Dear Editor,

please find enclosed a revised version of the paper "Lyapunov analysis of multiscale dynamics: the slow bundle of the two-scale Lorenz '96 model" that we wish to resubmit to your attention.
The remarks of the referees have been addressed in the modified version of the manuscript (and our manuscript title slightly changed according to the first referee remarks) and a detailed answer to all the points raised in the reports is provided below.

All changes of note we have made to the manuscript have been marked in red for easy reference. Additionally, an indexing mistake has been corrected in Table 1 and its LEs estimates have been slightly improved by newer numerical data.

Sincerely yours,

The authors

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Reply to Referee 1

We wish to thank the referee for carefully reading our manuscript and for judging our results novel, interesting and highly relevant.

We believe, however, that some criticisms have been induced by a misunderstanding due to our careless use of the term "slow manifold" while referring to the sub-space spanned by the effectively slow variables. For this reason, we have decided to change our terminology, and renamed the subspace "**slow bundle**".

In any case, we wish to make clear that the "wide spectral band", whose covariant Lyapunov vectors project strongly onto the slow variables, is not "close to a neutral spectrum" in an *absolute* sense. With the only exception of the single zero LE associated to the flow, the absolute value of all other LEs of the non-conservative, full L96 model is strictly larger than zero; there is no trace of any band characterized by a sub-exponential grow, which would correspond to the central manifold and make the system only partially hyperbolic.
In fact, in our numerical analysis we are able to perfectly discriminate the single zero exponent from the rest of the spectrum with a precision of one or two orders of magnitude. See for instance the example in the first figure included in this reply.

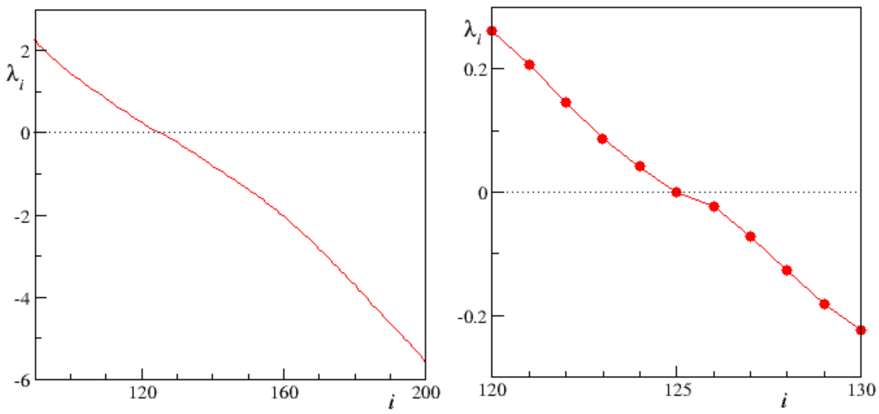

Fig. A -- Details of the Lyapunov spectrum for h=1/2 and the parameters used in Fig. 3 of our manuscript. In the left panel we focus on the part roughly corresponding to the slow bundle. On the right, we zoom closer to the 0 LE, with the index i=125, which measured to be 0 with an accuracy of 10^{-4} .

Moreover, most Lyapunov exponents (LEs) corresponding to the slow bundle are typically of order one, and – as it can be appreciated from table 1 -- the corresponding band extends roughly between +2 and -5. Therefore, these LEs are small only in the *relative* sense, that is, when compared with the larger LEs of the entire spectrum, typically one order of magnitude larger.

Furthermore -- as it can be appreciated by the example illustrated in Fig. B of this reply – all the vectors of the slow bundle but the 0 CLV are characterized by essentially the same probability distribution of the X-projection, peaked near 0 and with an exponential tail. The only exception, other than the "special" 0-CLV is represented by the two closest vectors (green curves), whose probability distribution show some sign of "hybridization" between the two shapes.

