Overall reactions/comments

The referees were both positive and helpful, the manuscript has been considerably improved, notably by a significant rewrite of the introduction and conclusion. The number of references has nearly tripled and the introduction now better situates the paper into a large body of literature. The updated conclusions underline the specific contributions of this paper and better point to future directions.

I should also mention that when the paper was written, I was on the verge of submitting another paper on the deterministic FEBE and its physical basis. Instead, the time was spent on the more urgent task of deriving the special H = 1/2 case – the Half ordered EBE (HEBE) from the classical energy transport equations. Although that paper will be submitted shortly, it still has not been done and therefore – with the referee's encouragement - I have given much more discussion of the geophysics aspects of the problem.

Referee 1:

Anonymous Referee #1 Received and published: 17 November 2019

The manuscript is devoted to a stochastic fractional model for the Earth's energy balance. The author developed a framework for handling fractional equations driven by white noise forcing, and analytically determined both the small and large scale limits for fractional relaxation motions (fRm) and fractional relaxation noises (fRn). He derived the main statistical properties of both fRm and fRn, including Green function, autocorrelation function, Haar fluctuations, spectra and sample process. These are extensions of fractional Gaussian noise and fractional Brownian motion. Furthermore, he examined the prediction of fRn, fRm with a past value problem by the minimum square skill score, which are needed for forecasting the Earth's temperature. The main points of the manuscript is expressed concisely and the paper is overall well organized. I recommend acceptance by NPG, after a revision taking into account the following comments and suggestions.

Au: I thank the referee for the positive evaluation.

1. The main case studied in this manuscript is $a = -\infty$, i.e., Weyl fractional derivative. It would be nice to give more explanation on how to understand the $a = -\infty$ in the fractional derivative operator ${}_aD^H{}_t$, and what is the significance of the application in the model?

Au: The key motivation and consequence of choice $a = -\infty$ *is that it is needed to obtain a statistically stationary fRn process and an fRm process with stationary increments. This is explicitly developed in detail in appendix A. In the introduction we have added more material on this.*

2. The caption of Fig.2 is not clear, such as "H increasing in units of 1/10 starting at a value 1/20 above the plot minimum". Perhaps it is better to say like this "H increasing in units of 1/10 starting at a value 1/20 (upper left) to 39/20 (bottom right)" ?

Au: Thanks, this has been done.

3. In Fig. 5a, 5b, 6a, 6b about fRn and fRm simulations, it would be nice to add the sample path simulation of fractional Gaussian noise and fractional Brownian motion for comparison, which will help reader to better understand the fractional relaxation noise.

Au: As indicated in the captions, fig. 5a, 6a already do show the fGn and fBm processes: their bottom lines. The wording in the captions has been improved to make this clear, and a reference to this has been added to the text.

4. Fig. 7 and Fig. 8 present the simulation for fGn and fRn forecasts. Is it possible to say or predict the Earth temperature T based on fractional energy balanced equation?

Au: Yes, indeed it was the success of macroweather predictions and climate projections that using the exponents from the FEBE that led to the initial proposal to use that the FEBE be a good model. Although this was mentioned in the introduction and conclusion, I have added a new paragraph detailing this.

5. It would be nice to give a summary of the main differences and links between fRn (fRm) and fGn (fGm) in conclusions.

Au: This has been done.

Referee #2
Anonymous Referee #2 Received and published: 20 November 2019

The paper introduces a novel class of stochastic processes, called "fractional relaxation noises and motions" (fRn, fRm) and discusses their particular application to a stochastic relaxation equation (Eq. 1), used to describe the Earth energy balance. The principal motivation behind the proposed processes is to synthesize the stochastic differential equation and scaling modeling approaches, each of which has been thoroughly explored (and shown efficient) in mathematics and applied literature.

The paper starts (Section 2) with introducing the fractional Langevin equation (Eq. 1) for the Earth energy balance, and uses it as a motivational example to develop the proposed theory, in parallel to what has been done for fractional Gaussian noises and Brownian motions (fGn, fBm). Section 3 discusses specific technical details, including derivations of process spectra and illustrating sample trajectories. This section notices a practically important aspect of not being able to distinguish between fRn, fRm, and fBm for specific ranges of parameters and for a finite range of observations. Section 4 examines the classical prediction problem, emphasizing that here one deals with the past value rather than initial value problem. Section 5 concludes.

The paper is clearly written (in particular, given its heavy math content). The materials is novel and of clear importance for the nonlinear geosciences community. I trust the paper will be of interest to the broad NPG readership and urge its publication.

Au: Thank you for your positive evaluation.

I have a comment related to the paper organization. Given the discussed material and the publication venue, I expect there will be two main categories of readers (that of course overlap) – those who are more interested in the math details, and those who are mainly interested in qualitative findings and applications. I think the paper will benefit from reorganization that first will clearly list the main proven facts (process definition, statement of stationarity, correlation function, spectra, sample path, prediction), and then will present the underlying derivations. The current version is math heavy and makes it hard to clearly see the key points of the presented material. Also, it is very important to explicitly list the differences between the fRn, fRm and their counterparts fGn, fBm.

Au: I apologize for the heavy math details. The organization of a paper takes into account many factors, in this case mostly the needed to systematically, logically develop the material, but remains somewhat subjective. Yet referee #1 states: "The main points of the manuscript is expressed concisely and the paper is ***overall well organized****." In order to answer this criticism as well as possible, I made numerous additions to the introduction including a new final paragraph detailing the main results and how they fit in the paper. In the conclusions, I have also compared and contrasted fRn, fRm and their counterparts fGn, fBm.*

Other minor comments/typos:
1) l. 59-60: Rewrite to mention the authors (West et al.) outside of the reference. This is how the sentence was originally intended.

