Dear Editor, Dear Referees,

In the process of correcting the manuscript, we realized that the model qgs with which the numerical results were obtained contained small errors. These errors do not impact the dynamical core but are related to the orography and temperature profile. As such, by changing the model's parameter we have recovered a model's regime similar to what we previously had. We have performed the analysis again and found no qualitative differences with the previous one, underlying the robustness of our method. In fact, the method shows a good correction of the postprocessing up to 4 days instead of 3. This is due to the atmosphere being less unstable, and the Lyapunov time being slightly longer.

We have updated the figures and the manuscript with these new results. Note that we have also removed the figure 11 of the initial manuscript, since we concluded it has little relevance to the subject of the present manuscript.

We are now confident that the model is correct and bug-free.

We join our updated responses to the referees to this letter. The full list of changes to the manuscript as a latex diff pdf is also joined to this letter.

Sincerely yours,

Jonathan Demaeyer and Stéphane Vannitsem.

1 Response to referee 1

We thank the referee for his nice and constructive review and address its comments below:

1.1 General comments

We are glad that the referee recognize the potential of this approach. We note that indeed it would be better to already have a realistic application. This article was limited to the proof-of-concept part. The application to real cases is currently considered, but it might require some time before it is done. As such, we hope that the present publication will provide some guidelines and attract interest to work on this. We address now the specific comments:

1. **since this approach supposes the changes in the model are small, we can wonder if it will really be useful for operations, where small changes in the model may not really impair post-processing. Furthermore, it should be interesting to see whether this approach improves over the use of dynamic filtering of residual biases (via Kalman filtering, classically). Do you have any hint about this alternative?**

   In the present model considered, we note that small changes can really impair the postprocessing. See now Fig. 9(a) and 10(a). Furthermore, experiments showed that in some cases, linear response theory can be valid outside of its predicted range of validity (see Gottwald et al. (2016) and Wormell and Gottwald (2018)). Dynamic filtering could be corrected as well, it is a good point. For instance by adjusting the past predictions in the filtering window to the new model via the response theory presented here.

2. **the stated conditions for using this correcting seem very strong: a tangent model must be available, the model change has to be provided as an analytic function. These two conditions may not be observed or the models themselves may not be available. Would it be possible to follow the same procedure in a data-based approach? In other words, could it be possible to deduce the necessary correction if one has only the two sets of forecasts on a common period without access to the models themselves?**

   A tangent model is often available together with the forecasting model, at least to evaluate the sensivity to initial condition. It would cost less to develop than a reforecasting system, and several out-of-the-box automatic differentiation packages also exist to do it. At this stage of the research, we do not see how to apply the method in a purely data-driven context. This is indeed worth exploring in the future.

3. **the extend of post-processing methods that may be corrected in this way is not very clear. You state that The only requirement is that the outcome of the minimization of the cost functions uses averages of the systems being considered. Does this mean that post-processing methods that do not use cost functions (such as random forest) are not eligible to this approach?**

   We do not have expertise on methods such as random forest, but in any case it is an interesting question that could be addressed in a subsequent work. For instance, the present method could maybe be used in conjunction with random forest, using directly the trajectories of the tangent model instead of the reforecasts trajectories. The caveat here is then to deal with the problem of fat tails. Indeed, some trajectories (the ones forming the fat tails) will show extreme deviations from the others. So new methods should be designed to take care of this issue.

1.2 Specific comments

1. **Do you have a reference to support this claim that post-processing is useful only up to a lead time of 3 days?**

   We believe that figures 9(a) and 10(a) are sufficient to support that claim. The claim concerns only the highly truncated system at hand, and not the current state-of-the-art models.

2. **Could you develop about what kind of sophistication you are thinking about?**

   We are mainly thinking about other MOS schemes, but as said above, other kind of postprocessing could be considered (Random Forest, Machine Learning, ...). We are more specific in the text now, and write (page 7, line 14 of the final manuscript):

   > "More sophisticated approaches can be evaluated in the future (other MOS schemes, ensemble MOS, . . .)."

3. **For the article to be self-contained, may you add a more comprehensive definition (with equations) for this system, maybe in the appendix?**

   The equations of the model have been added in an appendix (Appendix B, pages 28-29 of the final manuscript).

4. **Is there a simple explanation why your system has only one blocking regime instead of two?**

   To our knowledge, it is not easy to explain intuitively such regime shift in nonlinear systems. A full understanding of the phenomenon should include a bifurcation analysis, and the analysis should then be translated to a physical description. In short, we do not have such a simple explanation for now.

5. **These pictures are not very clear. In panel (b) only model 0 seems to have two attractors but from the text (page9, line 12 and elsewhere) I understand that reality also should have two attractors. Please can you clarify either the text or the pictures?**

   The referee is right, the global attractor of both reality and model 0 contains two parts. It is now mentioned in the figure 2 caption :

   > "The attractors of the reality and model 0 are qualitatively similar, with two different parts which are indicated by ellipses."

6. **Later on, you say that the distribution of perturbation has been approximated with a gaussian distribution. In the conclusion, you propose to use the CLV method to get better corrections at farther lead times. I was wondering if we could improve the correction at long lead times by using a different distribution (with fatter tails) to approximate the perturbation distribution?**

   This is a possibility, but to our knowledge, estimating moments of fat tails distributions is quite difficult. This can be considered in a future work, together with the development of the approach in a more realistic environment.

7. **Based on the results, it seems this approach may require a very short period where both models are available. The length of the comon period (a few months?) seems to correspond to what may be available operationnaly in national weather services. This is a very good point, to check on real data, maybe in a future study.**

   There is no need of an overlapping period, except for verification of the method. In the future we indeed plan to use this approach in a realistic context.

1.3 Technical corrections

All the technical corrections have been addressed. We thank again the reviewer for his help.

2 Response to referees 2

2.1 General comments

We thank the referee for his/her nice review. We have tried to address his/her recommendations to improve the readability of the article.

2.2 Specific comments

- **The information in Eq. (13)-(14) and (16)-(17) is very redundant, since one only has to exchange the parameters. I suggest to compress this a little bit.**

  We understand the comment of the referee to avoid the redundancy of equations (13)-(14) and (16)-(17). However we do believe that this redundancy is necessary at this stage of the article, such that the reader understands clearly the method proposed here (impact of model change on post-processing). It is the critical point where the design of the experiments is clarified, and we do not want to loose the reader at this stage with too compact notations.

- **The application of the concept to the OU-processes is quite helpful. I would suggest to first formulate the response theory in general, and then in a separate section apply and explain the formulas for the OU case, followed by the post-processing.**

  and also

- **The application of the response theory here is very not very clear to me, and I still do not quite see, where and how the tangent model comes into play.**

  We concur with the reviewer concerning these points, and indeed it is a good suggestion to present the response theory first. We thus added a new section (Section 2, pages 3 to 5 of the final manuscript) to introduce this technique. We hope that it will be clearer now to a broader audience.

- **It would further be helpful to maybe see a figure of $\alpha(\tau)$ and $\beta(\tau)$.**

  We added a figure showing α and β (figure 8, page 21 of the final manuscript).

- **With respect to the QG model it would be helpful to, on the one hand, see the model equations probably in the Fourier space, but, on the other hand, simplify the Fourier expansion (e.g. using complex expansion), since this is a quite standard Galerkin approach (you have to mention the boundary conditions).**

With respect to the QG model, we prefer to follow the non-complex expansion, since it is the one used in the papers we cite, and it is also how the model was coded. We have provided information on the boundary conditions (page 11, lines 4-5 of the final manuscript):

> "Both fields are defined in a zonally periodic channel with no-flux boundary conditions in the meridional direction ($\partial \cdot / \partial x \equiv 0$ at $y = 0, \pi$). The fields are expanded in Fourier modes respecting these boundary conditions:..."

- **Regarding Fig. 2 (b) you may guide the reader a little bit, e.g., mention, that ψ_2 and ψ_3 represent the strength of the zonal wavenumber 1 anomalies, which are both small in the dashed line ellipse, and large in the dashed-dot ellipse, which also seems to have a strong zonal mean ψ_1 component.**

  We have added more explanations on Fig.2 according to the suggestion of the referee (page 12 last line, page 13 first line of the final manuscript):

  > "In the former case, the variables ψ_2 and ψ_3 characterising the strength of the meridional anomalies are small, while in the latter case they are large, indicating indeed a blocking situation."

- **What is the relevance of the equilibrium ploints?**

  The equilibrium points have no other utility than to help the reader have an idea about the general structure of the global attractor.

2.3 Technical corrections

In general, we agree with most of the corrections proposed by the referee. However, for some corrections, we have additional comments:

p. 1, l.5 "averages involved": I only understood after reading the paper, what is meant by this.

We have replaced "averages" by "parameters". This is indeed the parameters that contain the averages of the observables.

p. 1, l. 9 What is "in a more operational environment"? Maybe "in an environment more similar to the operational environment"

We changed the sentence into (last line of the abstract of the final manuscript):

> " The potential application in a operational environment is also discussed."

p 3., l. 17 square bracket "$\langle\rangle$" are not explained here, only later on. In general I find it confusing to refer here to averages, as these are to my understanding expectation values (which are indeed estimated via ensemble averages).

We understand your comment and the confusion that may arise, but we prefer to keep this notation as it is a traditional one in the context of stochastic process and response theory.

p. 3, l. 29 "Note that if $\kappa = 1$, the correction is perfect.": Not absolutely true, since forecast and model still differ in λ.

Thank you very much, the text has been changed to (page 6 line 20 of the new manuscript):

"Note that the best correction is obtained if $\kappa = 1$."

p. 6, l. 19 "It is also assumed that there is no interference due to initial condition errors in the problem.": Please clarify, what you mean by this.

We have removed the sentence, but an explanation is now provided in the new section 2 regarding response theory (last line of page 4 of the final manuscript):

"It is thus assumed that there is no interference due to initial condition errors in the perturbation problem. Note however that the effects on the trajectories of both the initial conditions perturbation and the $\boldsymbol{\Psi}$ perturbation can be investigated through this equation by setting $\boldsymbol{\delta y}(0) \neq 0$, altough we are not aware of any study of the response to both type of perturbations together."

Eq. (23) $f^\tau()$ is only given later and specific for the OU process. Please state here, what this is.

The definition is given just after in the text and Eq. (24). We felt that it was less artificial to introduce it right after its first appearance than before. However, the concept of dynamical system's flow is now also introduced in the new (and anterior) section 2.

Also Eq.(23) The term in the square brackes may be a scalar product in general, but as you have formulated ψ_y, it is not. Later on you use bold letters for vectors, right?

It is a simple multiplication, and thus we have removed the $\cdot$. Vectors are bold letters, as recommended by the NPG guidelines.

p. 7, l. 1 "its stochastic integrals ..." > This may need some explanation.

We think that the paper is already long enough so we prefer to avoid an introduction about stochastic integrals, but we now cite there Gardiner's book which is a reference on the subject (page 9, line 22 of the final manuscript).

Eq. (28) Check formula: Isnt there missing $A(f^\tau(x_o))$ and $A(f^\tau(x_o))$ should be $A(f^\tau(x_o))$? Again is this really a scalar product? Formula is right. This is a second-order formula in the perturbation $\mathbf{\Psi}$ which appears twice. On the other hand, the observable A should appears only once and the final time τ is correct. We comment on this now in a footnote. Please see Lucarini (2012) for more details. As before, you are right about the fact that it is not a scalar product and we removed the $\cdot$.

p.8, l.8+9 I assume, that the n in the sine and cosine should not be there.

n is the aspect ratio of the model domain which is $(0 \leq x \leq \frac{2\pi}{n}, 0 \leq y \leq \pi)$, therefore it must be included in the cosine and sine of x. We clarify now the domain in the model introduction (2 first lines of page 11 of the final manuscript):

"The horizontal adimensionalised coordinates are denoted x and y, the model's domain being defined by $(0 \leq x \leq \frac{2\pi}{n}, 0 \leq y \leq \pi)$, with $n = 2L_y/L_x$ the aspect ratio between its meridional and zonal extents L_y and L_x."

p. 12, Eq. (39) Use either scalar product or transpose ^{T} .

The transpose indicates that a row vector is used (rather than a column one), and the $\cdot$ indicates the kind of product that is being taken (here the scalar/matricial product indeed). This notation is quite standard in physics (see Gaspard (2005)), this is ultimately a matter of convention and we want to keep it the way it is.

Eq. (40) Since it is an approximation, use $\approx$ instead of $=$ here.

The referee is right that this is an approximation. But we make the quite standard abuse of notation here that $\boldsymbol{\delta x}$ is directly replaced by its first order approximation. We mention that it is the first order equation. With this abuse of notation, Eq. (40) is an equality and as it is an ODE, we prefer it like that.

p.21, l. 6ff Please revise paragraph.

We have changed the paragraph into:

> "To test this approach, we have focused on the EVMOS statistical postprocessing method, but other methods could be considered as well. The only requirement is that the outcome of the minimisation of the cost function uses averages of the systems being considered. For instance, member-by-member methods that correct both the mean square errors and the spread of the ensemble while preserving the spatial correlation (. . .) could be considered. These methods generally use the covariance between the model forecasts and the observations as ingredient. Response theory can also be applied here since this covariance can be written as an average. This will be investigated in a future work, together with the applicability of the approach to parameters of probability distributions, as often used in meteorology (. . .)."

3 Manuscripts differences (latexdiff)

Please see next page.

Correcting for Model Changes in Statistical ~~Post-Processing~~ Postprocessing – An approach based on Response Theory

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Abstract. For most statistical ~~post-processing~~ postprocessing schemes used to correct weather forecasts, changes to the forecast model induce a considerable ~~reforecasting~~ reforecasting effort. We present a new approach based on response theory to cope with slight model ~~change~~changes. In this framework, the model change is seen as a perturbation of the original forecast model. The response theory allows us then to evaluate the variation induced on the ~~averages~~ parameters involved in the statistical ~~post-processing~~postprocessing, provided that the magnitude of this perturbation is not too large. This approach is studied in the context of simple Ornstein-Uhlenbeck models, and then on a more realistic, yet simple, quasi-geostrophic model. The analytical results for the former case ~~allow for posing~~ help to pose the problem, while the application to the latter provide a proof-of-concept ~~of~~ and assesses the potential performances of response theory in a chaotic system. In both cases, the parameters of the statistical ~~post-processing~~ postprocessing used – an *Error-in-Variables* Model Output Statistics (EVMOS) – are appropriately corrected when facing a model change. The potential application in a ~~more~~ operational environment is also discussed.

1 Introduction

A generic property of the atmospheric dynamics is its sensivity to initial conditions. This implies that probabilistic forecasts will always be needed to adequately describe this behaviour (Wilks, 2011). Indeed, these methods represent a way to go beyond the natural predictability barrier that the chaotic atmospheric models exhibit (Vannitsem, 2017). These forecasts are at the same time subject to the impact of the presence of structural uncertainties, also known as *model errors*. Such errors degrade the forecasts as well, and their impact needs to be mitigated.

