Nonlin. Processes Geophys. Discuss., https://doi.org/10.5194/npg-2018-38
Manuscript under review for journal Nonlin. Processes Geophys.
Discussion started: 24 September 2018

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Exploring the effects of missing data on the estimation of fractal and multifractal parameters based on bootstrap method

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Keywords: bootstrap, multifractal, interpolation method, time series, missing values

HIGHLIGHTS

- Estimation of accuracy for the parameters of fractal and multifractal series containing missing values in the collection processes
- Bootstrap statistical analysis of the fractal and multifractal parameters
- A new resampling mechanism based on randomly gliding boxes

Competing interests statement: The authors declare that they have no competing financial interests.

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ABSTRACT

A time series collected in the nature is often incomplete or contains some missing values, and statistical inference on the population or process with missing values, especially the population or process having multifractal properties is easy to ignore. In this study, the simulation and actual data were used to obtain the probability distributions of fractal parameters through a new bootstrap resampling mechanism with the aim to statistically infer the estimation accuracy of the time series containing missing values and four kinds of interpolated series. Firstly, the RMS errors results showed that compared with the four interpolation methods for one parameter H required for fBm the direct use of the series with missing values has the highest estimation accuracy, while it shows certain instability in the estimations of the multifractal parameters C_1 and α, especially at higher missing levels, however, the accuracy of the parameters estimated by preprocessing of piecewise linear interpolation method can be improved; in addition, it is also concluded that α is more sensitive to the changes caused by these processing than another parameter C_1. Secondly, the effects on the ability of statistical inference for a population caused from the data losses are explored through the estimation of confidence intervals and hypothesis testing by proposing a new bootstrap resampling mechanism, and the conclusions showed that whether it is a mono-fractal parameter or multifractal parameters, the large deviations from the estimates of original series occur on the series with missing values when the losses are serious, while the defects can be compensated by the preprocessing using PLI and PBI methods; similarly, although the results of the incomplete series at the low missing levels are close to the original and PLI series, while at the high missing levels, the probabilities of Type Ⅱ Errors of the neighboring values are unable to ignore, but the PLI or PBI method can avoid the erroneous judgments.

1 Introduction

The scale invariance property has been known as a basic feature of natural phenomena, which is always associated with the inherent properties of the physical world such as complexity, non-smoothness, and irregularity. These phenomena with scale invariance are easily found in the natural world, such as pulsation of turbulent flow in pipelines, accumulation of minerals in the crust of the earth, volatility of financial market prices and winding coastline. As for the universality of this law, as pointed out by Cheng (2008) in the evolutions of a metallogenic system, various physical, chemical, and biological processes that take place in the earth system are interrelated, influenced and restricted mutually, which constitute a self-organizing structure, so as to make the system vary from equilibrium to far from balance, then to a critical state, and finally until a new cycle. For these phenomena, the traditional linear theory lacks the physical basis, and the process-based differential models are slightly less efficient, while fractal theory provides an effective method for describing this feature. So far, a variety of fractal models have been proposed, the most famous being the α-$f(\alpha)$ model based on measure theory and the co-dimensional model γ-$c(\gamma)$ based on probability theory (Evertsz and Mandelbrot, 1992; Schertzer and Lovejoy, 1987).

Observations of the natural world such as atmospheric environment, land cover and meteorological factors are usually recorded as time or spatial series to study the evolution of the earth system. As mentioned above, scale invariance is universal in the natural world, therefore the data collected from these natural phenomena such as $PM_{2.5,}$ surface temperature of the earth, and DEM generally possess fractal properties. Statistical inference of fractal parameters such as confidence interval estimation, hypothesis testing, etc, is important for accurate modeling. For example, the Hurst index H of Brownian motion is 1/2, while $0<H<1/2$ or $1/2<H<1$ indicate it is a fractional Brownian motion with negative or positive long-range dependence, in addition, for a generalized cascade series, when $\alpha=2$, it is a log-normal process, and when $1<\alpha<2$ or $0<\alpha=<1$, it is a log-Levy process, while $\alpha=1$, then it is a Cauchy process. A lot of studies about the statistical inferences of the parameters for time series with fractal or long-range dependence properties has been done. For example, Wendt et al. (2007) used the bootstrap resampling method to quantitatively study the estimated accuracy of multifractal parameters, and Gao et al. (2002) studied the estimation of the spectral function for non-stationary Gaussian process with stationary increments by constructing an estimator of asymptotic normality through the Gauss-Whittle function; in addition, on the basis of the in-depth analysis of the causes of sensitivity and deviation for calibrated spectral functions, a modified Jackknife estimator is tested to reduce the bias by Gaume et al. (2007). However, due to human or mechanical factors such as equipment maintenance, power failure, and improper human operations, the data obtained by users are often incomplete and there will be a large number of missing values. The effects on the estimates of parameters caused from the missing data for time series with fractal properties are easily ignored, therefore, it is necessary to carry out statistical inference on fractal parameters for the time series containing missing values.

Lack of data makes it difficult for a series of statistical methods to be used properly, because the preconditions for their use are compromised. Many imputation methods have been developed in the past to efficiently estimate a parameter of interest θ in a missing data situation, and to assess the variability of the estimates $\hat{\theta}$, i.e., multiple imputation, Bayesian imputation or commonly used interpolation methods. Considering singularity or irregularity is the essential feature of fractal or multifractal data, while most interpolation methods are implemented by using some linear or nonlinear models, and there is a problem, namely, it is whether the production of the relative regular data will have the positive or opposite effect on the parameter estimation. For example, researchers who are engaged in comparative politics or international relations, or others with the incomplete data, have been unable to complete the data because the best available imputation methods work poorly with the time series cross-section data structures common in these fields (Honaker and King, 2010). However, people always make the statistical inference after interpolating such data, and the common statistical inference methods cover parametric and non parametric methods. Robins and Wang (2000) have developed an estimator of the asymptotic variance of both single and multiple imputation estimators, especially for the variance estimator, which is consistent even when the imputation and analysis models are misspecified and incompatible with one another; considering the variance estimator proposed by Rubin, can be biased when the imputation and analysis models are misspecified and/or incompatible, Hughes et al. (2016) explored four common scenarios of misspecification and incompatibility through a full mechanism bootstrapping method and modified Rubin's multiple imputation procedure.