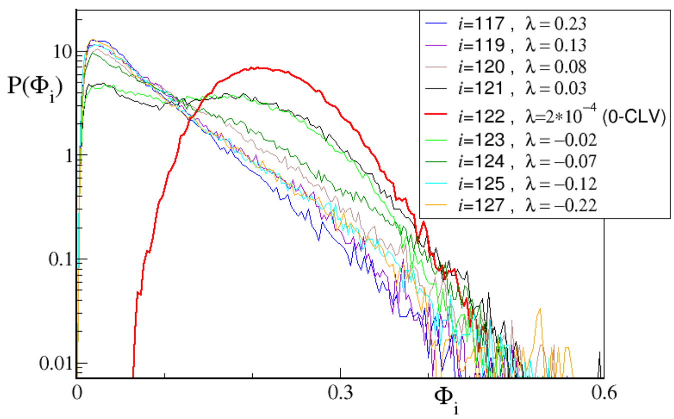

Fig. B – Probability distribution of the X-projection of the 0-CLV and its closest neighbours for h=1/4 and the typical parameters used in Fig. 5 of our manuscript. LE indices and values are reported in the legend.

To summarize, we apologize for the misunderstanding, probably induced by our usage of the term “slow manifold”, but we strongly remark that the tangent subspace we have identified is not related in any way to sub-exponential or weakly chaotic instabilities in the absolute sense. Its instability rates and associated timescales are those typical of the fully chaotic slow variables.
It is thus a “slow” subspace only in a relative sense, i.e. with respect to the time scales of the fast variables and of the maximum LE.

Minor comments

1. As already discussed, we agree with the referee that and we have changed accordingly our terminology.

2. Done
3. Done
4. Done

5. We have added a couple of comments, specifying that we always refer to normalized vectors and recalling the role of degeneracies.

6. (a) The role of Fig. 1 is not (yet) to emphasize the presence of the central band, but rather to show the way the Lyapunov spectra converge in the so-called thermodynamic limit.
We have selected the spectral region, where the convergence is slower and we still think it is the most appropriate to display these data for a generic reader.
(b) As we have clarified above, we do not claim to be in the presence of a close-to-neutral band. The central band, the way we define it, does not substantially change with K and J, as covering an instability range of a few units as, e.g. shown in Table 1.
(c) We have removed the thick lines overlapping the symbols in order to improve the readability of the figure. The inability to clearly distinguish the different symbols one from each other, however, has precisely the meaning of showing that the dependence is extremely weak. Further zooming wouldn’t help much, since we are not going to discuss the convergence of the spectrum in the thermodynamic limit.
(d) We are now recalling that the Pesin formula provides only an upper bound.

7. Since, we have not introduced the concept of “information dimension”, we don’t think it makes sense to define it to claim it is bounded from above by the Kaplan-Yorke formula. We have preferred to make explicitly reference to the Kaplan-Yorke dimension as it is often done in the literature.

8. & 9. These two observations originate from the same problem: an inaccurate definition of the thermodynamic limit. Section 2.6 has been thoroughly rewritten. This should solve all the objections raised by the referee.

10. Also following the minor remark n.2 from the second referee, to better demonstrate the symmetry we have added an inset comparing the difference between the original and the reflected spectra with our numerical precision. For the reasons already discussed, we do not believe important to particularly focus our analysis on the region closer to the 0 LEs.

11. Here, we disagree: no any change of scale can ever change the dynamics (if the corresponding equations are properly modified). We are just facing an unavoidable problem, when different physical observables are compared. The direction of the very same vector depends on which units are being used! This is a mathematical fact and the dependence of the results on the units of measure is something that should be stated as soon as possible. For this reason, we have not postponed the analysis: we have only made some changes to the text to make the point clearer.

12. We have now clarified our argument.

13. The shift is a standard practice in the presentation of Lyapunov spectra, especially in Hamiltonian models. In fact, it ensures a perfect symmetry of the spectrum around the midpoint 1/2 of the rescaled variable range, irrespective of the value of N. We have added a proper reference to explain it. Moreover, we now discuss this point earlier (footnote 3), since the shift is first used while showing the LS collapses of Fig. 1
Following comments n. 8-9, the thermodynamic limit is now better discussed in Section 2.6.

14. As we already discussed, the "central region" of CLVs with a non-negligible projection on the slow variables clearly extends up to LEs magnitude of the order one or larger. This can be already seen in Table 1, but we have now added a comment to this regard when first discussing Fig. 3.