Statistical ~~post-processing~~ postprocessing methods are used to correct the operational predictions of the atmospheric models. An important family of statistical techniques used to ~~post-process~~ postprocess the forecasts are linear regression techniques, with possibly multiple predictors (Glahn and Lowry, 1972; Vannitsem and Nicolis, 2008), also known as Model Output Statistics (MOS). This rather simple but very efficient technique can be adapted to ensemble forecasts (e.g. Vannitsem (2009); Johnson and Bowler (2009); Glahn et al. (2009); Van Schaeybroeck and Vannitsem (2015)). One of the first approaches that was proposed is called *Error-in-Variable* MOS (EVMOS) because it takes into account the presence of errors in both the observations and model observables (Vannitsem, 2009).

Despite their simplicity, most ~~post-processing~~ postprocessing schemes depend on the availability of a database of past forecasts, that allows one to "train" the regression algorithm by comparison with the observations database. ~~These operational~~ Operational models are however subject to frequent evolution cycles, which are needed to improve their representation of the atmospheric ~~behaviours~~processes. Therefore, there is a continuous need to recompute ~~past forecasts~~ forecasts starting from past initial conditions with the latest model version, to avoid a degradation of the ~~post-processing~~ postprocessing schemes due to model change. Such recomputation of the past forecasts are called *reforecasts*, and typically requires a huge data storage and management framework, as well as large computational resources (Hamill, 2018). For instance, the *European Center for Medium-range Weather Forecast* (ECMWF) and the *National Weather Service* (NWS) both produce hundreds of reforecasts every week (Hamill et al., 2013).

~~Recent research have studied~~ A recent research has investigated non-homogeneous regression with time-adaptative training scheme, ~~where~~ for which a trade-off ~~have to be considered~~ between large training data sets for stable ~~estimation~~ estimates and the benefit of ~~shorter training periods to adjust more rapidly to changes in the data~~ a shorter training period for faster adaptation to data changes is considered (Lang et al., 2020). These results can help mitigate the impact of model change on ~~post-processing~~ postprocessing and may call into question the need for reforecast systems. These systems do however help to better represent rare events, they increase the size of the training data sets and greatly improve sub-seasonal forecasts (Scheuerer and Hamill, 2015; Hamill, 2018), which can justify their prohibitive cost.

The present work investigates another research direction and considers a new technique to reduce the cost of adapting a linear ~~post-processing~~ postprocessing scheme to a model change. This method relies on the response theory for dynamical systems (Ruelle, 2009) and assumes that the model variation can be written as analytical perturbations of the model tendencies. In this context, parameters variations as well as new terms in the tendencies are potential model ~~change~~changes.

In section ~~3~~2, we start by ~~considering~~ introducing the Ruelle response theory that is used to adapt past postprocessing parameters to new models. A didactical example of such adaptation is considered with simple Ornstein-Uhlenbeck models in section 3. It is used to describe the methodology and the concept involved. We show that obtaining a new ~~post-processing~~ postprocessing scheme after a model change requires the computation of the response of the average of the involved predictors, seen as observables of the system. In the simple case considered, exact analytical results for the response can be obtained at any order. The correction of the model observables and the ~~post-processing~~ postprocessing parameters due to the model change only requires the response-theory corrections up to the second order.

In section 4, a more complex case is considered with a toy model of atmospheric variability in the form of a 2-layer quasi-geostrophic model with an orography. We compute the linear response of the predictors of the ~~post-processing~~ postprocessing for two model change experiments involving a modification of the friction and the horizontal temperature gradient of the model. The response theory approach provides an efficient correction of the ~~post-processing~~ postprocessing scheme up to a lead time of ~~3~~ 4 days, which matches the lead-time window where the scheme's correction is efficient.

In the last section, we discuss the implications that this new method could have on operational forecast ~~post-processing~~ postprocessing systems, as well as new research avenues to be explored.

2 Response theory

The systems used to produce the weather forecasts are typically non-linear dynamical systems whose time evolution is governed by multi-dimensional ordinary differential equations:

$$\dot{\boldsymbol{y}} = \boldsymbol{F}(t, \boldsymbol{y}). \tag{1}$$

The generic chaotic nature of these systems for some parameter values implies that they are sensitive to the initial data used to produce the forecasts. For such chaotic dynamical systems, one can assume that a well-defined time-invariant measure exists, and with which the averages are performed. However, the existence of such measures has been proved for systems that are *uniformly hyperbolic* and they are called *SRB measure* (Young, 2002), but rigorous proofs for other systems are rather difficult to obtain. A way to proceed is then to proceed as if physical systems were uniformly hyperbolic. This assumption is called the Gallavotti-Cohen hypothesis (Gallavotti and Cohen, 1995a, b). With this assumption, response theory has been succesfully used in various weather and climate-related problems (Demaeyer and Vannitsem, 2018; Vissio and Lucarini, 2018; Lembo et al., 2019; Bó
. Indeed, the systems used to produce weather forecasts are typically not *uniformly hyperbolic*, but thanks to the aforementioned hypothesis, one can still use what will follow and compare with the results obtained with experiments.[1] It is the rationale behind the formal presentation of the linear response theory for general systems like (1) in Ruelle (1998a). Main concepts that will be used in this article are now introduced.

[1]We point the reader to recent articles dealing with the validity of the response theory for weakly hyperbolic systems and time series (Gottwald et al., 2016; Wormell and Gottwald, 2018).

2.1 Perturbations of dynamical systems

Without loss of generality, we shall assume for simplicity that the system (1) is autonomous and given by

$$\dot{\boldsymbol{y}} = \boldsymbol{F}(\boldsymbol{y}). \tag{2}$$

In the general setting considered, let's assume that any given probability measure converges to a unique invariant measure ρ under the time evolution given by the Liouville equation of (2). This measure is used to compute the average of an arbitrary observable A (a smooth function of the state $\boldsymbol{y}$) of the system, which is given by

$$\langle A \rangle_{\boldsymbol{y}} = \int \rho(\mathrm{d}\boldsymbol{y})\, A(\boldsymbol{y}) \tag{3}$$

and assuming the ergodicity of the system, a time average of the observable A along a trajectory of the system on its attractor can be equivalently performed:

$$\langle A \rangle_{\boldsymbol{y}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}\tau\, A\left(\boldsymbol{y}(\tau)\right) \tag{4}$$

where $\boldsymbol{y}(\tau)$ is a solution of Eq. (2). If a perturbation $\boldsymbol{\Psi}$ of the dynamical system is introduced in the original system at the time $\tau = 0$:

$$\dot{\boldsymbol{y}} = \boldsymbol{F}(\boldsymbol{y}) + \boldsymbol{\Psi}(\boldsymbol{y}), \tag{5}$$

it induces a perturbation of the obervable's average which at first order is given by[2]

$$\delta\langle A(\tau)\rangle_{\boldsymbol{y}} = \int \rho(\mathrm{d}\boldsymbol{y}_0)\, \delta A\left(\boldsymbol{f}^\tau(\boldsymbol{y}_0)\right) = \int \rho(\mathrm{d}\boldsymbol{y}_0)\, \boldsymbol{\delta y}(\tau)^\mathsf{T} \cdot \boldsymbol{\nabla}_{\boldsymbol{f}^\tau(\boldsymbol{y}_0)} A \tag{6}$$

where $\boldsymbol{f}^\tau$ is the flow the system (2) mapping an initial condition $\boldsymbol{y}_0$ to the system's state at time τ: $\boldsymbol{y}(\tau) = \boldsymbol{f}^\tau(\boldsymbol{y}_0)$. $\boldsymbol{\delta y}$ is the perturbation of the trajectory of the system induced by the perturbation $\boldsymbol{\Psi}$. This formula gives the transient response to the perturbation, and long time average of the integrand of (6) gives the stationary response to the perturbation, i.e. the sensibility $\delta\langle A\rangle_{\boldsymbol{y}}$ of the system observables to the perturbation (Eyink et al., 2004; Wang, 2013). The higher-order corrections $\delta^k\langle A(\tau)\rangle_{\boldsymbol{y}}$ can in principle be computed as well but are quite complicate to obtain for chaotic dynamical systems, see for instance Lucarini (2009). We will show an analytically tractable case in Section 3.

2.2 The tangent linear model

The linear perturbation $\boldsymbol{\delta y}$ of the trajectories of (2) can be computed by introducing $\boldsymbol{y} + \delta\boldsymbol{y}$ in Eq. (5) to get at first order

$$\dot{\boldsymbol{\delta y}} = \boldsymbol{\nabla}_{\boldsymbol{y}} \boldsymbol{F} \cdot \boldsymbol{\delta y} + \boldsymbol{\Psi}(\boldsymbol{y}) \tag{7}$$

[2] When taking the gradient of a function A, the notation $\boldsymbol{\nabla}_{\boldsymbol{y}} A$ means taking the gradient at the point $\boldsymbol{y}$, i.e. evaluating $\boldsymbol{\nabla}_{\boldsymbol{y}} A(\boldsymbol{y})$.

where $\boldsymbol{y}$ is the solution of system (2) and $\boldsymbol{\nabla_y F}$ is the Jacobian matrix evaluated along this solution. Therefore, both Eqs. (2) and (7) have to be integrated simultaneously. In the weather forecasting context, this latter linearised equation is called the tangent linear model of system (2) (Kalnay, 2003) *plus* the perturbation term $\boldsymbol{\Psi}$. Without this latter term, it provides the linearised time evolution of a perturbation $\boldsymbol{\delta y}$ superimposed to the initial conditions. Here, Eq. (7) is initialised with $\boldsymbol{\delta y}(0) = 0$ and provides the linear "response" of the trajectory $\boldsymbol{y}(\tau)$ to the perturbation $\boldsymbol{\Psi}$. It is thus assumed that there is no interference due to initial condition errors in the perturbation problem. Note however that the effects on the trajectories of both the initial conditions perturbation and the $\boldsymbol{\Psi}$ perturbation can be investigated through this equation by setting $\boldsymbol{\delta y}(0) \neq 0$, altough we are not aware of any study of the response to both type of perturbations together.

The tangent model provides thus the tool through which we will evaluate the impact of the model change on the average used by statistical postprocessing schemes. In other words, the tangent model will allow us to take into account the information on the model change (viewed as a perturbation of the initial model) to modify the previous postprocessing scheme and adapt it to the new model. The solution to Eq. (7) with $\boldsymbol{\delta y}(0) = 0$ is given by

$$\boldsymbol{\delta y}(\tau) = \int_0^\tau \mathrm{d}\tau' \, \mathbf{M}\left(\tau - \tau', \boldsymbol{f}^{\tau'}(\boldsymbol{y_0})\right) \cdot \boldsymbol{\Psi}\left(\boldsymbol{f}^{\tau'}(\boldsymbol{y_0})\right), \tag{8}$$

where $\mathbf{M}$ is the fundamental matrix of solutions of Eq. (7) (Gaspard, 2005; Nicolis, 2016):

$$\mathbf{M}(\tau, \boldsymbol{y}) = \boldsymbol{\nabla_y f}^\tau \tag{9}$$

solution of the homogeneous equation $\dot{\mathbf{M}} = \boldsymbol{\nabla_y F} \cdot \mathbf{M}$. Using the chain rule, the response (6) is rewritten in term of the perturbation alone:

$$\delta\langle A(\tau)\rangle_{\boldsymbol{y}} = \int_0^\tau \mathrm{d}\tau' \int \rho(\mathrm{d}\boldsymbol{y}_0) \, \boldsymbol{\Psi}\left(\boldsymbol{f}^{\tau'}(\boldsymbol{y}_0)\right)^{\mathsf{T}} \cdot \boldsymbol{\nabla}_{\boldsymbol{f}^{\tau'}(\boldsymbol{y}_0)} A\left(\boldsymbol{f}^\tau(\boldsymbol{y}_0)\right) \tag{10}$$

where the causality of the perturbation acting on the system and perturbing the averaged observable appears (Lucarini, 2008), since $\tau' < \tau$. We will also use this alternative expression throughout the article. Note that when the initial perturbation $\boldsymbol{\delta y}(0)$ is not equal to 0, additional terms to Eqs. (8) and (10) will appear. These will not be addressed here but some hints can be found in Nicolis et al. (2009) and Nicolis (2016).

2.3 Non-stationary response theory

The equation (6) gives the transient, non-stationary response to the perturbation, evaluated for averages computed with the invariant measure. However, in this work, we need to evaluate response to perturbation for averages computed with non-stationary measures evolving in time. In that sense, it is a *non-stationary response theory*, done for arbitrary initial probability density. As such, all the formula presented are valid if the measure being used is the measure at the time when the perturbation is introduced ($\tau = 0$), as shown in Appendix A. In this case, other usual formula obtained through substitution, for instance to obtain an adjoint representation of (10), should be used with care since the measure is no longer invariant and an extra Jacobian term appears in the integrand.

We will thus also assume that the measures ρ_τ being used are absolutely continuous with respect to the Lebesgue measure. In this case, we can write $\rho_\tau(\mathrm{d}\boldsymbol{y}) = \rho_\tau(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$. We now present the problem of model change in the framework of postprocessing and show on a simple stochastic model[3] how response theory allows to tackle the issue.

3 A simple analytical example

In order to get a first impression of the impact of a model change on a ~~post-processing~~ postprocessing scheme, we consider two Ornstein-Uhlenbeck processes representing the reality $x(\tau)$ and a model $y(\tau)$ of the reality. These processes obey the following equations:

$$\dot{x}(\tau) = -\lambda_x\, x(\tau) + K_x + Q_x\, \xi_x(\tau) \tag{11}$$

$$\dot{y}(\tau) = -\lambda_y\, y(\tau) + K_y + Q_y\, \xi_y(\tau) \tag{12}$$

where ξ_x and ξ_y are Gaussian white noise processes such that

$$\begin{aligned}
\langle \xi_x(\tau) \rangle = \langle \xi_y(\tau) \rangle &= 0 \\
\langle \xi_x(\tau)\, \xi_x(\tau') \rangle &= \delta(\tau - \tau') \\
\langle \xi_y(\tau)\, \xi_y(\tau') \rangle &= \delta(\tau - \tau') \\
\langle \xi_x(\tau)\, \xi_y(\tau') \rangle &= 0
\end{aligned}$$

These are therefore uncorrelated Ornstein-Uhlenbeck processes with noise amplitudes Q_x and Q_y.

We then consider a change Ψ_y of the model $y(\tau)$, possibly improving or degrading the forecast performances:

$$\dot{\hat{y}}(\tau) = -\lambda_y\, \hat{y}(\tau) + K_y + Q_y\, \xi_y(\tau) + \Psi_y(\tau) \tag{13}$$

where

$$\Psi_y(\tau) = -\kappa \left(\delta K + \delta Q\, \xi_y(\tau) \right) \tag{14}$$

with $\delta K = K_y - K_x$ and $\delta Q = Q_y - Q_x$. It can represent, for example, a better parameterisation of subgrid-scale processes or an increase of the model resolution. Note that ~~if~~ the best correction is obtained if $\kappa = 1$~~, the correction is perfect~~.