Statistical inference involves making propositions about a population based on estimators constructed from some samples of the population. Compared with parametric methods, non-parametric methods require less assumptions about probability distributions made about a population or process. However, they are computationally intensive, and lie in using computers to resample a large number of new samples from one original sample, so as to obtain the estimates based on the sample distributions. For fractal or multifractal series formed by infinite subdivisions, the formulas of the parameter estimators for statistical inference are more complex and need to make some assumptions, but non-parametric methods can avoid the derivations of the formulas. Obviously, the accuracy of the non-parametric statistical inference depends on the degree of the resemblance to the original sample for the resampling samples. The study by Wendt et al. (2007) indicates that the parameter confidence intervals calculated by the bootstrap method well cover the intervals simulated by Monte Carlo method, i.e. the resampling samples can well reflect the characteristics of the population. When pursuing the accuracy by performing statistical inference, except directly using the series with missing data for estimation, in order to reduce errors, it is often necessary to perform interpolation or imputation preprocessing. Here we will adopt four kinds of interpolation methods to deal with the missing data including piecewise linear interpolation (PLI), piecewise cubic spline interpolation (PCSI), piecewise cubic Helmit interpolation (PCHI) and piecewise Bessel interpolation (PBI). Based on the five kinds of series generated from simulation and experimental data, the performances of statistical inference will be studied by proposing a new resampling mechanism with the purpose to obtain the quantitative study of the accuracy of the parameters for the series containing missing values with fractal or multifractal properties.

2 Methodology

In order to test the statistical performances of the parameters for the time series with missing values, here we will apply two kinds of simulation data with a priori known and controlled scaling properties, fractional Brownian motion and generalized cascade process, the generation and estimation of which are introduced as follows.

2.1 Simulation and estimation of fBm and cascade process

Fractional Brownian motion is a typical Gaussian process with self-similar property, long-range dependence, stationary increments and regularity. The self-similar feature is characterized by the parameter H, also called Hurst index, which is between [0, 1]. The larger the H value is, the smoother the process will be, on the contrary, the smaller the value is, the coarser the process will be. Except the special case H=1/2 when the process is Brownian motion, if H>1/2, the increments are positively correlated, and while H<1/2, the increments are negatively correlated.

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There are many kinds of simulation methods for fractional Brownian motion, which can be divided into two kinds: exact and approximate. The exact includes Hosking method (Hosking, 1981), Cholesky method and Davies and Harte method (McLeod and Hipel, 1978; Davies and Harte, 1987), while the latter has stochastic integral method, summation method, random displacement method and wavelet decomposition (Mandelbrot and Van Ness, 1968; Norros, et al., 1999; Meyer et al., 1999). Here we will adopt the simulation method based on the wavelet coefficients, which are synthesized from the decomposition terms of the Gaussian white noise using wavelet transformation.

The estimation methods commonly used for H are R/S analysis, regression residual variance, wavelet estimation, spectral analysis and periodogram. The regression residual variance method is widely used in the estimates of fractal features with the following specific steps: suppose there is a series x_t, where t=1, 2...N, firstly, the cumulative difference is calculated: $Y_t = \sum_{t=1}^{n} x_t - < x >$; Secondly, divide the series into subintervals with length N_s=int(N/s) that do not overlap, and in order to reduce the errors the series is flipped and repeated, so 2N_s subintervals are obtained, where s is the length of each segment; thirdly，each subinterval is subjected to the elimination of the trend from the regression operation to yield the series: $Y_s(t)=Y(t)-p_v(i)$, where $p_v(i)$ is the polynomial trend obtained by fitting $Y(t)$ using the least square method to the subinterval of the vth segment; finally, the variance $F_s^2(v)=<Y_s^2(t)>$ is calculated for each interval, and we can get the fluctuation function $F_s = [\frac{1}{N}\sum_{v=1}^{N_s} F_s^2(v)]^{1/2}$ in the whole interval. Then the parameter estimates can be obtained through the relationship between the fluctuation functions and the scale $F(s) \propto s^h$. In addition, $F^n(s)$ can be obtained by detrending using n-order polynomial fitting function, here n is set to 1.

From the view of the construction of a fractional Brownian motion, the process involves the iterative sums of random quantities, and the corresponding is the iterative products of random quantities with the following specific construction process shown in **Fig.1**: suppose there is a unit quantity I_0, which scale is viewed as l, first split I_0 and its scale become l/2, yielding two values, $I_0*\eta_{11}$ and $I_0*\eta_{21}$, and η_{11} and η_{21} are random variables followed a probability distribution with

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$<\eta>=1$, where <> means expectation; second, the analogy is continuously iterated by k times and the result $I_0\prod_{i=1}^{k}\eta_{f(j,i),i}$ is obtained, where j=1, 2, . . ., b^k is the positional index of the layer k, i is the number of a layer, where at this level the scale is b^{-k}, and $f(j,i)$ represents the position in the i-th layer, with the form $f(j,i)$=roundup($j/b^{(k-i)}$) (Gaume, et al., 2007). The series generated from the above process is called cascade process which is dominated by the multi-fractal behavior, and the simulation data in this study is generated by the method provided by Schertzer and Lovejoy (1987).

