langle A(\tau)\rangle_y = \int_0^\tau \mathrm{d}\tau' \int \mathrm{d}y\,\rho_{y,0}(y)\,\left\langle \Psi_y(\tau')\cdot\nabla_{f^{\tau'}(y)}\,A\left(f^\tau(y)\right)\right\rangle \tag{S30}$$

where $\rho_{y,0}$ is the initial distribution with which the model is initialized. As indicated by Eq. (S14), in the post-processing framework, it is typically taken as the initial distribution of the reality $\rho_{x,0}$. We have also $\langle\cdot\rangle$ which denotes the average over the realization of the stochastic process [1], and also the mapping f^τ is the stochastic "flow" of the unperturbed system (S2):

$$f^\tau(y) = y\,e^{-\lambda_y\tau} + \int_0^\tau \mathrm{d}\tau'\,e^{-\lambda_y(\tau-\tau')}\left[Q_y\,\xi_y(\tau') + K_y\right] \tag{S31}$$

1. *First order*

a. First moment: First the response for the shift of the mean of $y(\tau)$ due to the perturbation is evaluated:

$$\delta\langle y(\tau)\rangle = -\kappa\int_0^\tau \mathrm{d}\tau' \int \mathrm{d}y\,\rho_{y,0}(y)\,\left\langle\left[\delta K + \delta Q\,\xi_y(\tau')\right]\cdot\nabla_{f^{\tau'}(y)}\,f^\tau(y)\right\rangle \tag{S32}$$

With

$$\nabla_{f^{\tau'}(y)}\,f^\tau(y) = \nabla_{f^{\tau'}(y)}\,f^{\tau-\tau'}\circ f^{\tau'}(y) = e^{-\lambda_y(\tau-\tau')} \qquad , \tag{S33}$$

$$\int \mathrm{d}y \, \rho_{y,0}(y) = 1 \qquad , \tag{S34}$$

and $\langle \xi_y(\tau') \rangle = 0$ for all τ', we get:

$$\delta\langle y(\tau)\rangle = -\kappa \int_0^\tau \mathrm{d}\tau' \delta K \, e^{-\lambda_y(\tau-\tau')} \tag{S35}$$

and thus:

$$\langle \hat{y}(\tau)\rangle = \langle y(\tau)\rangle - \frac{\kappa}{\lambda_y} \, \delta K \, \left(1 - e^{-\lambda_y \tau}\right) \tag{S36}$$

and the exact relation (S21) is recovered.

b. Second moment: We now turn to the computation of

$$\begin{aligned}
\delta\langle y(\tau)^2\rangle &= -\kappa \int_0^\tau \mathrm{d}\tau' \int \mathrm{d}y \, \rho_{y,0}(y) \, \left\langle \left[\delta K + \delta Q \, \xi_y(\tau')\right] \cdot \nabla_{f^{\tau'}(y)} \, (f^\tau(y))^2 \right\rangle & \text{(S37)}\\
&= -\kappa \int_0^\tau \mathrm{d}\tau' \int \mathrm{d}y \, \rho_{y,0}(y) \, \left\langle \left[\delta K + \delta Q \, \xi_y(\tau')\right] \cdot \nabla_{f^{\tau'}(y)} \, f^\tau(y) \cdot \nabla_{f^\tau(y)} \, (f^\tau(y))^2 \right\rangle & \text{(S38)}\\
&= -2\,\kappa \int_0^\tau \mathrm{d}\tau' \int \mathrm{d}y \, \rho_{y,0}(y) \, \left\langle \left[\delta K + \delta Q \, \xi_y(\tau')\right] \cdot f^\tau(y) \cdot \nabla_{f^{\tau'}(y)} \, f^\tau(y) \right\rangle & \text{(S39)}
\end{aligned}$$

which gives, using the formula (S31):

$$\begin{aligned}
\delta\langle y(\tau)^2\rangle = &-2\,\kappa\,\delta K \int_0^\tau \mathrm{d}\tau' \int \mathrm{d}y \, \rho_{y,0}(y) \, \left[y \, e^{-\lambda_y\tau} \, e^{-\lambda_y(\tau-\tau')} + \frac{K_y}{\lambda_y} \, e^{-\lambda_y(\tau-\tau')} \left(1 - e^{-\lambda_y \tau}\right)\right] \\
&-2\kappa \, Q_y \, \delta Q \, \left\langle \int_0^\tau \mathrm{d}\tau' \, e^{-\lambda_y(\tau-\tau')} \, \xi_y(\tau') \int_0^\tau \mathrm{d}\tau'' \, e^{-\lambda_y(\tau-\tau'')} \, \xi_y(\tau'') \right\rangle & \text{(S40)}\\
= &-2\,\kappa\,\delta K \, \left[\frac{1}{\lambda_y} \, \langle y(0)\rangle \, e^{-\lambda_y\tau} \, \left(1 - e^{-\lambda_y\tau}\right) + \frac{K_y}{\lambda_y^2} \, \left(1 - e^{-\lambda_y\tau}\right)^2\right] \\
&-2\,\kappa\,\delta Q \, Q_y \int_0^\tau \mathrm{d}\tau' \, e^{-2\,\lambda_y(\tau-\tau')} & \text{(S41)}
\end{aligned}$$