We have thus the reality $x(\tau)$ and two different models of it: $y(\tau)$ and $\hat{y}(\tau)$. We now want to evaluate the difference between a ~~post-processing~~ postprocessing scheme constructed before the model change (with the past forecasts of the model $y(\tau)$), and one constructed after ~~it~~ (with the past forecasts of model $\hat{y}(\tau)$).

[3] Response theory is also valid for stochastic models with a well-defined stationary measure, as shown by Lucarini (2009).

3.1 The ~~post-processing~~ postprocessing method

We now consider a forecast situation where the model y is initialised at the time $\tau = 0$ with a perfect observation of the reality: $y(0) = x(0) = x_0$. We use the *Error-in-Variables Model Output Statistics* (EVMOS) ~~post-processing~~ postprocessing scheme (Vannitsem, 2009) to correct the forecasts of the model y based on these initial conditions. In this context, given ~~K past forecasts y_k and observations x_k~~ N past forecasts y_n and observations x_n, the correction of the univariate EVMOS ~~post-processing~~ postprocessing of variable x from a new forecast $y(\tau)$ is provided by the linear regression

$$\cancel{x}y_C(\tau) = \alpha(\tau) + \beta(\tau)\cancel{\cdot}\, y(\tau) \tag{15}$$

The coefficients α and β are obtained by minimising the functional

$$J(\tau) = \sum \cancel{{}_{k=1}^{K} \frac{[\{\alpha(\tau) + \beta(\tau) y_k(\tau)\} - x_k(\tau)]}{\sigma_x^2(\tau) + \beta^2(\tau)\,\sigma_y^2(\tau)}}\, {}_{n=1}^{N} \frac{[\{\alpha(\tau) + \beta(\tau) y_n(\tau)\} - x_n(\tau)]^2}{\sigma_x^2(\tau) + \beta^2(\tau)\,\sigma_y^2(\tau)}, \tag{16}$$

and are thus given by the equations:

$$\alpha(\tau) = \langle x(\tau)\rangle - \beta(\tau)\,\langle y(\tau)\rangle \tag{17}$$

$$\beta(\tau) = \sqrt{\frac{\sigma_x^2(\tau)}{\sigma_y^2(\tau)}} \tag{18}$$

where

$$\sigma_x^2(\tau) = \left\langle \left(x(\tau) - \langle x(\tau)\rangle\right)^2 \right\rangle \tag{19}$$

$$\sigma_y^2(\tau) = \left\langle \left(y(\tau) - \langle y(\tau)\rangle\right)^2 \right\rangle \tag{20}$$

The averages $\langle\cdot\rangle$ are taken over an ensemble of past forecasts and observations. This approach has been developed to ~~allow for~~ obtain a correct climatological forecast calibration. It constitutes a simple setting in which the impact of model changes can be evaluated and corrected. More sophisticated approaches can be evaluated in the future (other MOS schemes, ensemble MOS,...).

Since we are dealing with simple analytical models here, we can compute the theoretical values of the coefficient α and β with an infinite ensemble of past forecasts, and the averaged quantities involved in this computation are then given by the averages of an infinite number of realisations of the Ornstein-Uhlenbeck processes, as if we had an infinite ensemble of past forecasts.

3.2 Averaging the Ornstein-Uhlenbeck processes

For the reality x and the model y, we directly get the averages (Gardiner, 2009)

$$\langle x(\tau)\rangle = \langle x_0\rangle\, e^{-\lambda_x \tau} + \frac{K_x}{\lambda_x}\left(1 - e^{-\lambda_x \tau}\right) \tag{21}$$

$$\sigma_x^2(\tau) = \sigma_{x_0}^2\, e^{-2\lambda_x \tau} + \frac{Q_x^2}{2\,\lambda_x}\left(1 - e^{-2\,\lambda_x \tau}\right) \tag{22}$$

and

$$\langle y(\tau)\rangle = \langle x_0\rangle\, e^{-\lambda_y\,\tau} + \frac{K_y}{\lambda_y}\,\left(1-e^{-\lambda_y\,\tau}\right) \tag{23}$$

$$\sigma_y^2(\tau) = \sigma_{x_0}^2\, e^{-2\lambda_y\,\tau} + \frac{Q_y^2}{2\,\lambda_y}\,\left(1-e^{-2\,\lambda_y\tau}\right) \tag{24}$$

where we note that the model is initialised with the same initial conditions as the reality:

$$\langle y(0)\rangle = \langle x(0)\rangle = \langle x_0\rangle \qquad , \qquad \sigma_y^2(0) = \sigma_x^2(0) = \sigma_{x_0}^2 \tag{25}$$

We get the ~~post-processing~~ postprocessing coefficients before the model change $\alpha(\tau)$ and $\beta(\tau)$ by inserting these expressions in the equations (17) and (18).

Similarly, we get the same kind of results for the model $\hat{y}$, after the model change Ψ_y:

$$\langle \hat{y}(\tau)\rangle = \langle x_0\rangle\, e^{-\lambda_y\,\tau} + \frac{K_y-\kappa\,\delta K}{\lambda_y}\,\left(1-e^{-\lambda_y\,\tau}\right) \tag{26}$$

$$\sigma_{\hat{y}}^2(\tau) = \sigma_{x_0}^2\, e^{-2\lambda_y\,\tau} + \frac{(Q_y-\kappa\,\delta Q)^2}{2\,\lambda_y}\,\left(1-e^{-2\,\lambda_y\tau}\right) \tag{27}$$

and we also obtain the ~~post-processing~~ postprocessing coefficients after the model change $\hat{\alpha}(\tau)$ and $\hat{\beta}(\tau)$ (see also the analysis in Vannitsem (2011)). We can also compute the variation of the bias α:

$$\hat{\alpha}(\tau) - \alpha(\tau) = \delta\alpha(\tau) = \beta(\tau)\,\langle y(\tau)\rangle - \hat{\beta}(\tau)\,\langle \hat{y}(\tau)\rangle \tag{28}$$

The ratio between the parameters β is given by

$$\frac{\hat{\beta}(\tau)}{\beta(\tau)} = \sqrt{\frac{\sigma_y^2(\tau)}{\sigma_{\hat{y}}^2(\tau)}} \tag{29}$$

For $\tau \gg \max(1/\lambda_x, 1/\lambda_y)$, we note that this ratio tends to

$$\cancel{\frac{\hat{\beta}(\tau)}{\hat{\beta}(\tau)}}\,\frac{\hat{\beta}(\tau)}{\beta(\tau)} \approx \frac{1}{1-\kappa\,\delta Q/Q_y} \tag{30}$$

and the difference between the biases α of the two models is approximatively given by:

$$\delta\alpha(\tau) \approx -\beta(\tau)\,\frac{K_y}{\lambda_y}\left[\frac{1-\kappa\,\delta K/K_y}{1-\kappa\,\delta Q/Q_y} - 1\right]. \tag{31}$$

Let us now assume that the model change Ψ_y can be considered as a perturbation of the initial model y. Using response theory, the averages $\langle\hat{y}\rangle$ and $\sigma_{\hat{y}}^2$ can be estimated using the initial model y instead of the perturbed model $\hat{y}$. In turn, ~~the use of~~ these new estimated averages ~~allows for computing the new post-processing~~ give us the new postprocessing scheme coefficients $\hat{\alpha}$ and $\hat{\beta}$. We now detail the results obtained by using this method.

3.3 Model Change and Response theory

After the model change, the forecasts are provided by the model $\hat{y}$ and their time-evolution is given by Eq. (13). This model can be seen as a perturbation of the model y by the term Ψ_y given by Eq. (14). In such case, given an observable A, its average after the model change can then be related to its original average by

$$\langle A(\tau)\rangle_{\hat{y}} = \langle A(\tau)\rangle_y + \delta\langle A(\tau)\rangle_y + \delta^2\langle A(\tau)\rangle_y + \ldots \tag{32}$$

where the averages on the right-hand side are taken over the forecasts of model y. Response theory allows us to obtain the average over the model $\hat{y}$ forecasts (the left-hand side) based solely on the average over ~~model the~~ the model y forecasts. The $\hat{y}$ model forecasts are therefore not required to estimate the new ~~post-processing~~ postprocessing scheme.

The observables depend on the lead time τ of the forecast, as do the parameters α and β which determine the ~~post-processing~~ postprocessing correction for every lead time. This reflects the fact that the ~~post-processing~~ postprocessing problem is typically a non-stationary initial value problem, since the initial conditions of the model Eqs. (12) and (13) are typically not chosen on their respective model attractor, but rather as observations[4] of the reality (11). As a consequence, the model averages (32) relax toward the stationary response in the long-time limit, and the stationary response theory (Ruelle, 2009; Wang, 2013) cannot provide us their short-time relaxation behaviour. Instead, the Ruelle time-dependent response theory should be used (Ruelle, 1998a). It follows that, if the perturbation (14) is small, then the first order is given by (see ~~Appendix A):~~ section 2):

$$\delta\langle A(\tau)\rangle_y = \int_0^{\tau} \mathrm{d}\tau' \int \mathrm{d}x_0\, \rho_0(x_0) \left\langle \Psi_y(\tau')\cancel{\cdot}\nabla_{f^{\tau'}(x_0)} A(f^{\tau}(x_0)) \right\rangle \tag{33}$$

where ρ_0 is the distribution of the initial conditions (observations) used to initialise the models. ∇_x is the gradient evaluated at the point x, and here it is the simple derivative. As indicated by Eq. (25), in the ~~post-processing framework, it~~ postprocessing framework, ρ_0 is taken as the stationary/invariant distribution of the reality. ~~It is also assumed that there is no interference due to initial condition errors in the problem.~~ As shown in Appendix A, Eq. (33) can be obtained through a Kubo-type perturbative expansion (Lucarini, 2008). We remark that this example deals with stochastic models, due to which we have to perform an additional averaging over the realisations of the stochastic processes, denoted here as $\langle\cdot\rangle$ (Lucarini, 2012). Finally the mapping f^{τ} which appears in Eq. (33) is the stochastic flow:

$$f^{\tau}(x_0) = x_0\, e^{-\lambda_y \tau} + \int_0^{\tau} \mathrm{d}\tau'\, e^{-\lambda_y(\tau-\tau')} \Big[Q_y\, \xi_y(\tau') + K_y\Big]. \tag{34}$$

This maps an initial condition x_0 of the model y to the state $f^{\tau}(x_0)$ of a realisation of this model at the later lead time τ. The principle of causality is thus implicit in Eq. (33), which estimates the impact of the perturbation Ψ_y on the subsequent perturbed model time-evolution by developing around the unperturbed model y trajectories.

[4] Here we consider that the observation are perfectly assimilated in the models, and that there is no observation errors. However in operational setups, such errors are of course to be taken into account.

Evaluating Eq. (33) and its stochastic integrals (Gardiner, 2009) gives us the variation of the averages $\langle y(\tau)\rangle$ and $\langle y(\tau)^2\rangle$ to the perturbation Ψ_y:

$$\delta\langle y(\tau)\rangle_y \;=\; -\kappa\int_0^\tau \mathrm{d}\tau'\delta K\, e^{-\lambda_y(\tau-\tau')} = -\frac{\kappa}{\lambda_y}\,\delta K\,\left(1-e^{-\lambda_y\tau}\right) \tag{35}$$

$$\begin{aligned}\delta\langle y(\tau)^2\rangle_y \;&=\; -2\,\kappa\,\delta K\int_0^\tau \mathrm{d}\tau'\left[\langle x_0\rangle\, e^{-\lambda_y(2\tau-\tau')}+\frac{K_y}{\lambda_y}\,e^{-\lambda_y(\tau-\tau')}\left(1-e^{-\lambda_y\tau}\right)\right]-2\,\kappa\,\delta Q\,Q_y\int_0^\tau \mathrm{d}\tau'\,e^{-2\,\lambda_y(\tau-\tau')}\\ &=\; -2\,\kappa\,\frac{\delta K}{\lambda_y}\,\langle y(\tau)\rangle\,\left(1-e^{-\lambda_y\tau}\right)-\frac{\kappa}{\lambda_y}\,\delta Q\,Q_y\,\left(1-e^{-2\,\lambda_y\tau}\right)\end{aligned} \tag{36}$$

Rearranging these two terms, we also get the following expression for the variation of the variance (24):

$$\delta\sigma_y^2(\tau) = -\frac{\kappa}{\lambda_y}\,\delta Q\,Q_y\,\left(1-e^{-2\,\lambda_y\tau}\right)-\frac{\kappa^2}{\lambda_y^2}\,\delta K^2\,\left(1-e^{-\lambda_y\tau}\right)^2 \tag{37}$$

Note that the variation (35) corresponds to the exact difference between the average of the two models $\langle\hat{y}(\tau)\rangle-\langle y(\tau)\rangle$. On the other hand the variation given by Eq. (37) lacks the term of order κ^2 involving δQ that appears in the exact difference $\sigma^2_{\hat{y}}(\tau)-\sigma^2_y(\tau)$ given by Eqs. (24) and (27). Instead, another term of order κ^2 and involving δK is present, indicating that ~~we need to consider the~~ higher-order ~~term~~ terms of response theory ~~Ruelle (1998b)~~ need to be considered to correct it (Ruelle, 1998b). The second-order term is given by the expression[5] (Lucarini, 2012):

$$\delta^2\langle A(\tau)\rangle_y = \int_0^\tau \mathrm{d}\tau'\int_{\tau'}^\tau \mathrm{d}\tau''\int \mathrm{d}y\,\rho_0(x_0)\,\left\langle \Psi_y(\tau')\cdot\nabla_{f^{\tau'}(x_0)}\Psi_y(\tau'')\cdot\nabla_{f^{\tau''}(x_0)}A\left(f^\tau(x_0)\right)\right\rangle. \tag{38}$$

Applying this to the first moment of the y models directly yields

$$\delta^2\langle y(\tau)\rangle_y = 0. \tag{39}$$

On the other hand, integrating the stochastic integrals present in this expression for the moment $\langle y(\tau)^2\rangle$ gives

$$\delta^2\langle y(\tau)^2\rangle_y = \frac{\kappa^2\,\delta K^2}{\lambda_y^2}\,\left(1-e^{-\lambda_y\tau}\right)^2+\frac{\kappa^2\,\delta Q^2}{2\lambda_y}\,\left(1-e^{-2\lambda_y\tau}\right) \tag{40}$$

which corrects the $\kappa^2\delta K^2$ term in Eq. (37) and makes the response theory up to order 2 exactly match the difference $\sigma^2_{\hat{y}}(\tau)-\sigma^2_y(\tau)$, for every lead time τ. In fact, the subsequent orders of the response vanish due to the linearity of the simple Ornstein-Uhlenbeck models, which ~~allows~~ enables us to truncate the response Kubo-like expansion to the second order. Finally, this shows that the (non-stationary) response theory can be used to estimate the ~~post-processing~~ postprocessing parameters after the model change based on the forecasts of the initial model. Indeed, instead of the averages $\langle\hat{y}(\tau)\rangle$ and $\langle\sigma^2_{\hat{y}}(\tau)\rangle$, the approximate averages $\langle y(\tau)\rangle+\delta\langle y(\tau)\rangle_y$ and ~~$\sigma^2_y(\tau)+\delta\langle\sigma^2_y(\tau)\rangle_y+\delta^2\langle\sigma^2_y(\tau)\rangle$~~ $\sigma^2_y(\tau)+\delta\sigma^2_y(\tau)+\delta^2\sigma^2_y(\tau)$ can be used to compute $\hat{\alpha}$ and $\hat{\beta}$. We emphasise that the second order contribution had to be considered in order to obtain the exact result. Nevertheless, the

[5]This expression is equivalent to the second term of Eq. (1) in Lucarini (2012) upon a time transformation. It can also be obtained by computing explicitly the second order perturbation of the average in Eq. (A14) in Appendix A.

difference between the first and the second order response is of order κ^2, which implies that for a small perturbation (model change), the first order will generally be a sufficiently good approximation. A more detailed derivation of the results obtained in this section can be found in the supplementary material.