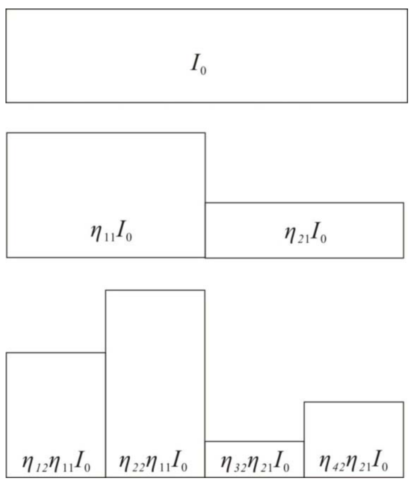

Fig.1. The generation of cascade process

Compared to mono-fractals, characterizing multi-fractals requires many parameters. Suppose there is a parameter γ, there is $\Pr(x>\lambda^{\gamma})\sim\lambda^{c(\gamma)}$, where γ is a singular value, and $c(\gamma)$ is the co-dimension function which is concave. Another widely used parameter is scaling function $K(q)$, which is related to $c(\gamma)$ by Legendre transformation pairs. In order to simplify the multifractal parameters, Schertzer and Lovejoy (1987) introduced an universal generalized multifractal model and simplified the parameters into two parameters: α and C_1, where the former represents the strength of multifractal, and the latter expresses sparse features. The accurate estimates of C_1 and α are an important task for multifractal analysis, usually the methods of which have the moment method and the double-trace moment method. Because involving lots of operations, the moment estimation method is used in the following study with the process which is as follows: firstly calculate the scaling function $K(q)$: supposing a series x_t, divide the series into different interval lengths, $x_{s,n}$, where s is the length of each segment, n=1, 2..., Ns, Ns=int(N/s), and next calculate $x_{s,n}{}^{q}$ for different length s respectively, then the $K(q)$ function can be estimated by the relation $<x_{s,n}{}^{q}>\sim\lambda^{K(q)}$, where λ=length(x_t)/s, length(x_t) is the length of the series; secondly, use the formula $K(q)=\frac{C_1}{\alpha-1}(q^{\alpha}-q)$, $0\le\alpha\le 2$ for the nonlinear fitting to get the parameter estimates.

2.2 Interpolation methods

The interpolation methods involved in this paper include piecewise linear interpolation (PLI), piecewise cubic spline interpolation (PCSI), piecewise cubic Helmit interpolation (PCHI) and piecewise Bessel interpolation (PBI). For PLI method, the missing values can be achieved by constructing linear functions between the adjacent known points around them, which does not consider the derivative values of the known points, and the accuracies of the interpolation points is related to the distance between the known points. PCHI method not only requires that the values at the two known points is equal to the values of the interpolation function, but also that the derivative values are equal, thereby improving the smoothness of the interpolants. Unlike the above interpolation methods, PCSI method requires that the second derivative be continuous at the known points in each interval, meaning that the interpolant is smoother than the previous two methods. Since there are many research on the above three kinds of interpolation methods, only the calculation formulas of PBI method are given here.

Unlike above interpolation methods, with the fact that the derivative values at the known points are not required to be equal to that of the interpolant for PBI method, its shape is controlled through the control points, which must be on the tangent line to the fitted curve at the known points. Let us take a look at the calculation process of PBI with $a\leqslant x_1<x<x_k\leqslant b$, where a and b are respectively the beginning and the end point of the series, x_1 and x_k are the two known points, and x are the missing points. In addition to the two known points, it requires to know two control points located between the known points. Denoting the two known points and the two control points as P_0, P_3 and P_1 , P_2 respectively, there are P_0=y_1 and P_3=y_k, and P_1 and P_2 are related to the tangent and can be written as functions of the derivatives at the known points, thus, there are

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$$P_1 = P_0 + 1/3(x_k - x_1) f'(x_1), \quad P_2 = P_3 + 1/3(x_k - x_1) f'(x_k), \tag{1}$$

where $f'(x_1)$, $f'(x_k)$ can be obtained by the differences between the known points and their neighboring points. After the four values at the four points are obtained, the Bezier curve can be written as a basis function form $B(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^3 P_3$, where $t=(x-x_1)/(x_k-x_1)$.

2.3 A resampling mechanism based on randomly gliding boxes

The traditional statistical inference is based on the normal distribution, while multifractal series usually have thick-tailed features, and the robust estimation can not be obtained by using the normal distribution test. The parametric statistical inference is based on the theoretical probability distribution of a population, that is, it is necessary to obtain the certain probability distribution in advance, and from the generation process of the cascade we can get it is difficult to obtain the probability distribution function of the cascade series. On the contrary, the non-parametric method need not to know the probability distribution function of the statistic, and can perform hypothesis testing on the condition that the actual distribution information of the series is poorly known, and can guarantee the robustness of the estimation. The key to the non-parametric method is to obtain the resampling distribution of the parameters from the original sample through resampling methods, and then use the resampling distribution to perform statistical inference, where the common resampling methods are bootstrap, jackknife, and Monte Carlo, etc..

In order to obtain the resampling distribution of an estimator, a new bootstrap resampling mechanism is introduced (**Fig.2**). As mentioned above, the parameter estimation process involves the series being divided into multiple non-overlapping intervals at different scales, and each interval contains the same number of the values. Imagine that if the starting point of the calculation for the series is different, and the estimation results obtained will be different. In practice, the origin-based estimation were used in many studies, and this caused the waste of the estimated information from original series. If a large number of random numbers are used as the starting points, then multiple estimates of the parameters can be obtained to form the resampling distribution. Obviously, this resampling method is achieved by the glide of a group of boxes controlled by a series of random numbers, therefore we could call it randomly gliding boxes (RGB) method. It is easy to see that the resampling mechanism can ensure that the resampling distribution function is similar to the original distribution, of course, and this method will be limited by the length of the series, the same idea can be seen in the Cheng's study (1999).

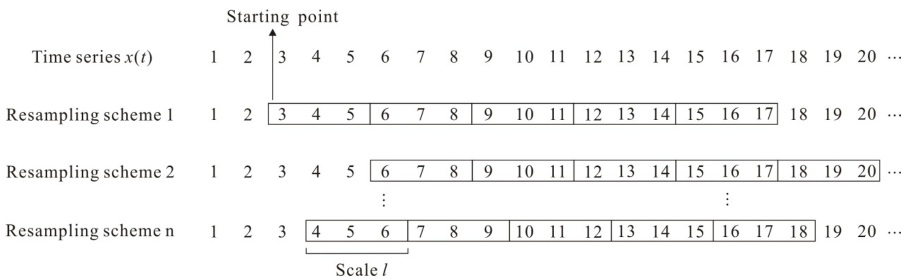

Fig.2. A new resampling mechanism realized by constantly determining the starting points controlled by a series of random numbers, also called randomly gliding boxes method.