15. This is of course true. Our comment however, regarded the accuracy of our algorithm when computing the two uncoupled spectra at once. Anyhow, we agree that this comment is unnecessary and we have removed it.

16. We have now made explicit our approximation and commented that the terms ignored in tangent space are linear in the coupling parameter *h*.

17 & 18. We have followed the referee suggestion and we now report the root mean squared difference between the full and approximate spectra in Fig. 6c.
Global LS indicators such as the upper bound on the KS entropy or the KY dimension are reproduced with great accuracy. For instance, for *h*=0.5, our approximation recovers the upper bound to KS with a 0.3% accuracy and the KY dimension with even a greater accuracy.

19. We now detail explicitly our procedure in a footnote.

20. We reworded our sentence to avoid the use of the term proof.

21. As we now show explicitly in Fig. 6c, the original and reconstructed spectra are very close one to each other. Therefore, we do not expect variations of note in either the spectral interval or in the number of LE contained in such interval if one would decide

instead to characterize the boundary of the slow bundle using the fully coupled spectrum (and a direct visual inspection of the projection pattern). Therefore, the data report in Table 1 already gives access to the relevant information on the amplitude of the slow bundle spectral band.
Moreover, note that the Lyapunov spectrum in the slow bundle region is approximately linear. As a consequence, the corresponding LE distribution will be approximately uniform, and we do not expect it to provide any additional information.

22. Finite-size Lyapunov exponents are a well-known tool, and more information on them can be obtained from the excellent literature cited in section 5.
They measure the typical growth rate of a finite size perturbation of amplitude δ in terms of the average time τ needed to grow by a factor $\sigma > 1$.
The logarithm of σ in the definition (28) assures that the corresponding FSLE does not depend on the choice of σ, as long as σ is small enough not to bridge over qualitatively different length scales. Our choice, $\sigma = 2^{1/2}$, is a typical value used in the literature that has been proved to be fully adequate for the L96 model.
On the other hand, the choice of δ_0 only determines the lower bound of the amplitude range to be explored via FSLE. As we explain in the text, our algorithmic procedure measures the FSLE at the set of amplitude thresholds δ_n for n=0,1,2,.., but FSLE can of course be measured starting from any arbitrary initial scale δ.
When evaluated at a size δ sufficiently small (formally in the limit δ -> 0), the FSLE coincides by construction with the largest LE, for any finite σ.
For the sake of clarity, we now label δ_n the horizontal axis of Fig. 8.2.

All changes we have made to the manuscript have been marked in red for easy reference. Figs. A and B above have been included for the benefit of the referee, but we have not judged necessary to include them in our manuscript.

Reply to Referee 2

We wish to thank the second referee for his constructive remarks and for finding our results interesting.
According to the analysis of Sec. 5, the central band defining the slow bundle (in the original version: slow manifold, see below) arises due to the overlap of the Lyapunov spectra of the two sub-systems in the central part of the spectrum. Since this latter occurrence should be generic for chaotic multiscale systems, we also believe our findings to be generic and not specific to the Lorenz 96 model. This intuition will of course need to be confirmed explicitly in other multiscale systems, but we now comment on this point in the conclusions.

We provide below answers to all comments raised by the second referee. Note that following a remark of the first referee, we have decided to substitute the term “slow manifold” with the more generic “slow bundle”.

Major comments

1. While the absolute value of the CLVs projection onto the slow bundle can be altered by a proper rescaling, what really defines the slow bundle is the ratio between the projection of the CLVs in the central band and the ones outside it (notice the more than two order of magnitude difference between the vertical scale in Figs. 2a-2b), which does not depend on the rescaling suggested by the referee.
Moreover, note that the red PDF in the inset of Fig. 5a belongs to the 0-CLV. As we comment in the text, this represent a unique exception due to its strongly delocalized nature.
As we discuss more extensively in our reply to the first referee, essentially all other CLVs in the slow bundle show the same projection behaviour, with an intermittent behavior and a PDF analogous to the black and green curves in the inset of Fig. 2a.