In order to investigate this research avenue on a case closer to those encountered in reality, we will now consider the application of ~~post-processing~~ postprocessing and response theory to a low-order atmospheric model displaying chaos.

4 Application to a low-order atmospheric model

A 2-layer quasi-geostrophic atmospheric system on a β-plane with an orography is considered (Charney and Straus, 1980; Reinhold and Pierrehumbert, 1982). This spectral model possesses well-identified large-scale flow regimes, such as *zonal* and *blocked* regimes. The horizontal adimensionalised coordinates are denoted x and y, the model's domain being defined by $(0 \le x \le \frac{2\pi}{n}, 0 \le y \le \pi)$, with $n = 2L_y/L_x$ the aspect ratio between its meridional and zonal extents L_y and L_x. The two main fields of this model are the 500 hPa pressure anomaly and temperature, which are proportional to the barotropic streamfunction $\psi(x,y)$ and the baroclinic streamfunction $\theta(x,y)$, respectively. ~~These~~ Both fields are defined in a zonally periodic channel with no-flux boundary conditions in the meridional direction ($\partial \cdot / \partial x \equiv 0$ at $y = 0, \pi$). The fields are expanded in Fourier modes respecting these boundary conditions:

$$
\begin{aligned}
F_1(x,y) &= \sqrt{2}\cos(y), \\
F_2(x,y) &= 2\cos(nx)\sin(y), \\
F_3(x,y) &= 2\sin(nx)\sin(y), \\
F_4(x,y) &= \sqrt{2}\cos(2y), \\
&\vdots
\end{aligned}
$$

such that

$$\nabla^2 F_i(x,y) = -a_i^2 F_i(x,y) \tag{41}$$

with eigenvalues $a_1^2 = 1,\ a_2^2 = a_3^2 = 1 + n^2,\ a_4^2 = 4, \ldots$ ~~, where n is the aspect ratio of the x-y domain~~. We have thus the following decomposition

$$\psi(x,y) = \sum_{i=1}^{n_\mathrm{a}} \psi_i F_i(x,y) \tag{42}$$

$$\theta(x,y) = \sum_{i=1}^{n_\mathrm{a}} \theta_i F_i(x,y). \tag{43}$$

where n_a is the number of modes of the spectral expansion. The partial differential equations controlling the time evolution of the fields $\psi(x,y)$ and $\theta(x,y)$ can then be projected on the Fourier modes to finally give a set of ordinary differential equations for the coefficients ψ_i and θ_i:

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}) \qquad , \quad \boldsymbol{x} = (\psi_1, \ldots, \psi_{n_\mathrm{a}}, \theta_1, \ldots, \theta_{n_\mathrm{a}}) \tag{44}$$

that can be solved with usual numerical integrators. All variables are ~~adimensionalized~~adimensionalised. The ordinary differential equations of the model are detailed in Appendix B.

In the version proposed by Reinhold and Pierrehumbert using the 10 first modes, beyond a certain value of the zonal temperature gradient, the system displays chaos and makes transitions between the blocked and zonal flow regimes embedded in its global attractor. Here, we ~~adopt their notations and main adimensionalized~~ use their main adimensionalised parameters values: the friction at the interface between the two layers ~~$k = 0.05$~~$k_d = 0.1$, the friction at the bottom surface ~~$k' = 0.01$~~$k'_d = 0.01$, and the aspect ratio of the domain $n = 1.3$. The β-plane lies at mid-latitude (~~45~~50°) and the Coriolis parameter f_0 is set accordingly.

In the present work, the parameter ~~h~~h_d, the Newtonian cooling coefficient is fixed to ~~0.1~~ 0.3 instead of the value found in Reinhold and Pierrehumbert (which is ~~$h = 0.045$~~$h_d = 0.045$). Two additional fields have to be specified on the domain: $\theta^\star(x,y)$, the radiative equilibrium temperature field, and ~~$\hbar(x,y)$~~$h(x,y)$, the topographic height field. These fields can be decomposed by projecting them onto the eigenfunctions of the Laplacian as before. The corresponding coefficients $\theta_i^\star$ and ~~$\hbar_i$~~ h_i then allow for writing these fields as sums of weighted eigenfunctions:

$$\theta^\star(x,y) = \sum_{i=1}^{n_\mathrm{a}} \theta_i^\star \, F_i(x,y) \tag{45}$$

$$\cancel{\hbar} h(x,y) = \sum_{i=1}^{n_\mathrm{a}} \cancel{\hbar} h_i \, F_i(x,y). \tag{46}$$

In the present case, we consider that the only non-zero coefficients are $\theta_1^\star = 0.2$ and ~~$\hbar_2 = 0.2$~~$h_2 = 0.4$, meaning that the radiative equilibrium profile is given by the zonally varying function $\sqrt{2}\cos(y)$ and the orography is made of a mountain and a valley shaped by the function $2\cos(nx)\sin(y)$. Again, the value of the temperature gradient $\theta_1^\star$ is larger than the one chosen in Reinhold and Pierrehumbert (which is $\theta_1^\star = 0.1$) to increase the chaotic variability in the system. Trajectories of variables θ_1 and ψ_4 are depicted in Fig. 1, for the reference system (reality) and a model version (model 0) for which the friction coefficient has been slightly modified.

These parameter changes induce slight modifications of the dynamics. In particular the system possesses two distinct weather regimes, depicted in Fig. 2(b): one characterised by a zonal circulation (see Fig. ~~1~~2(c)), and another characterised by a blocking situation (see Fig. ~~1~~2(d)). In the former case, the variables ψ_2 and ψ_3 characterising the strength of the meridional anomalies are small, while in the latter case they are large, indicating indeed a blocking situation. This is different from the ~~case~~ situation considered in Reinhold and Pierrehumbert (1982), where two different blocking regimes coexist with the zonal regime.

4.1 ~~Post-processing~~ Postprocessing experiments

The model described above with 10 modes ($n_\mathrm{a} = 10$) is used and two different ~~post-processing~~ postprocessing experiments are performed, one involving the Newtonian cooling parameter ~~h~~ h_d and another involving the friction parameter ~~k~~ k_d between the two atmospheric layers. The parameter values detailed above correspond to the long-term reference (i.e. the reality). A first model is defined (model 0) which is a copy of the 2-layer quasi-geostrophic model defining the reality, but the parameters ~~h or k~~ h_d or k_d are slightly changed, i.e. the model error of the forecasting system lies in either the Newtonian cooling or the friction parameter. Then, as in Section 3, a model change is imposed, leading to another forecasting model (model 1) that can

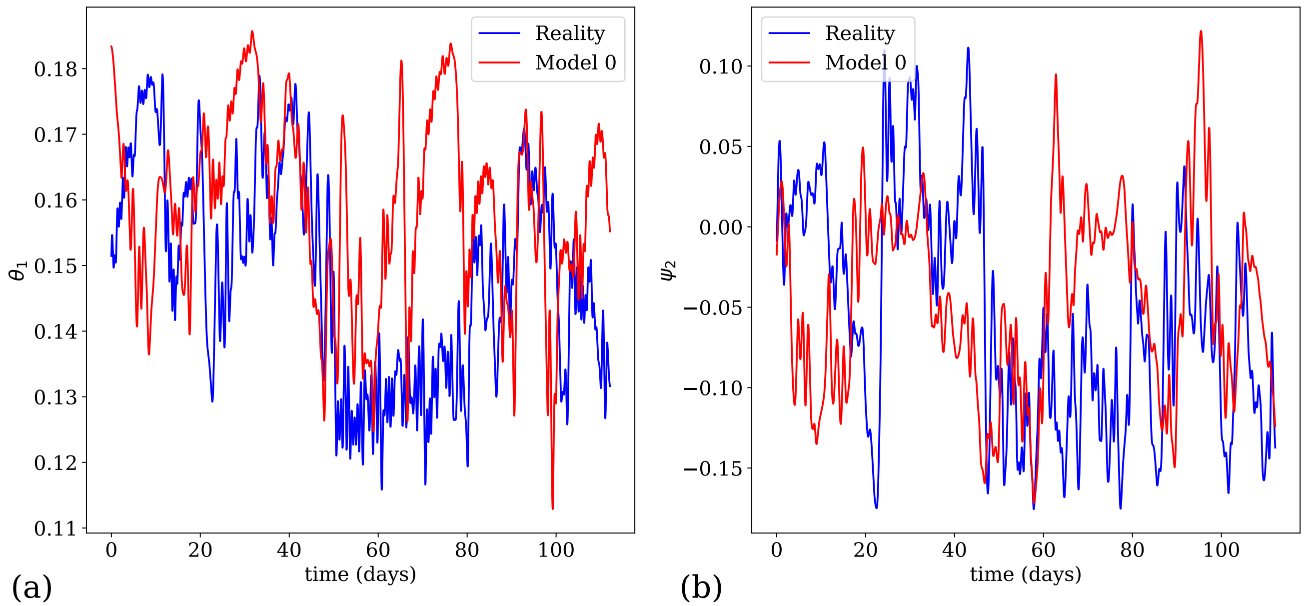

Figure 1. Dynamics of reference system and model 0 of the ~~post-processing~~ postprocessing experiment with change of friction (see Table 1), for: (a) time evolution of the variable θ_1, (b) time evolution of the variable ψ_4.

either improve or degrade the model error by a factor κ. The parameter variations involved in these experiments are detailed in Table 1. Without loss of generality, we consider model changes that improve the representation of reality, in the sense that the amplitude of the model errors in model 1 is smaller than in model 0. The effect of the model change is depicted in Figs. 3 and 4 for the friction parameter experiment. These figures display the mean and the standard deviation of the model forecasts and observations coming from the reference forecasts, as a function of the lead time τ. We have used a set of one million trajectories of each system to compute these averages.

In the framework of the EVMOS ~~post-processing~~ postprocessing scheme, the predictors and the predictands are the same nominal variable and no other predictors are used. In both experiments considered, the ~~post-processing~~ postprocessing parameters α and β of the EVMOS for model 0, as well as $\hat{\alpha}$ and $\hat{\beta}$ for model 1, are computed. The main objective here is then to estimate the difference between the former and the latter using Ruelle response theory. The approach in a multivariate setting is presented below.

4.2 Model change, Response theory and the ~~tangent linear~~ Tangent Linear model

Let us consider again the response theory described in ~~Section~~section 3.3, but in the general multivariate deterministic case described in section 2. In the ~~post-processing~~ postprocessing framework, models 0 and 1 evolve in time from a set of initial conditions taken outside of their respective attractors. Response formulae found in Ruelle's work have to be adapted to take this into account. One therefore has to consider the density of initial conditions as the measure. For a system with a time-

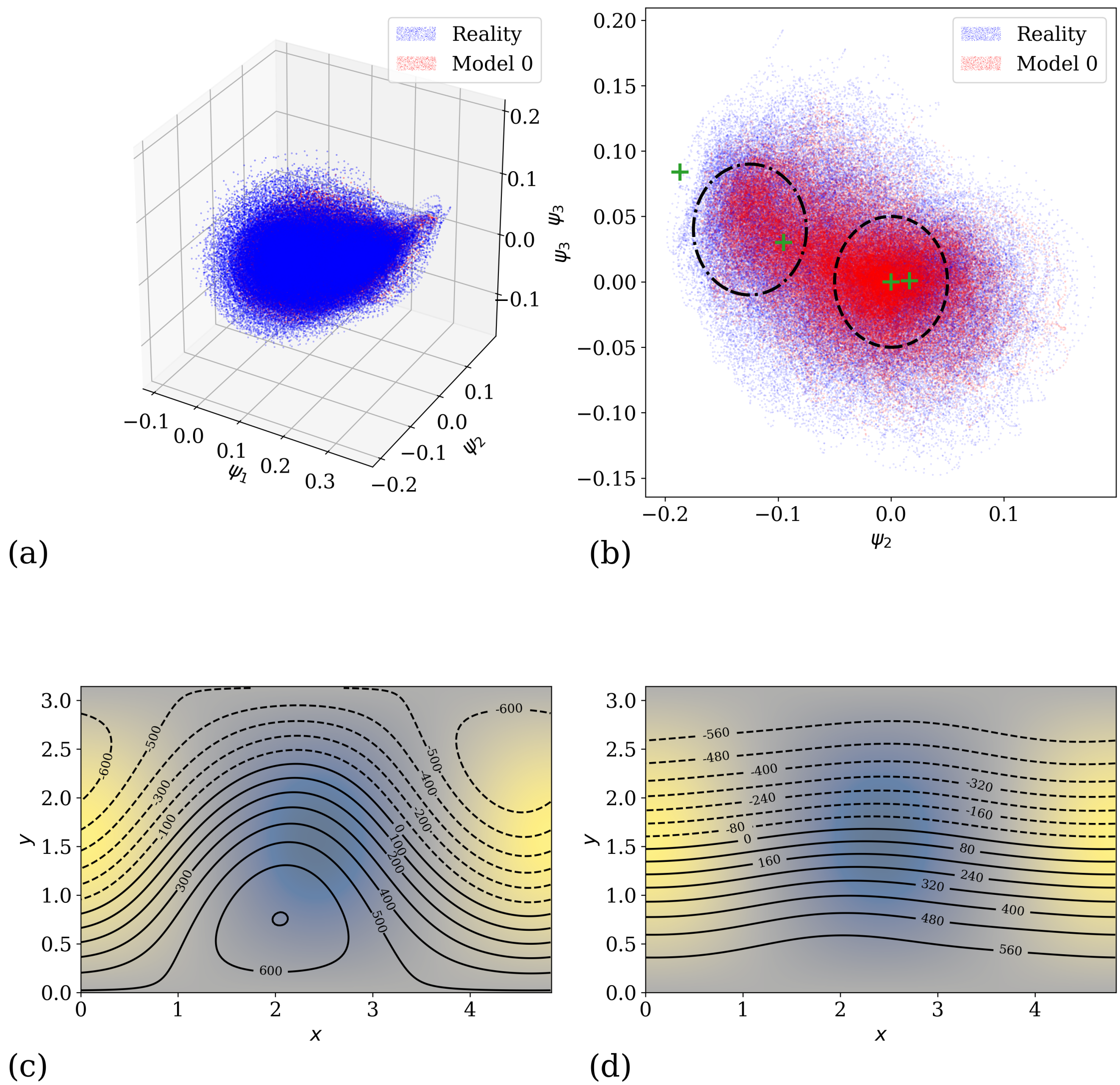

Figure 2. (a) 3D scatter plot of the attractors along the variables ψ_1, ψ_2 and ψ_3, (b) 2D scatter plot of the attractors along the variables ψ_2 and ψ_3. ~~Two different parts~~ The attractors of the ~~attractors~~ reality and model 0 are qualitatively similar, with two different parts which are indicated by ellipses. The green crosses correspond to equilibrium points of the reference model, the reality. The dashed ellipse corresponds on average to a zonal circulation depicted on panel (c). The dashed-dotted ellipse corresponds on average to a blocking situation depicted in panel (d). In both panels (c) and (d), the underlying colour map denotes the orography on the domain, and the contours the geopotential height anomaly at 500 hPa.