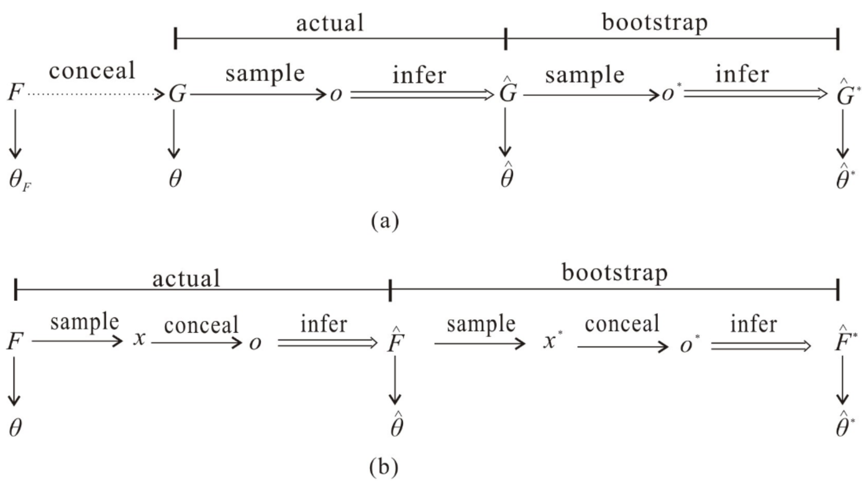

Fig.3. Two kinds of bootstrap logics by Efron (1994)

The main purpose of various sampling logics or mechanisms is to reduce the bias of an estimator θ^* of θ, which depend on the adequacy of the use of population information, concealment, and a sampling procedure, etc. As shown in **Fig.3(a)**, suppose there is a population F, $F=\{X_j, j=1,\cdots,N\}$,where X_j denotes a random sample having one or several random variables, then the population G with some concealment in some members is generated through a concealment process $o=c(x_i)$. $\theta_F=s(F)$ is the inference we need, where in our study $s(F)$ is a fractal or multifractal parameter. A sample o with size n is obtained from the population G, and after imputation or interpolation the empirical distribution $\hat{G}$ can be got from it, so we can get $\hat{\theta} = t(\hat{G})$ as an estimate of θ_F. The bootstrap inference begins with $\hat{G}$ instead of G, and repeat the above procedure to get a bootstrap estimate $\hat{\theta}^* = t(\hat{G}^*)$ of $\hat{G}^*$ corresponding to each replication o*. This mechanism does not fully use the population information, merely based on the establishment of the first sampling, that is, there is a small connection in the method with the required knowledge of θ_F or θ. The full bootstrap mechanism diagrammed in **Fig.3(b)** is more directly than in Fig.3(a), in which a random sample x is drawn and processed by a concealment $o=c(x_i)$ to obtain the observed data o, and then the parameter $\theta=s(F)$ is estimated by $\hat{\theta} = s(\hat{F})$. To

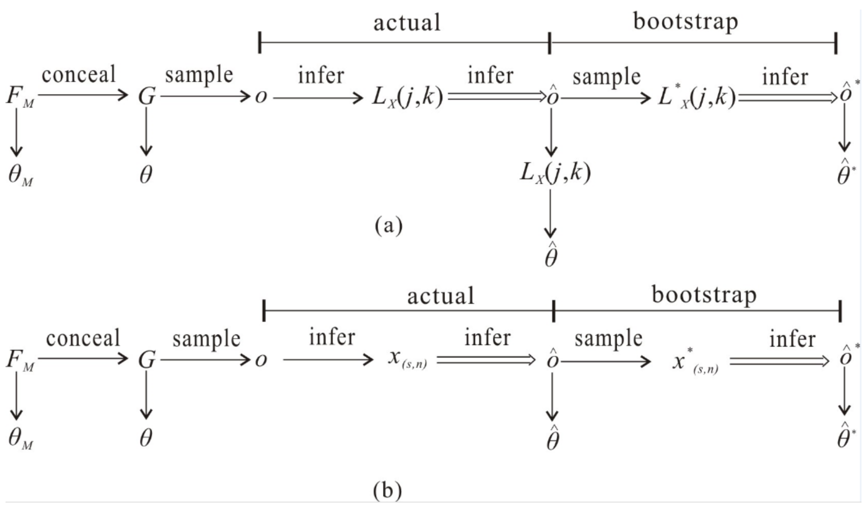

Fig.4. Our proposed bootstrap logic and another by Wendt

be different of this method is by repeating the whole process, i.e., sampling, concealment and inference to yield bootstrap replications $\hat{\theta}^* = s(\hat{F}^*)$. In our study, for the fractal or multifractal estimation, use the above simulation method to generate a sample set with a capacity of N from the known parameter values, denoted as F_M, and an concealment procedure is used to eliminate a part of data for each sample to obtain the sample set G containing missing values, which can be

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considered as observations. Another four sample sets are generated when apply the four interpolation methods to the sample set G, and therefore *N* estimates are available for each sample set, which are also called Monte Carlo estimates. Before using our proposed bootstrap method, given the uniqueness of the acquired time series one of the Monte Carlo samples is selected for statistical inference. The key to our proposed method shown in **Fig.4(a)** is to obtain the probability distributions of the estimated values by multi-segmenting the time series by random numbers, which is different from the resampling of the moment estimators at different scales on the basis of one-time segmentation shown in **Fig.4(b)**. The advantage is that it is easy to control the biases of multifractal parameters, especially the time series with higher sparseness.

2.4 Bootstrap confidence Intervals and hypothesis testing

After obtaining the bootstrap resamlpling distribution, and then we can perform the interval estimation and hypothesis testing on the estimators. Common bootstrap confidence interval methods are normal approximation method, percentile method, bias-corrected percentile method, and percentile-*t* method, and this study will use the percentile and percentile-*t* methods. Supposing the function of bootstrap distribution of $\hat{\theta}$ is $F^*(\hat{\theta}^*)$, there is $P\{\theta^*_{a/2} < \hat{\theta} < \theta^*_{(1-a)/2}\} \geq 1-a$, and the confidence interval of $\hat{\theta}$ is $[\theta^*_{a/2}, \theta^*_{(1-a)/2}]$, also known as the confidence interval with a confidence level of 1-α, where α is quantile. The meaning of the confidence interval is that if there are multiple samples, the probability that the confidence interval calculated by each sample contains the true value is 1-α. The percentile confidence interval has two shortcomings, one is that when the sample size is small, its performance is poor; the second is the need to make assumption that the bootstrap distribution is an unbiased estimate in advance. In order to avoid the above problems, the percentile-*t* method proposed is as follows: firstly transform $\hat{\theta}^*$ into a standard variable $t^*: \hat{t}^* = (\hat{\theta}^* - \hat{\theta})/\hat{\sigma}^{**}$, and the bootstrap distribution of t_B* is determined through resampling, where *R* is the number of resampling times; secondly, like the percentile method, we need determine the statistical values of *t** at α/2 and 1-α/2, and then combining with the parametric test we can get *t*-test confidence interval $[\hat{\theta} - \hat{\sigma}^{**}\hat{t}^*_{B,1-\alpha}, \hat{\theta} - \hat{\sigma}^{**}\hat{t}^*_{B,\alpha}]$, where $\hat{\sigma}^{**}$ is the standard deviation obtained by double sampling, that is, we need to take the second sampling for *S* times after the first sampling, *S* is the number of times of the secondary sampling.