2. We acknowledge that our reference to the asymptotic behavior of FTLEs is confusing. In fact, the point is that FTLEs are well defined observables (i.e. independent of the set of variables used to describe the dynamics) only when the time is long enough to kill correlation so that one can construct a coordinate-independent large-deviation function. Accordingly, by “asymptotic” we mean long enough so as to ensure “universal” fluctuations. Surely, the instantaneous growth rate would not be a proper observable, as its fluctuations would be strongly coordinate-dependent. We have rephrased the entire paragraph on page 18.

3. We are only showing the finite-time fluctuations and angle distribution of a couple of nearby vectors for illustrative purposes, but we have indeed verified that this behavior is absolutely generic. This is to be expected in non hyperbolic systems (see the brief discussion of the existing literature before Eq. (25)), but to clarify the matter we have added a specific comment to this regard.

4. We first discuss our claim that the FSLE provides an estimate of the leftmost border of the slow bundle central band. On this regard, we have to respectfully disagree with the referee objection. A careful analysis of Fig. 8(b) shows that the vertical lines marking the FSLE position in the LS do align with a clear change of slope in the projection pattern Φ_i. This beginning of a steep descent towards a negligible projection provides a good boundary for the central band. This boundary is sharper for small coupling values h, but it can be confidently identified up to at least h=1/2.
Considering the lack of a very sharp boundary for the central band, we have nevertheless slightly toned down our claim (reasonable, instead of convincing).

Coming to the first objection, our numerical analysis shows that the strongest linear instability of the slow bundle CLVs roughly coincides with the second plateau in the FSLE analysis.

Thus, infinitesimal perturbations fully contained in the slow bundle tangent subspace will grow as finite perturbations of a sufficiently large scale.
Different linear instabilities are characterized by different saturation scales, and we remark that finite but sufficiently small perturbations will initially grow according to the strongest linear instabilities they align with. Our analysis show that, as they grow in size, they pass the saturation thresholds of the fastest CLVs lying to the left of the slow bundle. After this threshold (before the final saturation) their growth is then determined by the slow bundle CLVs, i.e. the only non-saturated instabilities at this relatively large scale.
Our analysis essentially shows that the saturation scales of the faster instabilities are well separated from the ones of the slow bundle instabilities.

Moreover, we remark that our intent is not to identify the FSLE plateau with a single well defined LE, but rather with the largest expansion rate of the slow bundle subspace. We realize that our original text could have been misleading, and therefore we have slightly modified it to make our point clearer.

We are anyhow grateful to the second referee for his comment that prompted us to better clarify the implications of our finite size analysis. We have modified accordingly the final paragraph of Section 5 and the relative conclusions in Section 6.

Minor comments

1. We thank the referee for pointing out this reference**.** We now cite this work in our manuscript.

2. We agree with the referee.
We have performed more accurate numerical simulations and checked that the symmetry is indeed verified within our numerical precision.
We have now added an inset in Fig. 2b showing the relative absolute difference between the original and reflected spectra compared with the standard deviation of our numerical estimates. An analogous inset also shows that the symmetry between the positive and negative coupling case is verified within numerical accuracy. We have modified the text accordingly.
(In Fig. 2 note that spectrum has been re-calculated because of a previous error in the determination of the simulation parameters)

3. What we want to point out is that simplifications where only a limited range of scales is explicitly represented, and rest completely ignored, are often inadequate. We have modified the sentence in order to clarify our argument.

4. The shift is a standard practice in the presentation of Lyapunov spectra, especially in Hamiltonian models. In fact, it ensures a perfect symmetry of the spectrum around the midpoint 1/2 of the rescaled variable range, irrespective of the value of N. We have added a proper reference to explain it.
Moreover, we now discuss this point earlier (footnote 3), since the shift is first used while showing the LS collapses of Fig. 1

5. Thank you for pointing out these misprints, we have corrected points (a)-(g). Concerning point (h), we note that our current usage of in text-citations is in agreement with the journal style guidelines:

<<In general, in-text citations can be displayed as "[…] Smith (2009) […]", or "[…] (Smith, 2009) […]".>>

All changes we have made to the manuscript have been marked in red for easy reference.