where we have used $\langle \xi_y(\tau) \, \xi_y(\tau')\rangle = \delta(\tau - \tau')$. Therefore, we finally get:

$$\delta_t\langle y(\tau)^2\rangle = -2\,\kappa \, \frac{\delta K}{\lambda_y} \, \langle y(\tau)\rangle \, \left(1 - e^{-\lambda_y\tau}\right) - \frac{\kappa}{\lambda_y} \, \delta Q \, Q_y \, \left(1 - e^{-2\,\lambda_y\tau}\right) \tag{S42}$$

which gives

$$\begin{aligned}
\sigma^2_{y,\Psi}(\tau) &\approx \langle y(\tau)^2\rangle + \delta\langle y(\tau)^2\rangle - \left(\langle y(\tau)\rangle + \delta\langle y(\tau)\rangle\right)^2 \\
&= \sigma_y^2(\tau) + \delta\langle y(\tau)^2\rangle - 2\langle y(\tau)\rangle \, \delta\langle y(\tau)\rangle - \left(\delta\langle y(\tau)\rangle\right)^2 \\
&= \sigma_y^2(\tau) - 2\,\kappa \, \frac{\delta K}{\lambda_y} \, \langle y(\tau)\rangle \, \left(1 - e^{-\lambda_y\tau}\right) - \frac{\kappa}{\lambda_y} \, \delta Q \, Q_y \, \left(1 - e^{-2\,\lambda_y\tau}\right) \\
&\qquad + 2\langle y(\tau)\rangle_0 \, \frac{\kappa}{\lambda_y} \, \delta K \, \left(1 - e^{-\lambda_y\tau}\right) - \frac{\kappa^2}{\lambda_y^2} \, \delta K^2 \, \left(1 - e^{-\lambda_y\tau}\right)^2 & \text{(S43)}\\
&= \sigma_y^2(\tau) - \frac{\kappa}{\lambda_y} \, \delta Q \, Q_y \, \left(1 - e^{-2\,\lambda_y\tau}\right) - \frac{\kappa^2}{\lambda_y^2} \, \delta K^2 \, \left(1 - e^{-\lambda_y\tau}\right)^2 & \text{(S44)}
\end{aligned}$$

which does not match Eq. (S22) and thus the second order terms should be computed.

2. *Second order*

The second order response for an observable A is given by:

$$\delta^2\langle A(\tau)\rangle = \int_0^\tau \mathrm{d}\tau' \int_{\tau'}^\tau \mathrm{d}\tau'' \int \mathrm{d}y \, \rho_{y,0}(y) \, \left\langle \Psi_x(\tau') \cdot \nabla_{f^{\tau'}(y)} \Psi_y(\tau'') \cdot \nabla_{f^{\tau''}(y)} A\left(f^\tau(y)\right)\right\rangle \tag{S45}$$

Since here the perturbations do not depend on the state variable y

$$\delta^2 \langle A(\tau) \rangle = \int_0^\tau \mathrm{d}\tau' \int_{\tau'}^\tau \mathrm{d}\tau'' \int \mathrm{d}y \, \rho_{y,0}(y) \, \left\langle \Psi_x(\tau') \Psi_y(\tau'') \nabla_{f^{\tau'}(y)} \nabla_{f^{\tau''}(y)} A\left(f^\tau(y)\right) \right\rangle \tag{S46}$$

and applied to the first moment of the $y(\tau)$ dynamics, it leads to

$$\delta^2 \langle y(\tau) \rangle = 0 \tag{S47}$$

since thanks to the Eq. (S33)

$$\nabla_{f^{\tau'}(y)} \nabla_{f^{\tau''}(y)} f^\tau(y) = 0 \qquad . \tag{S48}$$