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Another problem of statistical inference is hypothesis testing, for example, if H=1/2, the series is a Brownian motion, while for $H \neq 1/2$ it is a fractional Brownian motion; for cascade series, when α=2, it follows log-normal distribution, otherwise it follows log-Levy distribution. Hypothesis testing of a parameter is firstly to set a null hypothesis, H_0: $\theta=\theta_0$, and an alternative hypothesis H_1: $\theta \neq \theta_0$, and then construct a hub statistic that contains the test parameter, where percentile and percentile-*t* statistics are used with the form $\hat{t}_B = \hat{\theta} - \theta_0$ and $\hat{t}_s = \frac{\hat{\theta} - \theta_0}{\hat{\sigma}*}$ respectively. When the condition $\Pr\{t \in T \,|\, P_t^{H_0}\} = 1 - \alpha$ is met, the original hypothesis is accepted, otherwise the original hypothesis is rejected, where T is the accepted domain, and α is a quantile, which is called the significance level. However, since the decision is made from a sample, when H_0 is actually true, it may be possible to make a decision that rejects H_0, this is the error denoted by $\alpha_{error} = P_t\{x \in T \,|\, H_0\}$, which is also called Type Ⅰ Error; similarly, the error $\beta_{error} == P_t\{x \in \bar{T} \,|\, H_1\}$ is called Type Ⅱ Error, where α_{error} and β_{error} are the probabilities of Type Ⅰ Error and Type Ⅱ Error respectively. If the theoretical probability distribution function of the estimators is not available, there is no way to calculate the probability that the statistic falls within the accepted domain. But the acceptance domain can be obtained through the bootstrap method, thus the test statistic become $\hat{t}_B^* = (\hat{\theta}^* - \hat{\theta})$ and $\hat{t}_s = \frac{\hat{\theta} - \theta^*}{\hat{\sigma}^{**}}$, and the corresponding accepted domains for the percentile and percentile-*t* methods are $[\hat{t}_{B,\alpha}^*, \hat{t}_{B,1-\alpha}^*]$ and $[\hat{t}_{S,\alpha}^*, \hat{t}_{S,1-\alpha}^*]$.

3 Results and discussions

For the bootstrap estimation, the relevant parameters will be set as follows: R=500, S =300, sample size N=1000, and the quantile α is set to 0.05, if it is a bilateral test, then the half is 0.025. Firstly, the simulation data of fractional Brownian motion and generalized multifractal series with the known values for parameters are generated based on above mentioned generation methods, here we adopt H=0.6 for the fBm data and α=1.6 and C_1=0.2 for the multifractal series, both of them having 2048 data points (**Fig.5**). Since these series are not got any processing, so they are labeled as the original series. Secondly, the series containing missing values are generated, which

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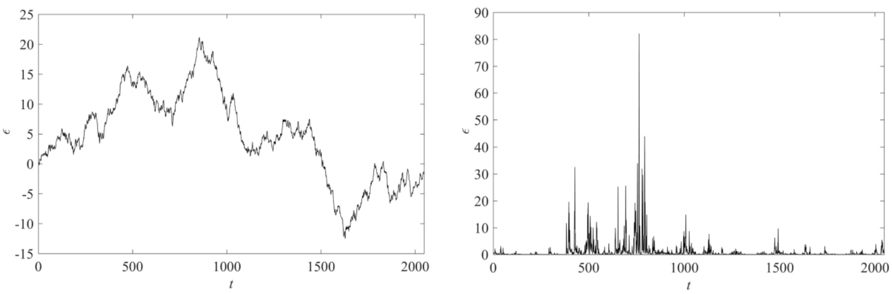
will be controlled by the missing degree which is represented by five levels, and the higher the level is, the more the missing values will be. The specific implementations are to randomly remove *T* data segments from the original series, where *T* in turn takes the values of 30, 50, 70, 100, and 150 respectively, and the number of each data segment is controlled by a random number ranging from 1 to 50. Thirdly, once the series with missing values are generated, then the four types of interpolated series are generated by the four interpolation methods.



fBm of *H*=0.6 Generalized cascade process of α=1.6 and C_1=0.2

Fig.5. The sample data generated by fBm and cascade models

3.1 Validation of mono-fractal data with missing values

Table 1 RMS errors of parameter *H* calculated for the series containing missing values (SCMV) and the four kinds of interpolated series at five missing levels

Data Types	Level1	Level2	Level3	Level4	Level5
SCMV	0.001	0.0013	0.0006	0.0011	0.0009
PLI	0.0042	0.0071	0.0125	0.016	0.0194
PCSI	0.078	0.0923	0.114	0.1198	0.1371
PCHI	0.0468	0.0565	0.074	0.0784	0.0942
PBI	0.0007	0.0003	0.0052	0.0106	0.0144

For a biased estimation statistic, the root mean square (RMS) error is widely used to quantitatively describing the estimation accuracy of the parameter. The RMS errors listed in **Table 1**, except the RMS error of the original series being 0.0028, calculated for the given five kinds of series are compared to show the accuracy of the parameter *H*. It can be seen that the series interpolated by PBI method and the direct use of the series containing missing values work best, while for the series with fewer missing values, the accuracy of PBI method is even higher. The

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next best performances are PLI and PCSI methods, while PCHI has the largest deviations. In general, with the increasing number of the missing values in the sample series, the accuracy of the estimates gradually decrease. From the estimates directly using the incomplete data, we can see that the change of the original series caused by the interpolation methods may be one of the reasons for the increase of the errors.