Au: OK

2) l. 68:"martingales" should not be capitalized.
Au: OK

3) l. 72 and everywhere: "Earth" should be capitalized.
Au: OK

4) Please use proper punctuation in all equations (commas, periods).
Au: OK

5) Eq. (1): Why not give refs to the classical EBM of Budyko and Sellers before diving into a fractional version?
Au: This has now been done.

6) Eq. (2): Define Gamma

Au: OK

7) l. 167: "standard Brownian motion"(instead of "usual Brownian motion").
Au: OK

Referee #3

Anonymous Referee #3 Received and published: 11 December 2019

This article deals with the resolution and the mathematical properties of the solutions of a stochastic fractional relaxation equation. The main motivation is supplied by the fact that, to model the earth's energy balance, this equation presents several advantages over previously considered equations. Physics requirements have implications which are used in the derivation of this equation, and lead to its particular form: integer order derivatives in the equation lead to unrealistic Green functions so that fractional derivatives are required- Solutions must be stationary.

This paper certainly is an important contribution to the existing literature on fractional processes. In particular, many divergence issues are discussed and handled convincingly. A natural question which is not addressed (though it is an important issue in real-life applications) is the discussions of Gaussian vs. non Gaussian modelling of the noise part in the equations. Many references in this paper are (rightly) to Benoit Mandelbrot, who was extremely careful to discuss this matter with great care, and I would have expected this issue to be discussed (even briefly) in such a review paper (the mathematical study of the processes solutions of equations with non Gaussian noise, though it started much later than in the Gaussian case, now starts to be substantial). The nature of this paper may seem surprising: This volume will contain review papers on geophysics issues. The present paper is mainly constituted of lengthy explicit computations, which are often hard to follow, partly because the paper is not self-contained: at many key-points in the proofs, the reader is just referred to another paper.

Au: Thank you for your positive evaluation. I have augmented the introduction and conclusion to include brief discussion of the question of Levy and multifractal forcing that are indeed important, but outside the present scope. The main aspects of the paper that are not self-contained are the fractional relaxation Green's functions, as well as their series expansions. Adding this standard (albeit not widely known) material would lengthen the paper without improving its clarity.

Additionally, very little physical intuition is given to back these formal computations. When this happens (see e.g. lines 445-454), the explanations are sketched and can give light only to readers that already are well acquainted with these questions. However, here, the review part, and in particular the geophysical motivations, are barely sketched and the reader is mostly advised to consult references. A very positive point is that simulations are welcome and convincing: they clearly show that different qualitative behaviors can occur. Here too, I think that the geophysical implications of these differences would deserve to be discussed in more details. The issue of prediction only is discussed for its physics implications.

Au: I have added several paragraphs especially in the introduction giving more physical justification, discussion. Several papers with this physical side are either under review or are in advanced stages of preparation.

As it is, this paper is more an exploratory paper in applied mathematics (as can be found in applied math journal, such as e.g. SIAM review). In my opinion, substantial rewriting would be needed to make it a review paper in geophysics.

Au: It is a research, not a review paper.

Referee #4
Anonymous Referee #4 Received and published: 11 December 2019

The author proposes here new models, built using fractional derivatives in a stochastic framework, to model geoscience dynamics in a way compatible with scaling properties and long-range correlations. Since there are many equations I regret that the author is not using LateX, which would be useful for a correct display of all the complex notations (see e.g. line 307, 320, 322,333 etc. where alignment is not correct).

Au: I have improved these. The final version will of course be re-typeset by NPG.

There are many mathematical expressions and the narrative is not clear. I suggest to explicitly indicate what is the process studied here, and what are its properties. A section on this seems really necessary: either in the beginning or at the end, as a kind of summary. Also, what is precisely the novelty, why is it necessary to have an infinite memory, what is new with respect to fBm. Also is the modeled process multifractal or monofractal ?

Figures 5 and 6 display some simulation realizations for various parameter values. What are precisely the equations used for these simulations? It would be useful to provide the code to the community.

Au: The numerical details were stated but too briefly; I have added more details and references, at the beginning of section 3.6.

Line 110: the author cites Lovejoy et al 2019 as the original introduction of the idea presented here. However this paper is indicated in the references as "in preparation". Hence I suggest to remove it from the reference list, since it cannot be consulted and is not yet published. The same applies to Hebert et al 219, which is under revision.

Au: I removed these references. In the meantime, a new paper on the half-order FEBE (the HEBE) analytically derived from the standard energy transport equations was submitted, this delayed the deterministic FEBE paper that was referred to in the text.

This is a long paper with relatively few references. I recommend to add more references. For example equation (4) for the canonical Weyl relaxation equation: if this equation is classical, a reference is here welcome. Another example: equation (16) seems to be a mathematical result and hence reference to relevant mathematical literature is needed. There is a rather vast literature on fractional dynamics, or continuous time fractional random walks, which is only superficially discussed. I recommend to provide a link with this literature, at least in the introduction and discussion.

Au: Thank you for the suggestions. I added these and many others, there are now 55 references – nearly triple the previous number.

For example the following might be relevant to discuss (this list is not exhaustive):
M. M. Meerschaert, A. Sikorskii, Stochastic Models for Fractional Calculus, De Gruyter Studies in Mathematics 43, 2012
R. Hilfer (Ed.), Applications of Fractional Calculus in Physics, World Scientific, 2000.
J. Klafter, S. Lim, R. Metzler (Eds.), Fractional Dynamics: Recent Advances, World Scientific, 2011.

D. Baleanu, K. Diethelm, E. Scalas, J. Trujillo, Fractional Calculus, Models and Numerical Methods, 2nd Edition, 2016