Parameter description \ Experiment	Symbol \ System	Reality	Newtonian cooling modification: Model 0	Newtonian cooling modification: Model 1	Friction coefficient modification: Model 0	Friction coefficient modification: Model 1
Newtonian cooling coefficient	h_d	0.3	0.33	0.315	0.3	
Atm. layers friction	k_d	0.1	0.1		0.12	0.11
Bottom layer friction	k'_d	0.01				
Domain aspect ratio	n	1.3				
Meridional temperature gradient	$\theta_2^\star$	0.2				
Mountain ridge altitude	h_2	0.4				

Table 1. The main parameters used and modified in the experiments. Model 0 and model 1 are respectively the forecast model of the reality before and after the model change.

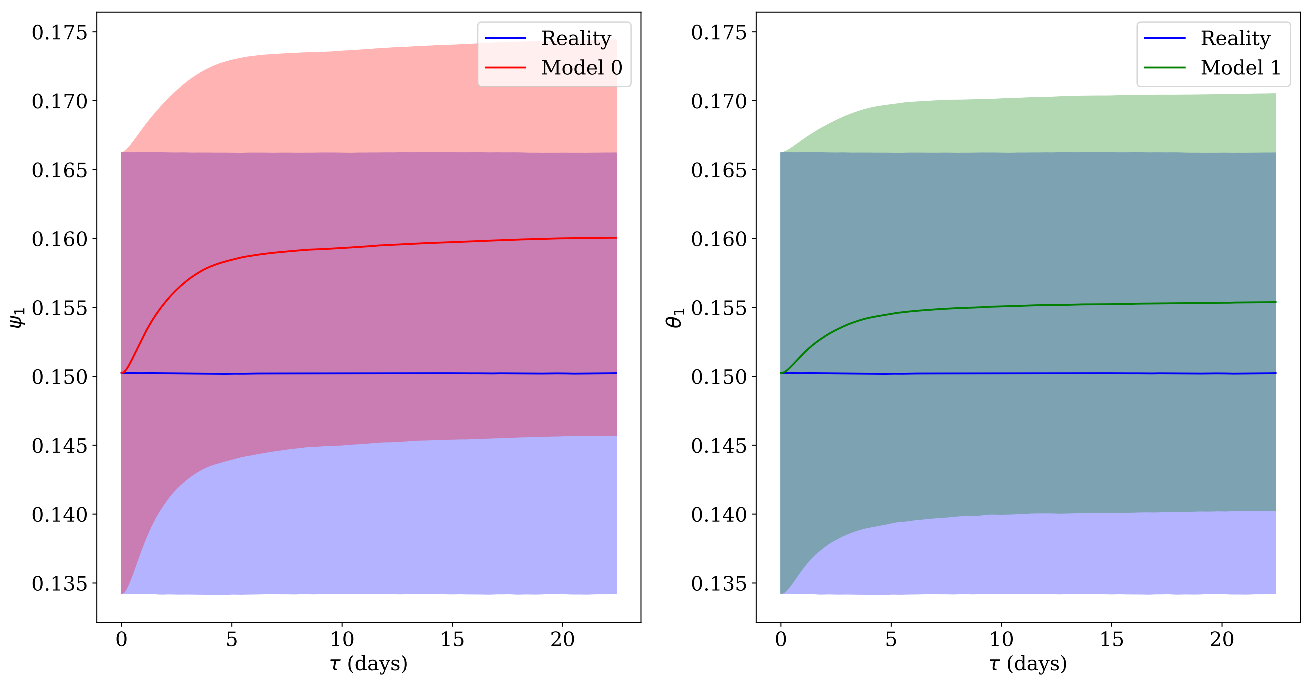

Figure 3. Behaviour of the averages as a function of the lead time τ in the reality and the forecast models before (left panel) and after (right panel) the model change, in the case of the friction ~~post-processing~~ postprocessing experiment (see Table 1). The variable considered is the temperature meridional gradient θ_1. The solid lines denote the mean while the shaded areas denote the one standard deviation interval.

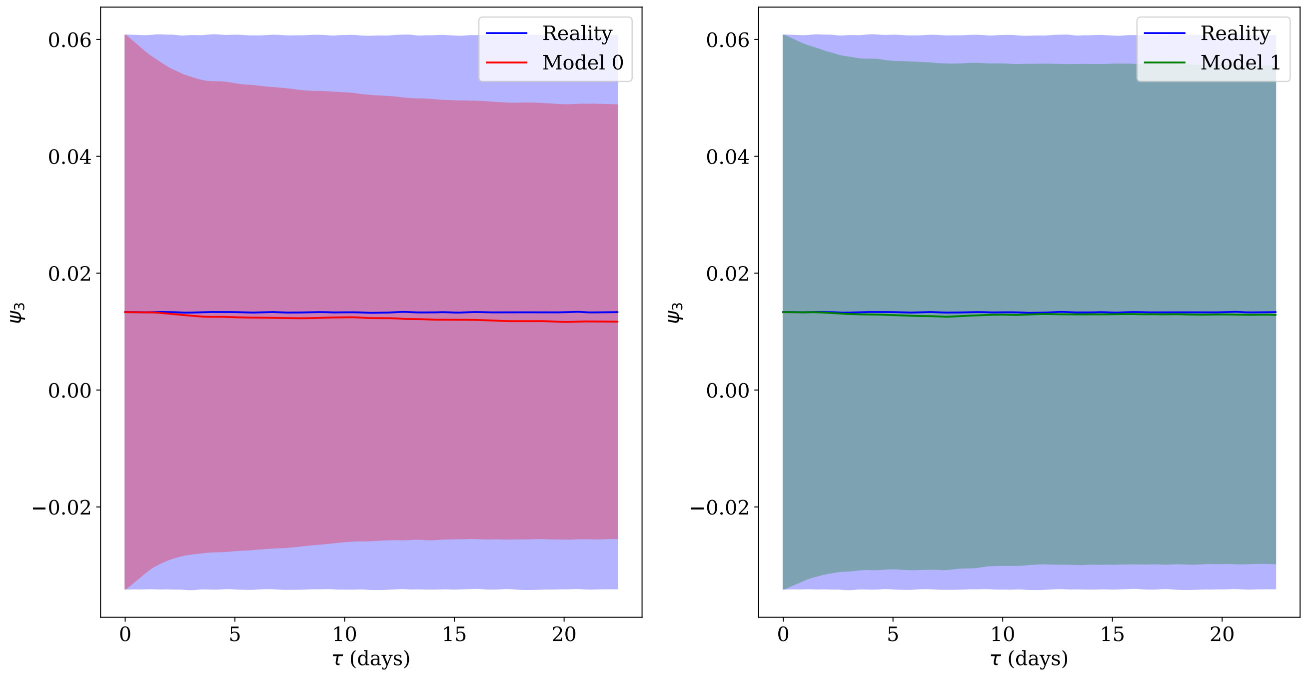

Figure 4. Same as Figure 3, but for the variable ψ_3 of the streamfunction ψ.

independent perturbation $\boldsymbol{\Psi}(\hat{\boldsymbol{y}})$,

$$\dot{\hat{\boldsymbol{y}}} = \boldsymbol{F}(\hat{\boldsymbol{y}}) + \boldsymbol{\Psi}(\hat{\boldsymbol{y}}) = \hat{\boldsymbol{F}}(\hat{\boldsymbol{y}}), \tag{47}$$

an observable A with average $\langle A(\tau)\rangle_{\boldsymbol{y}}$ at the lead time τ for the system

$$\dot{\boldsymbol{y}} = \boldsymbol{F}(\boldsymbol{y}) \tag{48}$$

has a first order response of

$$\delta\langle A(\tau)\rangle_{\boldsymbol{y}} = \int \mathrm{d}\boldsymbol{y}_0\, \rho_0(\boldsymbol{y}_0)\, \boldsymbol{\delta y}(\tau)^{\mathsf{T}} \cdot \boldsymbol{\nabla}_{\boldsymbol{f}^{\tau}(\boldsymbol{y}_0)} A \tag{49}$$

where $\boldsymbol{f}^{\tau}$ is the flow of the unperturbed system (48), ρ_0 is the distribution of initial conditions, and $\boldsymbol{\delta y}(\tau)$ is the solution of the equation $\dot{\boldsymbol{y}} + \dot{\boldsymbol{\delta y}} = \hat{\boldsymbol{F}}(\boldsymbol{y} + \boldsymbol{\delta y})$ which can be approximated at first order by the following linear inhomogeneous differential equation

$$\dot{\boldsymbol{\delta y}} = \boldsymbol{\nabla}_{\boldsymbol{y}} \boldsymbol{F} \cdot \boldsymbol{\delta y} + \boldsymbol{\Psi}(\boldsymbol{y}). \tag{50}$$

where $\boldsymbol{y}(\tau)$ is solution of the unperturbed equation (48) with initial condition $\boldsymbol{y}(0) = \boldsymbol{y}_0$ and we see that the systems (48) and (50) have to be integrated simultaneously (Gaspard, 2005). The homogeneous part of Eq. (50) is the well-known tangent linear model of the system and here it has to be solved with an additional boundary term which is the perturbation itself.

Equation (49) is derived in Appendix A, and can be computed in the same way as the averages depicted in Figs. 3 and 4, by averaging over multiple initial conditions of the reference system. Since we initialise the unperturbed (model 0) and ~~the~~ perturbed systems (model 1) with the same initial conditions, the initial state of the tangent model (50) is $\boldsymbol{\delta y}(0) = 0$. Therefore we do not estimate the impact of the observation or assimilation errors, but rather the direct impact of the model errors viewed as time-independent perturbations. The formulation of the problem and Eq. (50) can be adapted to take these errors into account, as described for instance by Nicolis (2016).

In what follows, we will numerically integrate Eq. (50) to evaluate the response on the average due to the perturbation induced by the model change. This will in turn, as in Section 3, ~~allow~~ enables us to compute the ~~post-processing~~ postprocessing parameters for the new model.

4.3 Main results

For each of the two experiments detailed in Table 1, we start by obtaining one million observations of the reality that will be used to initialise the forecast models. For each observation, this is done by starting the model $\boldsymbol{x}$ (the reference) with a random initial condition and running it for a very long time (100000 ~~adimensionalized~~ adimensionalised time units) to achieve convergence to its global attractor. Once the observations have been obtained, we run the reference model, model 0 and model 1 over 200 time units (corresponding to roughly 22 days) to obtain the reality and the forecasts. The systems have been integrated using the fourth-order Runge-Kutta integration scheme with a time-step of 0.1 time unit corresponding to 16.15 minutes. The averaging over the one million trajectories of the reality and of the forecasts at each lead time ~~allow for computing the post-processing~~ is used to compute the postprocessing coefficients α and β of the EVMOS by using formulas (17) and (18). For each ~~variable, the variable itself~~ predictand, the corresponding model variable is used as the unique predictor.

The response-theory approximations of the averages of the model $\hat{\boldsymbol{y}}$ (model 1) averages are obtained by integrating the linearised equations of model 0 along its trajectories with the perturbation $\boldsymbol{\Psi}$ as inhomogeneous term. This is done by integrating Eq. (50) over a lead time of 200 time ~~unit~~ units with a zero initial condition, using the same integration scheme as before. It ~~allows us to obtain~~ gives us the integrand of Eq. (49) for each trajectory, and the integral is then approximated as the average of this integrand over the whole set of trajectories. The result of this integration and averaging is shown in Figs. 5 and ~~7~~ 6 for the first and second moment of the variable θ_1. The results for other variables are available in the supplementary material. The black curve shows the moments of model 0 with the addition of their linear response $\delta\langle\theta_1\rangle$ and $\delta\langle\theta_1^2\rangle$ to the perturbation Ψ. This curve agrees well with the green curves of the model 1 moments up to a lead time of ~~3 or 4~~ 4-5 days, showing the efficiency of response theory. Note that in contrast with the calculation of the averages shown in Figs. 3 and 4 and computed with one million trajectories, we have here considered a limited subset of 10000 trajectories of model 0 and its tangent to compute the corrections to these averages. The correction of the moments of model 1 are accurate until ~~3~~ 4 days for both experiments. After this critical lead time, obtaining a good accuracy requires a huge increase in the number of forecasts and tangent model integrations to perform the averaging. This problem is well-known (Nicolis, 2003; Eyink et al., 2004) and is due to the appearance of fat-tails in the distribution of the perturbations $\boldsymbol{\delta y}$ in the integrand of Eq. (49). As it can be seen in Fig. ~~9~~7, the problem worsens with the increase of the lead time: initially the distributions are near-Gaussian and fat-tails appears progressively. Therefore, the number of samples of $\delta \boldsymbol{y}$ needed to converge to the correct mean up to a certain precision increases exponentially as the lead time increases. This problem has consequences on the method used to perform the average. Indeed, to avoid rare and unrealistic extreme ~~response~~ responses of the system located far in the tails of the distributions, ~~we have removed~~ outliers above a certain threshold (set to ~~3. adimensional unit)~~ 3 adimensional units) have been removed from the averaging.