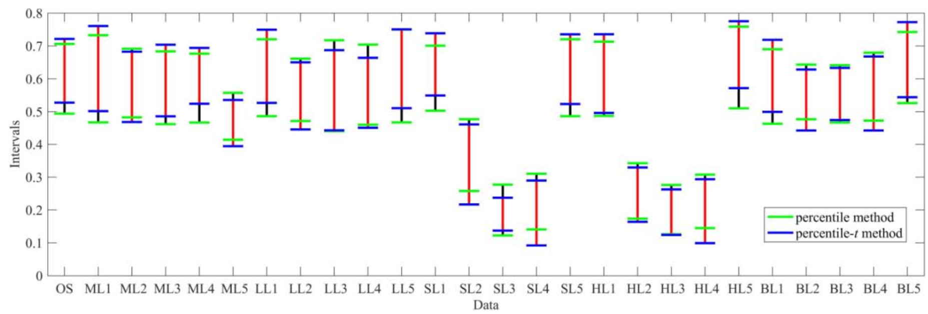


Fig.6. The percentile and percentile-*t* intervals of *H* estimated for the six kinds of time series at five levels based on RGB resampling mechanism (OS denotes the selected sample from the original samples, ML denotes the series containing missing values and the number 1 means the level, and LL, SL, HL and BL denote the series interpolated by PLI, PCSL, PCHL and PBL methods respectively)

Because the sample series collected or obtained in the real world usually is unique, therefore firstly we need choose one sample from the Monte Carlo simulations and then perform statistical inference, and the percentile and percentile-*t* intervals estimated for various series using the RGB method are shown in **Fig.6**. From the percentile or percentile-*t* perspective, it can be seen that as the amounts of the missing data increase, the estimated values for the series with missing values gradually decrease and constantly deviate from the original estimate, and visually the correct estimate of *H* is not obtained by using directly the incomplete data when the amount of the missing data increases to a certain extent. The endpoint estimates of the confidence intervals are also random variables, based on the comparisons of the percentile estimates of the two endpoints in each interval for the five kinds of series, we can see that the PLI and PBI series have the best performances, especially at the high missing levels, which obviously compensate for the impacts caused by the missing data, and the same conclusion can be obtained from the percentile-*t* method, therefore this indicates that the estimation accuracy of incomplete data can be improved by the preprocessing using PLI and PBI methods. In addition, it can be seen from the estimates of the original series that the estimated intervals using percentile method shift to the left compared with the *t*-method, which are experimentally by Monte Carlo method found to be caused by the variances of the series. Moreover, from the estimated results of *t*-method, it can be also concluded that the stability using PLI method for the right endpoint estimation is robust.

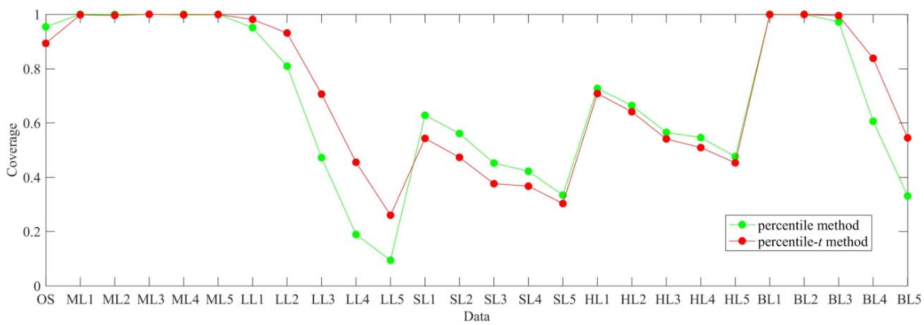
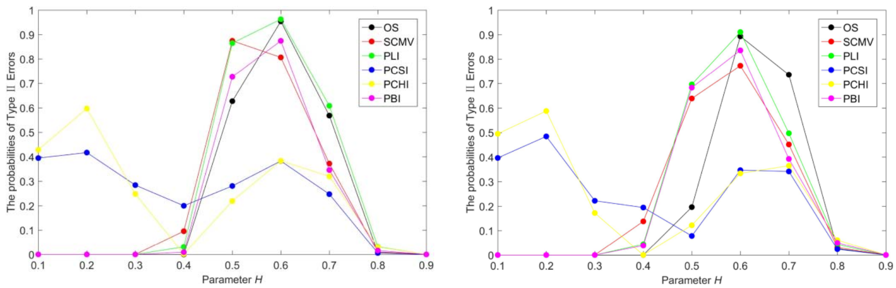


Fig.7. The coverage ratios of *H* estimated for the six kinds of time series at five levels by using percentile and percentile-*t* methods, i.e., the probabilities of the estimates of six kinds of simulation data with N=1000 fall within the intervals of the selected sample from original samples by RGB mechanism

The reliability of the parameter *H* estimated is also evaluated through the coverage of the estimated intervals for the incomplete or processed samples, i.e., the probabilities of the Monte Carlo estimates for the incomplete and interpolated time series falling within the bootstrap intervals of the selected sample extracted from the original samples (**Fig.7**). The results in the figure clearly show that the percentile estimated coverage for the original series is approximately equal to 0.95, which is very close to the theoretical range of the significance level, while the coverage of the percentile *t*-method interval is a little small, which prove that this sampling method can be used to statistically and reliably infer the parameter. Regardless of the level for the missing data, the estimated results of directly using the series with missing values are located in the regions of the selected sample, this indicates that for the parameter *H* stable estimation results can be obtained without any preprocessing for the raw time series. Moreover, it also shows that both the PLI and PBI series have higher coverage ratios when there are fewer missing values, but when faced the higher levels, a considerable part of estimates for all interpolated series fall outside the percentile or percentile-*t* intervals.