The computation of the derivatives for the second moments leads to

$$\begin{aligned} \nabla_{f^{\tau'}(y)} \nabla_{f^{\tau''}(y)} \left(f^\tau(y)\right)^2 &= 2 \, \nabla_{f^{\tau'}(y)} \, f^\tau(y) \, \nabla_{f^{\tau''}(y)} \, f^\tau(y) \\ &= 2 \, e^{-\lambda_y (\tau - \tau')} \, e^{-\lambda_y (\tau - \tau'')} \qquad , \end{aligned} \tag{S49}$$

and thus the correction to the second moment of $y(\tau)$ can be obtained by computing

$$\delta^2 \langle y(\tau)^2 \rangle = 2 \, \kappa^2 \int_0^\tau \mathrm{d}\tau' \int_{\tau'}^\tau \mathrm{d}\tau'' \, \left\langle \left(\delta K + \delta Q \; \xi_y(\tau')\right) e^{-\lambda_y (\tau - \tau')} \left(\delta K + \delta Q \; \xi_y(\tau'')\right) e^{-\lambda_y (\tau - \tau'')} \right\rangle \tag{S50}$$

which thanks to the averaging properties of ξ_y gives

$$\begin{aligned} \delta^2 \langle y(\tau)^2 \rangle &= \kappa^2 \, \delta K^2 \left[\int_0^\tau \mathrm{d}\tau' \, e^{-\lambda_y (\tau - \tau')} \right]^2 + 2\kappa^2 \, \delta Q^2 \int_0^\tau \mathrm{d}\tau' \int_{\tau'}^\tau \mathrm{d}\tau'' \, \left\langle \xi_y(\tau') \right) e^{-\lambda_y (\tau - \tau')} \, \xi_y(\tau'') \, e^{-\lambda_y (\tau - \tau'')} \right\rangle \\ &= \kappa^2 \, \delta K^2 \left[\int_0^\tau \mathrm{d}\tau' \, e^{-\lambda_y (\tau - \tau')} \right]^2 + \kappa^2 \, \delta Q^2 \int_0^\tau \mathrm{d}\tau' \, e^{-2\lambda_y (\tau - \tau')} \qquad (\mathrm{S51}) \\ &= \frac{\kappa^2 \, \delta K^2}{\lambda_y^2} \, \left(1 - e^{-\lambda_y \tau}\right)^2 + \frac{\kappa^2 \, \delta Q^2}{2\lambda_y} \, \left(1 - e^{-2\lambda_y \tau}\right) \qquad (\mathrm{S52}) \end{aligned}$$

3. *Final result*

By aggregating the two orders, we get:

$$\langle y(\tau) \rangle_{\hat{y}} \approx \langle y(\tau) \rangle + \delta \langle y(\tau) \rangle = \langle y(\tau) \rangle - \frac{\kappa}{\lambda_y} \, \delta K \, \left(1 - e^{-\lambda_y \tau}\right) \tag{S53}$$

$$\begin{aligned} \sigma_{\hat{y}}^2(\tau) &\approx \sigma_y^2(\tau) - \frac{\kappa}{\lambda_y} \, \delta Q \, Q_y \, \left(1 - e^{-2\,\lambda_y \tau}\right) - \frac{\kappa^2}{\lambda_y^2} \, \delta K^2 \, \left(1 - e^{-\lambda_y \tau}\right)^2 \\ &\quad + \frac{\kappa^2 \, \delta K^2}{\lambda_y^2} \, \left(1 - e^{-\lambda_y \tau}\right)^2 + \frac{\kappa^2 \, \delta Q^2}{2\lambda_y} \, \left(1 - e^{-2\lambda_y \tau}\right) \qquad (\mathrm{S54}) \\ &= \sigma_y^2(\tau) - \frac{\kappa}{\lambda_y} \, \delta Q \, Q_y \, \left(1 - e^{-2\,\lambda_y \tau}\right) + \frac{\kappa^2 \, \delta Q^2}{2\lambda_y} \, \left(1 - e^{-2\lambda_y \tau}\right) \qquad (\mathrm{S55}) \end{aligned}$$

These approximations are in fact exact for any time τ, even for the transient short-time $\tau < 1/\lambda_y$ and gives back the post-processing results of the section I B of the present note.

---

[1] Lucarini, V.: Stochastic perturbations to dynamical systems: a response theory approach, Journal of Statistical Physics, 146, 774–786, 2012.

[2] Ruelle, D.: General linear response formula in statistical mechanics, and the fluctuation-dissipation theorem far from equilibrium, Physics Letters A, 245, 220–224, 1998.

[3] Ruelle, D.: A review of linear response theory for general differentiable dynamical systems, Nonlinearity, 22, 855, 2009.

[4] Vannitsem, S.: A unified linear model output statistics scheme for both deterministic and ensemble forecasts, Quarterly Journal of the Royal Meteorological Society, 135, 1801–1815, 2009.

[5] Wang, Q.: Forward and adjoint sensitivity computation of chaotic dynamical systems, Journal of Computational Physics, 235, 1–13, 2013.