The moments obtained by the response theory approach are used to compute new EVMOS ~~post-processing~~ postprocessing α and β coefficients, thanks to the formulas (17) and (18). These corrected coefficients are shown in Fig. 8 and in the panels (c) and (d) of Fig. 11. In Figs. ~~6 and 8~~9 and 10, we compare the ~~three post-processing~~ four postprocessing schemes hence obtained: the ~~post-processing~~ postprocessing of model 0 (red curves) and 1 (green curves) obtained by averaging over their trajectories (forecasts), and the ~~post-processing~~ postprocessing of model 1 obtained with the past model 0 forecasts (green "+" crosses) and with the response theory approach (black "×" crosses). In the panel (a) of these figures, the mean square error (MSE) between the trajectories of the models and the reference trajectories is displayed by solid curves, while the MSE between both models correction and the reference is depicted by dash-dotted curves. ~~In general, the~~ A first comment is that, even if the model change here is small, the postprocessing using the past forecasts of model 0 completely fails to correct the model 1 forecasts, highlighting the need for an adaptation of the postprocessing to the model change in the present case. Then, we note that in addition, the statistical postprocessing corrections are efficient until ~~a lead time of 3-4~~ lead times of 4-5 days. In fact, the skill of the corrections decreases with the lead time, and thus the EVMOS schemes become not better than the original models after ~~3~~ 4 days. In the panels (b) and (c) of Figs. ~~6 and 8~~9 and 10, the mean and variance of the corrected forecasts is compared with those of the original models. Again, these corrections are efficient until ~~3 days for all the post-processing scheme considered. For all~~ 4 days for the postprocessing schemes. For the 3 panels of these figures, the correction of model 1 using the response-theory EVMOS is depicted by black crosses and it matches almost perfectly the score of the "exact" EVMOS obtained with the forecasts of model 1 (dash-dotted green curve), up to a ~~3~~ 4 days lead time. After that lead time, the errors due to the fat-tails in the response of the first moments of the statistics induce errors in the variance needed to compute the α and β coefficients (see Eqs. (17) and (18)). These coefficients therefore degrade sharply after ~~3~~ 4 days, as shown by the solid black curve in Fig. ~~10~~ 11 (c) and (d). This in turn induces a degradation of the response theory ~~post-processing~~ postprocessing scheme, as depicted in Figs. ~~6 and 8~~9 and 10.

Nevertheless, this limitation of response theory is not a concern for our present aim, since after the critical lead time of ~~3~~ 4 days, the EVMOS skill improvement vanishes anyway. In conclusion, obtaining accurate EVMOS coefficients is therefore critical up to ~~3~~ 4 days, which is precisely the timespan ~~where~~ during which the response theory is the most accurate.

5 Discussion and Conclusions

Statistical ~~post-processing~~ postprocessing techniques used to correct numerical weather predictions (NWP) require substantial past forecast and observation databases. In the case of a model change, which frequently occurs during the normal life cycle of an operational forecast model, one has to reforecast the entire database of past forecasts (Hagedorn et al., 2008; Hamill

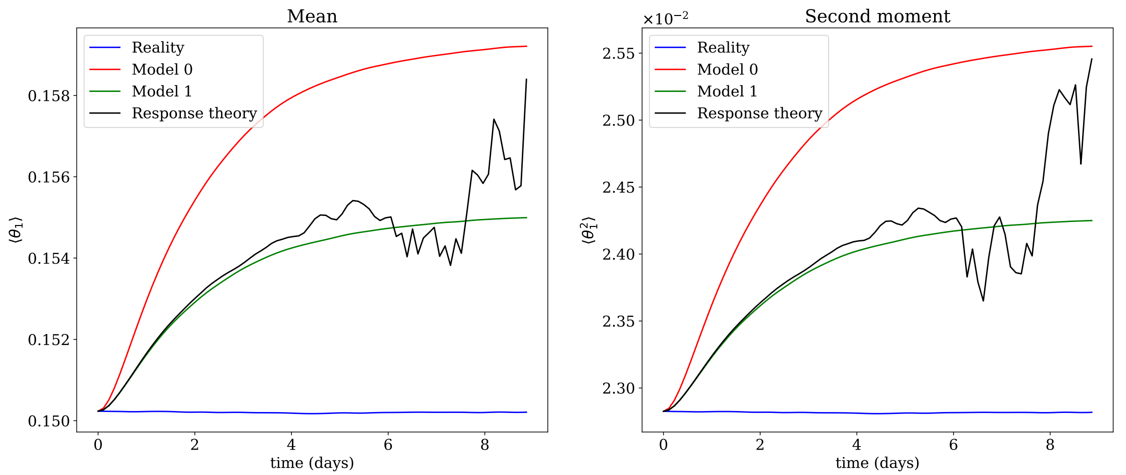

Figure 5. Corrections of the moments of θ_1 from model 0 to model 1 using the response theory formula (49), for the experiment with varying friction coefficient.

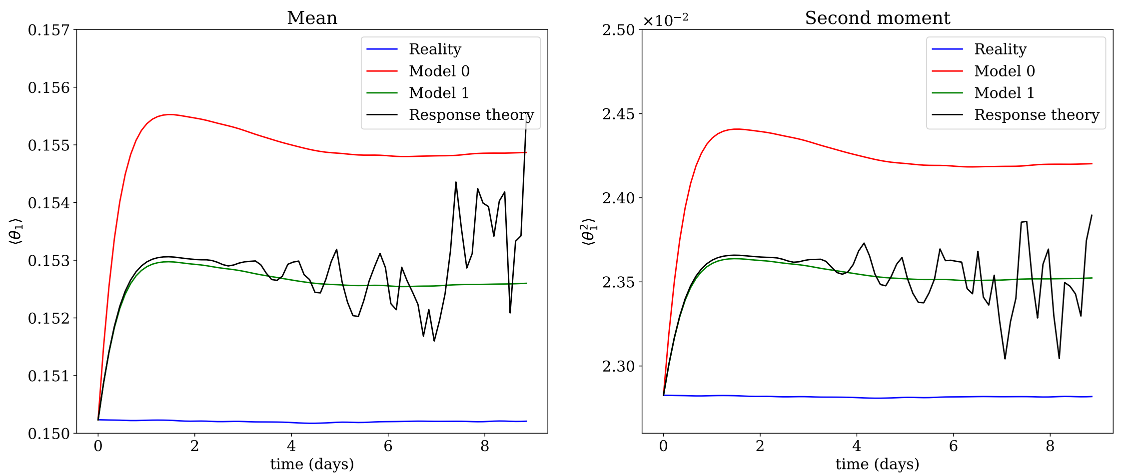

Figure 6. ~~Performance~~ Corrections of the ~~corrections on the variable~~ moments of θ_1 from model 0 to model 1 using the response theory formula (49), for the experiment with varying ~~the friction~~ Newtonian cooling coefficient. ~~(a) Mean square error (MSE) evolution between~~

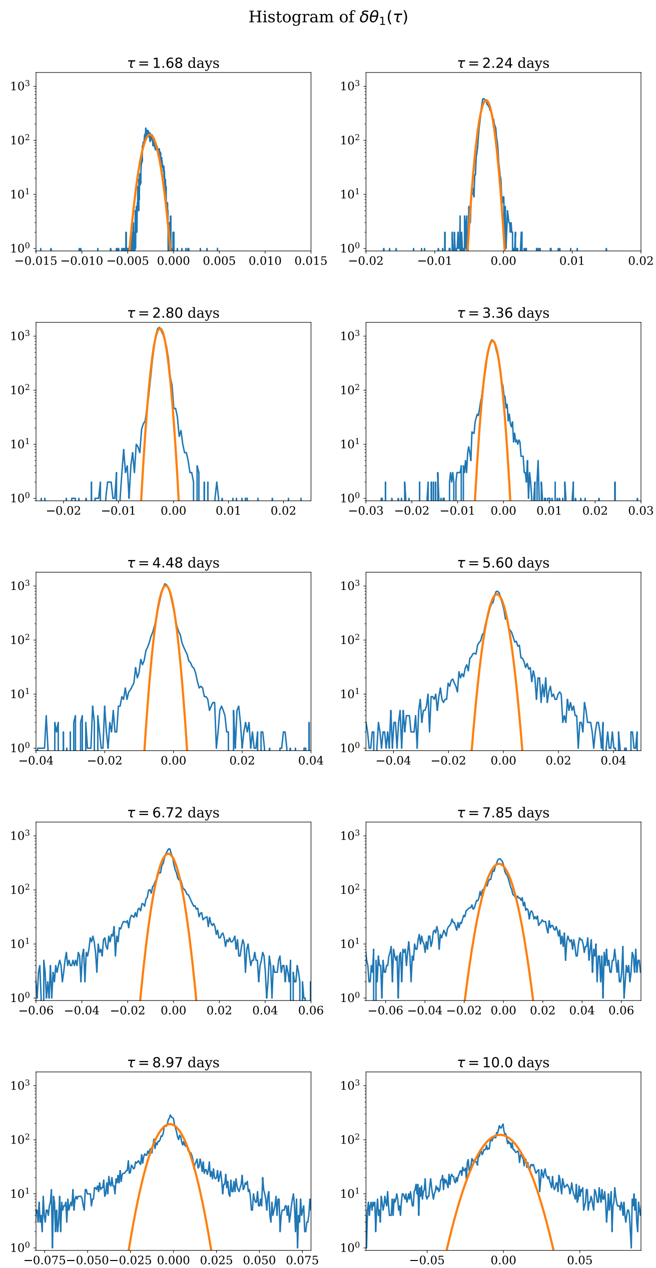

Figure 7. Histograms of the ~~different forecasts and their correction, and~~ solutions of the ~~reality.~~ equation (~~b~~50) ~~Mean of~~ for the perturbation $\boldsymbol{\delta y}(\tau)$ along the ~~different~~ trajectories ~~(reality,~~ of model 0~~and 1) and corrected forecasts~~, for different values of the lead time τ. ~~(c) Variance~~ The solid orange curves are fits of a Gaussian distribution function to the different ~~trajectories~~ histograms. The fat-tail phenomenon described

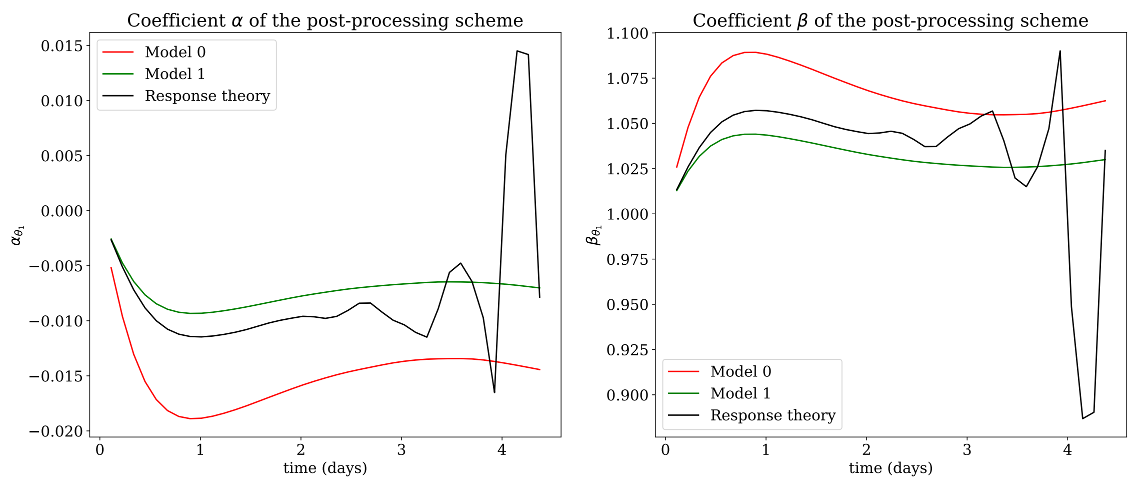

Figure 8. ~~Corrections~~ Coefficients α and β of the ~~moments of θ_1 from model 0 to model 1 using~~ postprocessing schemes and their correction by the response theory~~formula (49)~~, for the experiment with varying Newtonian cooling coefficient.

et al., 2008) to update the ~~post-processing~~ postprocessing coefficients and parameters. In the present work, we proposed a new methodology based on response theory to produce these new coefficients without having to reforecast. Instead, the database of past forecasts is reused to perform integrations in the tangent space of the model. It allows to obtain the new ~~post-processing~~ postprocessing coefficients as modifications of the older ones. These new coefficients were shown to be accurate enough within the lead time range for which the ~~post-processing~~ postprocessing corrections improve the forecast.

Figure ~~10~~ 11 summarises the main results of this work, with the quasi-geostrophic system described in Section 4, but using a different number m of trajectories of model 0 and its tangent model to compute the response-theory corrections. It shows that up to a lead time of 2 days, good ~~post-processing~~ postprocessing scheme coefficients are obtained even with a mere 20 integrations in the tangent space.

Note however that in the context of this conceptual quasi-geostrophic model, it turns out that one can also obtain good estimates of the ~~post-processing~~ postprocessing coefficients α and β simply by using a small set of reforecasts. For this it suffices to directly integrate the updated model 1, given by the non-linear equation (47), for only 20 trajectories. So the response theory approach in the present case cannot really compete with the simple ~~reforeasting~~ reforecasting method. How this can be improved in an operational context is an important question that should be addressed in the future. For instance, we can use a simplified tangent linear model to reduce the computational burden, as often used in data assimilation (Bonavita et al., 2017). This approach could also be implemented for short-range forecasts, say from 1 to 3 days.

The response-theory is efficient because the model changes are assumed to be small in comparison with the original parameterisation of the models. The method cannot improve a ~~post-processing~~ postprocessing scheme, but it can efficiently adapt it to a new model. As such, the success of this method also depends on the quality of the past ~~post-processing~~ postprocessing

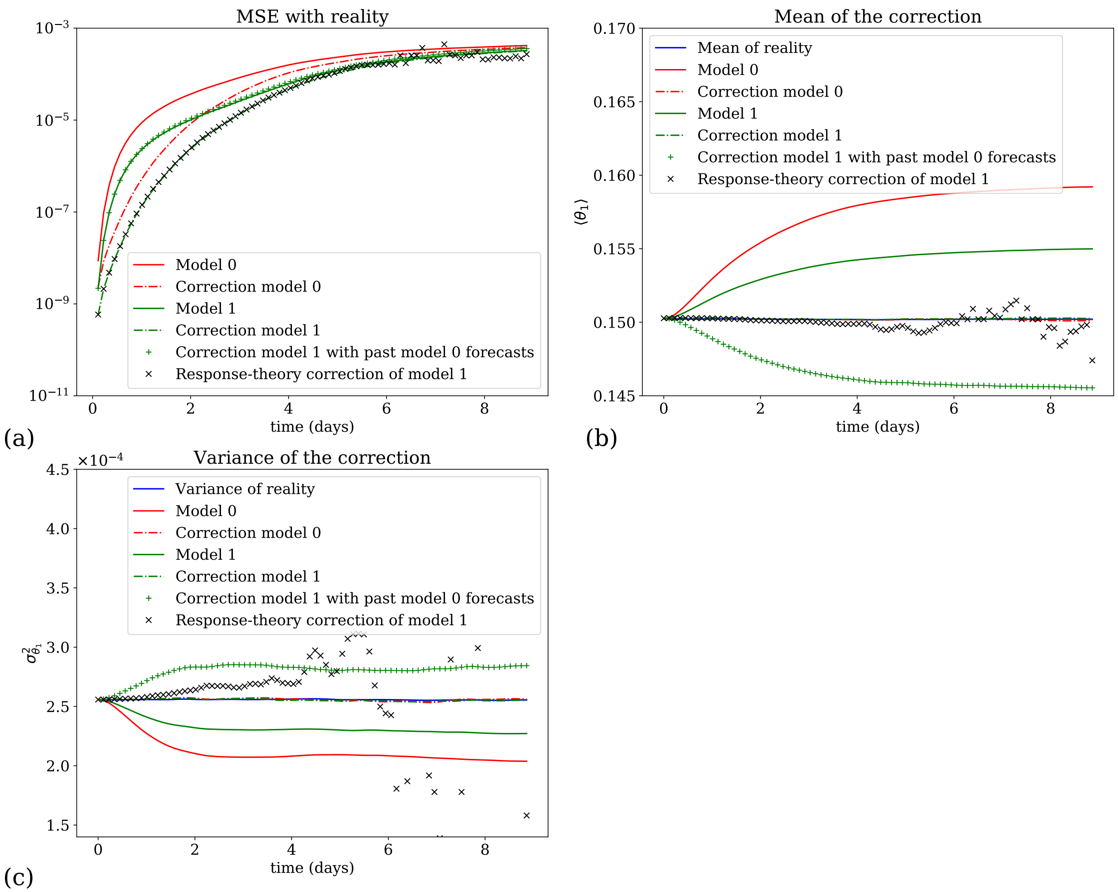

Figure 9. Performance of the corrections on the variable θ_1 for the experiment varying the ~~Newtonian cooling~~ friction coefficient. (a) Mean square error (MSE) evolution between the different forecasts and their correction, and the reality. (b) Mean of the different trajectories (reality, model 0 and 1) and corrected forecasts. (c) Variance of the different trajectories and corrected forecasts.