Another method for evaluating the efficiency in a hypothesis testing problem is the power function, the larger the power function is, the more effectively it can distinguish the null hypothesis from the alternative hypothesis. Since the probabilities of the two types of error are related to the theoretical distributions of the estimators, so they are not easily available, but the probabilities of them can be calculated by Monte Carlo simulations and bootstrap resampling methods. In general, the power function is continuous, which reflects the distributions of the two types of error for an estimator, and in order to easily manipulate in the study, here the discrete values are used for the investigation. In the following the calculation of the probability of Type Ⅱ Error is used as an example to illustrate the efficiency of the parameter estimates for the various time series, and prior to the comparisons the Monte Carlo simulations are used to generate a series of sample data with *H* values ranging within 0.1-0.9 with a step 0.1, where the size of each sample is set to 1000. **Figure 8(a)** and **8(b)** shows the probabilities of the estimates of different populations falling within the percentile and percentile-*t* estimated intervals of the incomplete and various interpolated series with the fixed parameter *H*=0.6, where the curves are averaged from the five missing levels. Compared with the other series, for the original series the percentile-based probabilities that the population parameters of *H*=0.5 and 0.7 are the smallest, that is, the probabilities of making Type Ⅱ Errors for the two neighbor populations are the smallest, which is obviously more efficient than the direct use of the incomplete data and various kinds of interpolated data for the hypothesis testing, but there are also some occurrences of errors for the populations of *H*=0.4 and 0.8. Note that there are the rises of the misjudgment ratios of θ=0.5 for the processed samples, especially obvious from the percentile-*t* estimation results. Let's take a closer look at the internal comparisons of the different degrees of deficiency, for the series with the missing values, when the alternative hypotheses θ=0.5 or 0.7 are true, as the missing data increase, the probabilities of making type Ⅱ Errors are gradually increased, even at the fifth level, the null hypothesis θ=0.6 will not get a correct test via the percentile or percentile-*t* method, however, the series processed by the PLI and PBI methods perform much better (**Fig.9**). Moreover, for all series at the higher missing levels, the probabilities of making Type Ⅱ Errors for the population parameters θ=0.4 and 0.8 are also increased, meaning that the more the missing data are, the lower the power test efficiency will be. For the PCSI and PCHI series, they exhibit the extreme performances, when the missing data are more serious, Type Ⅱ Errors mainly occur at the populations parameter *H*=0.1, 0.2, and 0.3, so the two interpolation methods should be used with caution.



(a) Hypothesis testing based on the intervals estimated by percentile method

(b) Hypothesis testing based on the intervals estimated by percentile-*t* method

Fig.8. Hypothesis testing of H=0.6 against the eight alternative hypotheses with the population parameters ranging within 0.1-0.9 with a step 0.1 except 0.6, where the size of each sample is set to 1000

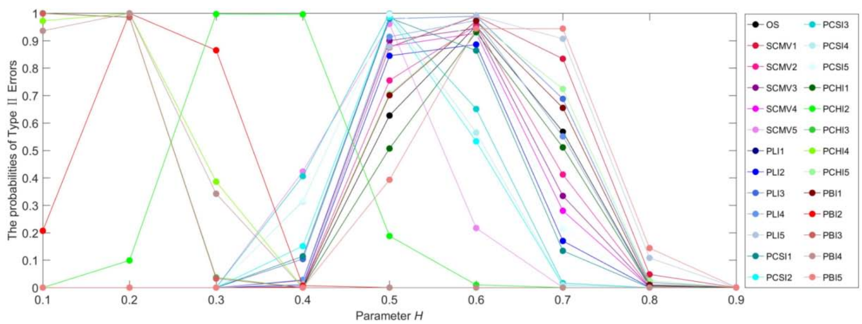


Fig.9. Hypothesis testing of H=0.6 against the eight alternative hypotheses with the population parameters ranging within 0.1-0.9 with a step 0.1 except 0.6 at five missing levels.

3.2 Validation of multifractal data with missing values

As mentioned above, for the simple case only one parameter H is required for the full characterization of scaling features for fractional Brownian motion, the estimation results directly using the incomplete data are less effective than using PLI method for completeness treatment, and the statistical inference of multifractal parameters will be more complex. From **Table 2** which shows the RMS errors of the two parameters C_1 and α, we can see, as a whole, that C_1 is less affected by the interpolation methods, and on the contrary α is affected more, however, α plays an important role in the identification and judgment of cascade models when multifractal analysis is used to seek the appropriate model for the natural time series. From the performances of the α

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estimates, the accuracy using both PLI and PBI methods are better than directly using the incomplete series, especially for PLI, at the five missing levels, the errors of the two parameters always oscillate around the values 0.0013 and 0.0831 obtained for the original series, while for the estimates of C_1, the estimation accuracy of the PBI method is higher, however, both PCSI and PCHI methods exhibit poor performances compared with PLI and PBI methods.

Table 2 RMS errors of parameters C_1 and α calculated for the generalized cascade process containing missing values (SCMV) and four kinds of interpolated series at five missing levels

Data	C_1					α				
	Level1	Level2	Level3	Level4	Level5	Level1	Level2	Level3	Level4	Level5
SCM	0.0026	0.0032	0.0048	0.0055	0.0058	0.0891	0.0999	0.1835	0.2165	0.2262
PLI	0.0031	0.0038	0.004	0.0048	0.0053	0.0806	0.0843	0.0825	0.0844	0.0714
PCSI	0.0043	0.0054	0.0082	0.0106	0.012	0.3579	0.4115	0.5413	0.6053	0.6918
PCHI	0.0042	0.0052	0.0072	0.0078	0.0082	0.2485	0.3162	0.4276	0.5125	0.5416
PBI	0.0013	0.0016	0.0019	0.0026	0.0033	0.0949	0.1118	0.1216	0.1365	0.1619

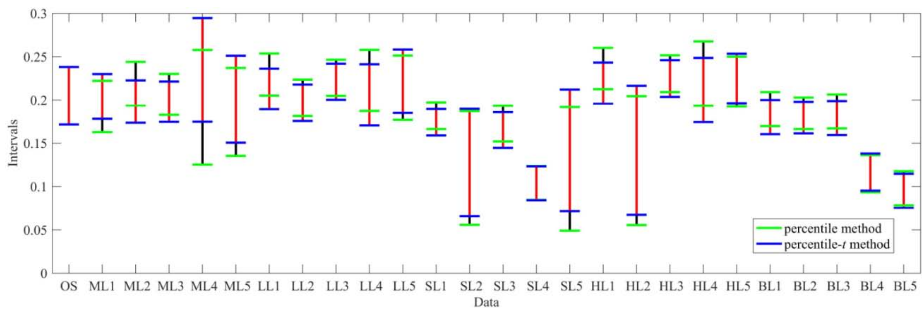

Fig.10. The percentile and percentile-*t* intervals of C_1 estimated for the six kinds of time series at five levels based on RGB resampling mechanism

Similarly, one sample selected from the 1000 Monte Carlo simulations is used to examine the RGB resampling mechanism for the estimation accuracy of multifractal parameters. **Fig.10** and **Fig.11** show the percentile and percentile-*t* estimated intervals of C_1 and α for various experimental data based on the RGB resampling mechanism. It can be seen that for the percentile and percentile-*t* estimates, the true value 1.6 of α is not located in the center of the estimated intervals of the selected series, and all the entire percentile intervals have leftward shifts, while the shifts for percentile-*t* method is slightly smaller, but the ranges of the estimated intervals are so large that this will be bound to influence the power tests. Compared with the estimated intervals of the selected sample, the left endpoints of the incomplete data for α and C_1 are estimated to have

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distinct left deviations at the high missing levels, while the PLI method can effectively compensates for this defect. The figure also shows large differences from the estimated values of the selected sample for the other three interpolation, except the estimation of the right endpoints having certain reference values.