~~Histograms of the solutions of the equation (50) for the perturbation $\boldsymbol{\delta y}(\tau)$ along the trajectories of model 0, for different values of the lead time τ. The solid orange curves are fits of a Gaussian distribution function to the different histograms. The fat-tail phenomenon described in Eyink et al. (2004) is apparent and becomes more prominent as the lead time increases.~~

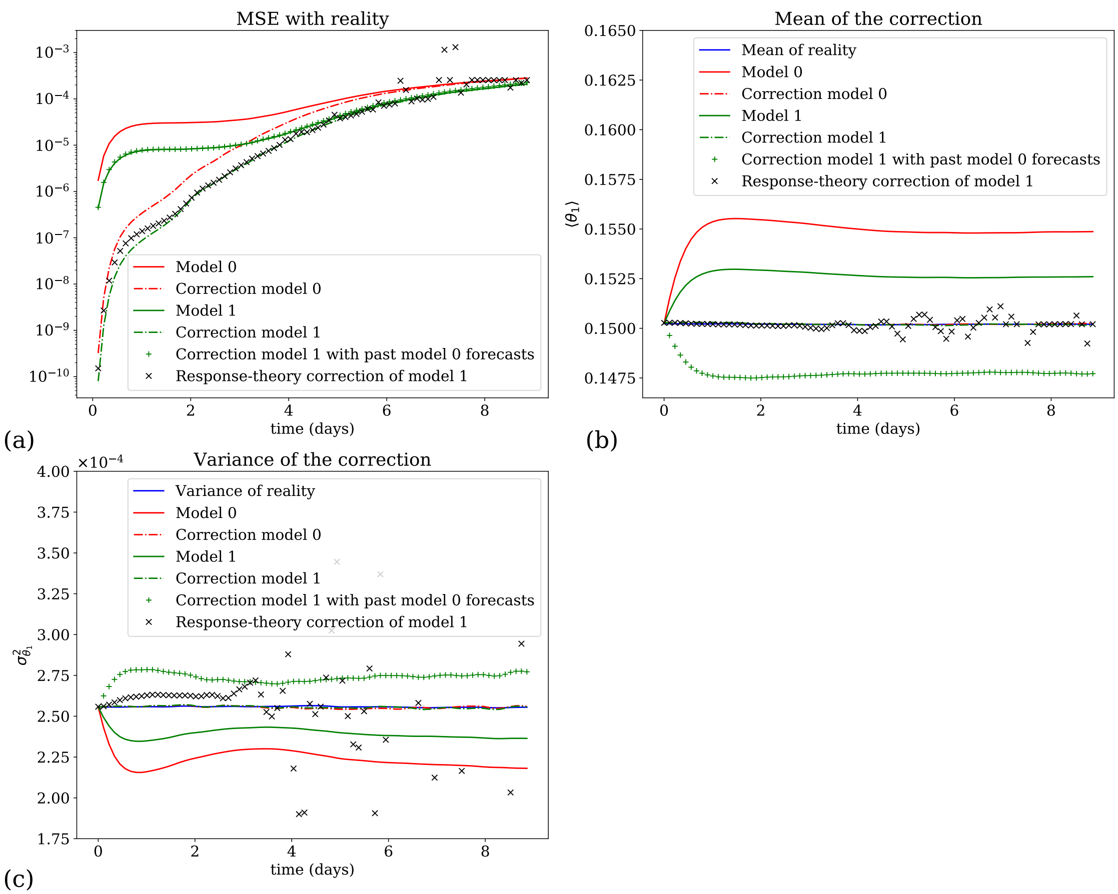

Figure 10. Performance of the corrections on the variable θ_1 for the experiment varying the Newtonian cooling coefficient. (a) Mean square error (MSE) evolution between the different forecasts and their correction, and the reality. (b) Mean of the different trajectories (reality, model 0 and 1) and corrected forecasts. (c) Variance of the different trajectories and corrected forecasts.

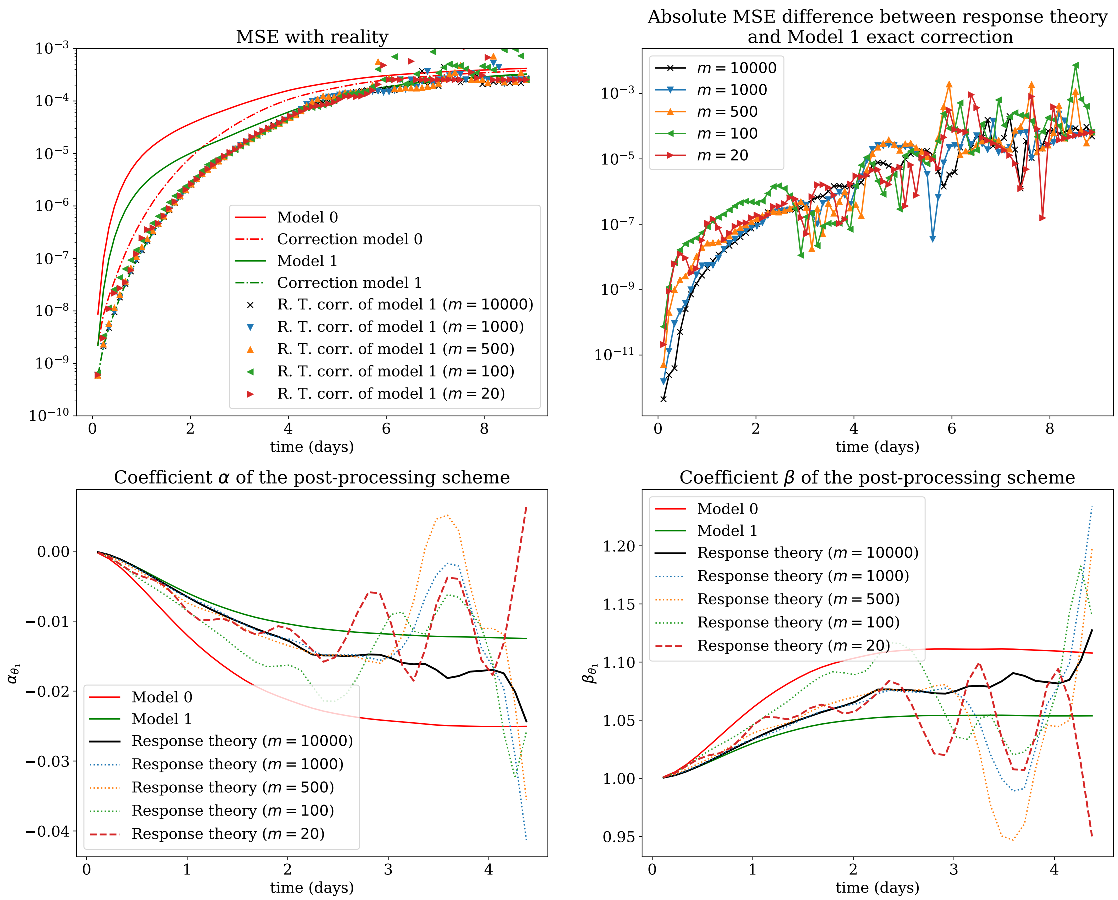

Figure 11. Comparison of the efficiency of the response theory correction for different numbers m of trajectories used to average Eq. (49), for the experiment of varying the friction coefficient: (a) Mean square error with reality, (b) Absolute difference between the response theory correction and the correction based on the forecast of model 1, (c) and (d) ~~Post-processing~~ Postprocessing coefficients α and β. On the panels (b), (c) and (d), the higher (100000) and the lower (20) numbers are depicted respectively by a solid black line and a dashed red line. The other cases in-between are depicted by dotted lines.

scheme. There are also situations where linear response theory is known to fail, but statistical tests which allow to identify its breakdown have been derived in Gottwald et al. (2016) and in Wormell and Gottwald (2018). In addition, the approach presented here applies only for models for which a tangent model is available. The model change itself has to be provided as an analytic function, which can in some circumstances limit ~~of~~ the applicability of the approach.

To test this approach, we have focused on the EVMOS statistical ~~post-processing~~ postprocessing method, but other methods could be considered as well. The only requirement is that the outcome of the minimisation of the cost ~~functions~~ function uses averages of the systems being considered. ~~In fact, we hypothesise that any statistical method using average quantities could be concerned by our approach.~~ For instance, member-by-member methods that correct both the mean square errors and the spread of the ensemble while preserving the spatial correlation (Van Schaeybroeck and Vannitsem, 2015) could be considered. These methods generally use the covariance between the model forecasts and the observations as ingredient~~, instead of their marginal variance for the EVMOS. However, it does not preclude the applicability of response theory~~ . Response theory can also be applied here since this covariance can ~~easily~~ be written as an ~~ensemble~~ average. This will be investigated in a future work, together with the applicability of the approach to parameters of probability distributions, as often used in meteorology (Vannitsem et al., 2018).

The impact of initial condition errors has not been addressed here, since the purpose was to demonstrate the applicability of the approach in a perfectly controlled environment. The main limiting issue of response theory in the present context is the presence of fat tails in the distribution of the perturbations $\boldsymbol{\delta y}$ in the tangent model. This implies that beyond a certain lead time, typically 2-3 days for the synoptic scale, the number of trajectories of the tangent model needed for the averages to converge increases exponentially. This renders the approach impractical at lead times beyond 2-3 days. This is a well-known problem, which is typically due to the trajectories passing close to the stable manifolds structuring the dynamics of chaotic systems (Eyink et al., 2004), generating an extreme response of the system to the perturbations $\boldsymbol{\Psi}$. This is possibly due to the exacerbated sensitivity of these manifolds to the perturbation of the system. ~~Indeed, as Fig. 11 suggests, the trajectories responsible for these extreme responses concentrate near some heteroclinic connection between the two regimes mentioned in the introduction of Section 4 and depicted in Fig. 2.~~ We see two possibilities to overcome this issue in the case where a long lead-time correction is needed.

- First, as suggested by Eyink et al. (2004), the problem should be studied in other systems. It might be resolved by itself in other systems. Indeed, in very large atmospheric systems, the encounter of such manifolds might become more rare. This could be related to the chaotic hypothesis (Gallavotti and Cohen, 1995a, b) which states that large systems can be considered to behave like Axiom-a hyperbolic systems for the physical quantities of interest, and thus Ruelle response theory (Ruelle, 2009) might get better as the dimensionality of a system increases. This hypothesis would be interesting to test in current state-of-the-art NWP systems.

- Secondly, another avenue would be to adapt the ~~very effective~~ techniques based on the Covariant Lyapunov vectors (CLVs) or on unstable periodic orbits (UPOs) to non-stationary dynamics. These techniques were recently introduced ~~(Wang, 2013; Ni and Wang, 2017; Ni, 2019)~~ (Wang, 2013; Ni and Wang, 2017; Ni, 2019; Lasagna, 2019; Lasagna et al., 201 to deal with stationary responses of chaotic systems, i.e. the response of a system that lies on its attractor.

The CLVs methods mentioned focus on finding an adjoint representation (Eyink et al., 2004) of the response, while in the present work the approach is based on forward integrations (direct method). The adjoint representation allows to change easily the perturbation function $\boldsymbol{\Psi}$ for a fixed observable A, while the direct method ~~allows for different observables to be considered,~~ enables to consider different observables while keeping the perturbation function fixed. The adjoint representation, however, requires one to integrate the tangent model backward in time. Therefore, its accuracy depends on the absolute value of the smallest Lyapunov exponent of the system, which might render its results less good than the direct forward representation.

~~2D scatter plot showing the phase-space locations of the perturbations composing the fat tails shown in Fig. 9. Blue dots: plot of the trajectories of the model 0 used to compute the average response, along the variables ψ_2 and ψ_3. Red dots: plot of the forecast trajectories of the model 0 leading to an extreme response $\delta\theta_1$ with an absolute deviation from the mean response greater than 100 standard deviations of the distribution shown in Fig. 9. Green crosses: Equilibrium points of model 0. White cross: Location of the maximum perturbation encountered in the data, across all the trajectories and lead times.~~

In conclusion, the response-theory approach developed here is an effective method to deal with the problem of the impact of model change on the ~~post-processing~~ postprocessing scheme. Its main advantage is to be computed on the past model version and does not require reforecasts of the full model. Its operational implementation, however, is still an open question that should be addressed in the future.

Code availability. The quasi-geostrophic model used is called QGS and was obtained by adapting the Python code of the MAOOAM ocean-atmosphere model (De Cruz et al., 2016), following the model description in Cehelsky and Tung (1987). It was recently released on Zenodo (Demaeyer and De Cruz, 2020) and is also available at https://github.com/Climdyn/qgs. The additional notebooks computing the response to model changes and generating the figures are also provided as supplementary material. They have been released on Zendo as well (Demaeyer, 2020), and are available at https://github.com/jodemaey/Postprocessing_and_response_theory_notebooks.

Appendix A: Non-stationary response theory

We consider a perturbed autonomous dynamical system

$$\dot{\hat{\boldsymbol{y}}} = \boldsymbol{F}(\hat{\boldsymbol{y}}) + \boldsymbol{\Psi}(\hat{\boldsymbol{y}}) = \hat{\boldsymbol{F}}(\hat{\boldsymbol{y}}) \tag{A1}$$

with a prescribed distribution of initial conditions ρ_0. For the unperturbed system

$$\dot{\boldsymbol{y}} = \boldsymbol{F}(\boldsymbol{y}), \tag{A2}$$

an observable A has the average at time τ

$$\langle A(\tau)\rangle_{\boldsymbol{y}} = \int \mathrm{d}\boldsymbol{y}_0\, \rho_0(\boldsymbol{y}_0)\, A(\boldsymbol{f}^\tau(\boldsymbol{y}_0)) = \int \mathrm{d}\boldsymbol{y}\, \rho_\tau(\boldsymbol{y})\, A(\boldsymbol{y}) \tag{A3}$$

where $\boldsymbol{f}^\tau$ is the flow of the unperturbed system (A2) and where ρ_τ is the distribution obtained by propagating the initial distribution ρ_0 with the Liouville equation (Gaspard, 2005). In this section, the variation of this average due to the presence of the perturbation is evaluated,

$$\langle A(\tau)\rangle_{\hat{\boldsymbol{y}}} = \langle A(\tau)\rangle_{\boldsymbol{y}} + \delta\langle A(\tau)\rangle_{\boldsymbol{y}} + \delta^2\langle A(\tau)\rangle_{\boldsymbol{y}} + \dots \tag{A4}$$

In other words, we compute the average of A in system (A1)

$$\langle A(\tau)\rangle_{\hat{\boldsymbol{y}}} = \int \mathrm{d}\boldsymbol{y}_0\, \rho_0(\boldsymbol{y}_0)\, A(\hat{\boldsymbol{f}}^\tau(\boldsymbol{y}_0)) \tag{A5}$$

as a perturbation of the average (A3) in the unperturbed system (A2). Here, $\hat{\boldsymbol{f}}^\tau$ is the flow of the perturbed system (A1). In the following, we will derive these corrections thanks to a Kubo-type perturbative expansion (Lucarini, 2008) that amounts to constructing a Dyson series in the interaction picture framework where the perturbation is seen as an interaction Hamiltonian (Wouters and Lucarini, 2012). We start by considering the time evolution of the observable A in the system (A1):

$$\frac{\mathrm{d}}{\mathrm{d}\tau} A\left(\hat{\boldsymbol{f}}^\tau(\boldsymbol{y}_0)\right) = (\mathcal{L}_0 + \mathcal{L}_1)\, A\left(\hat{\boldsymbol{f}}^\tau(\boldsymbol{y}_0)\right) \tag{A6}$$

with the operators

$$\begin{cases} \mathcal{L}_0\, A(\boldsymbol{y}) & = & \boldsymbol{F}(\boldsymbol{y})^\mathsf{T} \cdot \boldsymbol{\nabla}_{\boldsymbol{y}} A \\ \mathcal{L}_1\, A(\boldsymbol{y}) & = & \boldsymbol{\Psi}(\boldsymbol{y})^\mathsf{T} \cdot \boldsymbol{\nabla}_{\boldsymbol{y}} A \end{cases} \tag{A7}$$

and define an interaction observable as

$$A_I(\tau, \boldsymbol{y}_0) = \Pi_0(-\tau)\, A\left(\hat{\boldsymbol{f}}^\tau(\boldsymbol{y}_0)\right) \tag{A8}$$

with $\Pi_0(\tau) = \exp(\mathcal{L}_0\, \tau)$. It is easy to show that the interaction observable satisfies the differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} A_I(\tau, \boldsymbol{y}_0) = \mathcal{L}_I(\tau)\, A_I(\tau, \boldsymbol{y}_0) \tag{A9}$$

with the interaction operator $\mathcal{L}_I(\tau) = \Pi_0(-\tau)\, \mathcal{L}_1\, \Pi_0(\tau)$. The solution to this equation is

$$A_I(\tau, \boldsymbol{y}_0) = A_I(0, \boldsymbol{y}_0) + \int_0^\tau \mathrm{d}s_1\, \mathcal{L}_I(s_1)\, A_I(s_1, \boldsymbol{y}_0) = A(\boldsymbol{y}_0) + \int_0^\tau \mathrm{d}s_1\, \mathcal{L}_I(s_1)\, A_I(s_1, \boldsymbol{y}_0) \tag{A10}$$

which can be rewritten as

$$A\left(\hat{\boldsymbol{f}}^\tau(\boldsymbol{y}_0)\right) = \Pi_0(\tau) A(\boldsymbol{y}_0) + \int_0^\tau \mathrm{d}s_1\, \Pi_0(\tau - s_1)\, \mathcal{L}_1\, \Pi_0(s_1)\, A_I(s_1, \boldsymbol{y}_0). \tag{A11}$$

Iteratively replacing the interaction observable by the formula (A10) finally leads to the Dyson series:

$$\begin{aligned} A\left(\hat{\boldsymbol{f}}^\tau(\boldsymbol{y}_0)\right) \quad = \quad & \Pi_0(\tau) A(\boldsymbol{y}_0) + \int_0^\tau \mathrm{d}s_1\, \Pi_0(\tau - s_1)\, \mathcal{L}_1\, \Pi_0(s_1)\, A(\boldsymbol{y}_0) \\ & + \int_0^\tau \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2\, \Pi_0(\tau - s_1)\, \mathcal{L}_1\, \Pi_0(s_1 - s_2)\, \mathcal{L}_1\, \Pi_0(s_2)\, A(\boldsymbol{y}_0) + \dots \end{aligned} \tag{A12}$$

Using the definitions (A3) and (A5), as well as the fact that

$$g\left(\boldsymbol{f}^{\tau}(\boldsymbol{y}_0)\right) = \Pi_0(\tau) g(\boldsymbol{y}_0) \tag{A13}$$

for any smooth function g, we get finally a formula for the perturbations in Eq. (A4):

$$\langle A(\tau)\rangle_{\hat{\boldsymbol{y}}} = \langle A(\tau)\rangle_{\boldsymbol{y}} + \int_0^{\tau} \mathrm{d}s_1 \int \mathrm{d}\boldsymbol{y}_0\, \rho_0(\boldsymbol{y}_0)\, \Pi_0(\tau - s_1)\, \mathcal{L}_1\, \Pi_0(s_1)\, A(\boldsymbol{y}_0) + \ldots \tag{A14}$$

We will now focus on the first term of this expansion, but the subsequent orders of the response can be treated alike. We thus have

$$\delta\langle A(\tau)\rangle_{\boldsymbol{y}} = \int_0^{\tau} \mathrm{d}s_1 \int \mathrm{d}\boldsymbol{y}_0\, \rho_0(\boldsymbol{y}_0)\, \boldsymbol{\Psi}\left(\boldsymbol{f}^{\tau - s_1}(\boldsymbol{y}_0)\right)^{\mathsf{T}} \cdot \boldsymbol{\nabla}_{\boldsymbol{f}^{\tau-s_1}(\boldsymbol{y}_0)} A\left(\boldsymbol{f}^{\tau}(\boldsymbol{y}_0)\right) \tag{A15}$$

which with the change of variable $s_1 \to t - \tau'$ can be rewritten as

$$\delta\langle A(\tau)\rangle_{\boldsymbol{y}} = \int_0^{\tau} \mathrm{d}\tau' \int \mathrm{d}\boldsymbol{y}_0\, \rho_0(\boldsymbol{y}_0)\, \boldsymbol{\Psi}\left(\boldsymbol{f}^{\tau'}(\boldsymbol{y}_0)\right)^{\mathsf{T}} \cdot \boldsymbol{\nabla}_{\boldsymbol{f}^{\tau'}(\boldsymbol{y}_0)} A\left(\boldsymbol{f}^{\tau}(\boldsymbol{y}_0)\right) \tag{A16}$$

and then

$$\begin{aligned}
\delta\langle A(\tau)\rangle_{\boldsymbol{y}} &= \int_0^{\tau} \mathrm{d}\tau' \int \mathrm{d}\boldsymbol{y}_0\, \rho_0(\boldsymbol{y}_0)\, \boldsymbol{\Psi}\left(\boldsymbol{f}^{\tau'}(\boldsymbol{y}_0)\right)^{\mathsf{T}} \cdot \left(\frac{\partial \boldsymbol{f}^{\tau}(\boldsymbol{y}_0)}{\partial \boldsymbol{f}^{\tau'}(\boldsymbol{y}_0)}\right)^{\mathsf{T}} \cdot \boldsymbol{\nabla}_{\boldsymbol{f}^{\tau}(\boldsymbol{y}_0)} A & \text{(A17)}\\
&= \int_0^{\tau} \mathrm{d}\tau' \int \mathrm{d}\boldsymbol{y}_0\, \rho_0(\boldsymbol{y}_0)\, \boldsymbol{\Psi}\left(\boldsymbol{f}^{\tau'}(\boldsymbol{y}_0)\right)^{\mathsf{T}} \cdot \mathbf{M}\left(\tau - \tau', \boldsymbol{f}^{\tau'}(\boldsymbol{y}_0)\right)^{\mathsf{T}} \cdot \boldsymbol{\nabla}_{\boldsymbol{f}^{\tau}(\boldsymbol{y}_0)} A & \text{(A18)}
\end{aligned}$$

$\mathbf{M}$ is the fundamental matrix (Gaspard, 2005; Nicolis, 2016) of the homogeneous part of the linear differential equation

$$\dot{\boldsymbol{\delta y}} = \boldsymbol{\nabla}_{\boldsymbol{y}} \boldsymbol{F} \cdot \boldsymbol{\delta y} + \boldsymbol{\Psi}(\boldsymbol{y}) \tag{A19}$$

where $\boldsymbol{y}$ is solution of Eq. (A2) with initial condition $\boldsymbol{y_0}$, and we have the definition

$$\mathbf{M}(t, \boldsymbol{y}) = \frac{\partial \boldsymbol{f}^{t}(\boldsymbol{y})}{\partial \boldsymbol{y}}. \tag{A20}$$

Equation (A19) is the linearised approximation of equation (A1):

$$\dot{\boldsymbol{y}} + \dot{\boldsymbol{\delta y}} = \boldsymbol{F}(\boldsymbol{y} + \boldsymbol{\delta y}) + \boldsymbol{\Psi}(\boldsymbol{y} + \boldsymbol{\delta y}) \tag{A21}$$

that provides a tool to estimate Eq. (A18). Indeed, since the solution of Eq. (A19) can be written as

$$\boldsymbol{\delta y}(\tau) = \int_0^{\tau} \mathrm{d}\tau'\, \mathbf{M}\left(\tau - \tau', \boldsymbol{f}^{\tau'}(\boldsymbol{y_0})\right) \cdot \boldsymbol{\Psi}\left(\boldsymbol{f}^{\tau'}(\boldsymbol{y_0})\right), \tag{A22}$$

we can write the first order variation of the average of the observable A in term of these solutions:

$$\delta\langle A(\tau)\rangle_{\boldsymbol{y}} = \int \mathrm{d}\boldsymbol{y}_0\, \rho_0(\boldsymbol{y}_0)\, \boldsymbol{\delta y}(\tau)^{\mathsf{T}} \cdot \boldsymbol{\nabla}_{\boldsymbol{f}^{\tau}(\boldsymbol{y}_0)} A \tag{A23}$$

The interpretation of this equation is that the averaging of an observable over the trajectories of the linear approximation (A21) of the perturbation equation (A1) provides the first order response of the observable. It is the main ingredient used to compute the new ~~post-processing~~ postprocessing scheme in the present work. It is explained in detail in Sections 3.3 and 4.2.

Appendix B: The quasi-geostrophic model equations

The ordinary differential equations of the model are given by

$$\begin{aligned}\dot{\psi}_i &= -a_{i,i}^{-1} \sum_{j,m=1}^{n_{\mathrm{a}}} b_{i,j,m} \left(\psi_j\, \psi_m + \theta_j\, \theta_m\right) - \frac{a_{i,i}^{-1}}{2} \sum_{j,m=1}^{n_{\mathrm{a}}} g_{i,j,m}\, h_m \left(\psi_j - \theta_j\right) \\ &\quad - \beta\, a_{i,i}^{-1} \sum_{j=1}^{n_{\mathrm{a}}} c_{i,j}\, \psi_j - \frac{k_d}{2} \left(\psi_i - \theta_i\right) \end{aligned} \tag{B1}$$

$$\begin{aligned}\dot{\theta}_i &= -a_{i,i}^{-1} \sum_{j,m=1}^{n_{\mathrm{a}}} b_{i,j,m} \left(\psi_j\, \theta_m + \theta_j\, \psi_m\right) + \frac{a_{i,i}^{-1}}{2} \sum_{j,m=1}^{n_{\mathrm{a}}} g_{i,j,m}\, h_m \left(\psi_j - \theta_j\right) \\ &\quad - \beta\, a_{i,i}^{-1} \sum_{j=1}^{n_{\mathrm{a}}} c_{i,j}\, \theta_j + \frac{k_d}{2} \left(\psi_i - \theta_i\right) - 2\, k_d'\, \theta_i + a_{i,i}^{-1}\, \omega_i \end{aligned} \tag{B2}$$

$$\dot{\theta}_i = - \sum_{j,m=1}^{n_{\mathrm{a}}} g_{i,j,m}\, \psi_j\, \theta_m + \frac{\sigma}{2}\, \omega_i + h_d \left(\theta_i^* - \theta_i\right) \tag{B3}$$

where adimensional parameters values and description can be found in Table 1 and section 4. β is the meridional gradient of Coriolis parameter which has the adimensional value 0.21 at 50 degrees of latitude (Reinhold and Pierrehumbert, 1982; Cehelsky and Tung, . The vertical velocity ω_i can be eliminated, leading to equations (B2) and (B3) being reduced to a single equation for θ_i. The parameter σ is the adimensional static stability of the atmosphere set typically to 0.2. The coefficients $a_{i,j}$, $g_{i,j,m}$, $b_{i,j,m}$ and

$c_{i,j}$ are the inner products of the Fourier modes F_i defined in section 4:

$$a_{i,j} = \frac{n}{2\pi^2} \int_0^{\pi} \int_0^{2\pi/n} F_i(x,y)\, \nabla^2 F_j(x,y)\, \mathrm{d}x\, \mathrm{d}y = -\delta_{ij}\, a_i^2 \tag{B4}$$

$$g_{i,j,m} = \frac{n}{2\pi^2} \int_0^{\pi} \int_0^{2\pi/n} F_i(x,y)\, J\left(F_j(x,y), F_m(x,y)\right) \mathrm{d}x\, \mathrm{d}y \tag{B5}$$

$$b_{i,j,m} = \frac{n}{2\pi^2} \int_0^{\pi} \int_0^{2\pi/n} F_i(x,y)\, J\left(F_j(x,y), \nabla^2 F_m(x,y)\right) \mathrm{d}x\, \mathrm{d}y \tag{B6}$$

$$c_{i,j} = \frac{n}{2\pi^2} \int_0^{\pi} \int_0^{2\pi/n} F_i(x,y)\, \frac{\partial}{\partial x} F_j(x,y)\, \mathrm{d}x\, \mathrm{d}y \tag{B7}$$

where the coefficients a_i are given by Eq. (41) and where J is the Jacobian present in the advection terms:

$$J(S,G) = \frac{\partial S}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial S}{\partial y}\frac{\partial G}{\partial x}. \tag{B8}$$

Competing interests. Stéphane Vannitsem is a member of the editorial board of the journal. Jonathan Demaeyer declares that he has no conflict of interest.

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