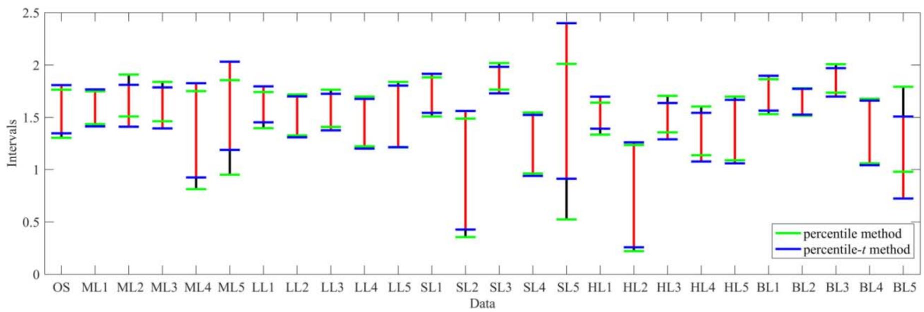

Fig.11. The percentile and percentile-*t* intervals of α estimated for the six kinds of time series at five levels based on RGB resampling mechanism

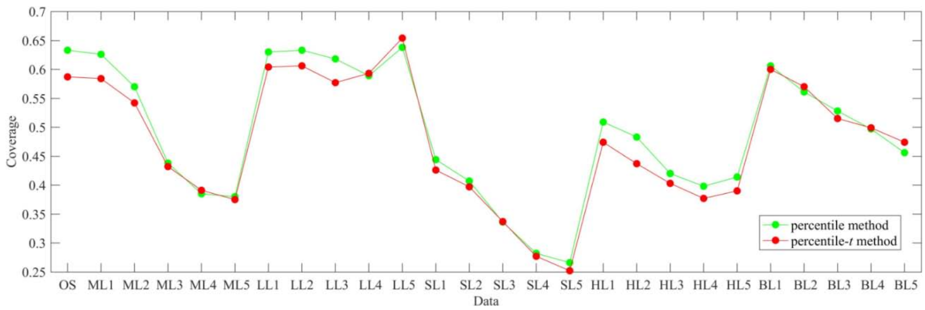

Fig.12. The coverage ratios of α estimated for the six kinds of time series at five levels by using percentile and percentile-*t* methods

The coverage of the original data, i.e., the percentage of unprocessed Monte Carlo simulation data falling within the estimated intervals of the selected sample, are approximately 0.71 and 0.63 for C_1 and α, and there is some deviations from the nominal quantile 0.95, which may be caused by the sample sizes, but it could constitute the main coverage of Monte Carlo estimations, therefore there are no fundamental impacts on the estimation accuracies of parameters in the statistical inference processes and the comparisons among interpolation methods. For the estimates of α shown in **Fig.12**, the accuracy of PLI method is higher than that of directly using the incomplete data, and the higher the level of missing, the higher the coverage. For the estimates of C_1 shown in **Fig.13**, the accuracy of the PBI, PLI and incomplete series are not much different,

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while the PCHI and PCSI series are inferior to the another two interpolations.

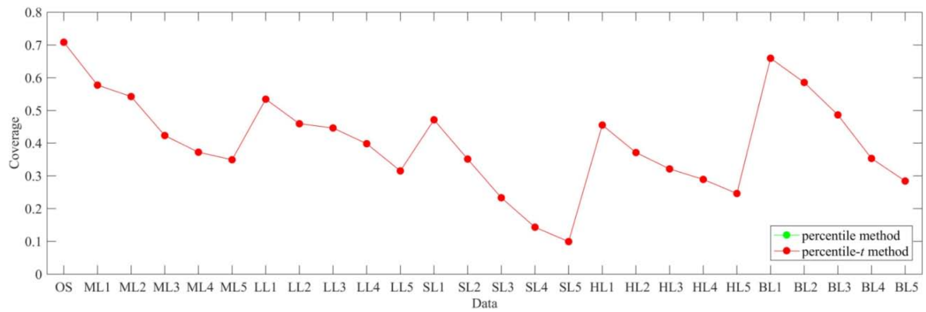
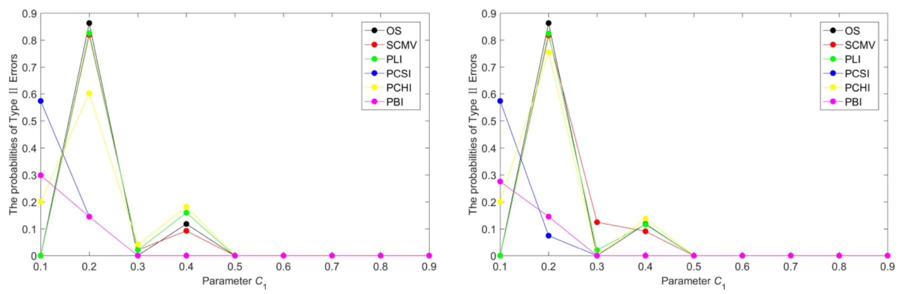


Fig.13. The coverage ratios of C_1 estimated for the six kinds of time series at five levels by using percentile and percentile-*t* methods



(a) Hypothesis testing based on the intervals estimated by percentile method

(b) Hypothesis testing based on the intervals estimated by percentile-*t* method

Fig.14. Hypothesis testing of C_1=0.2 against the eight alternative hypotheses with the population parameters ranged within 0.1-0.9 with a step 0.1 except 0.2, where the size of each sample is set to 1000

In order to avoid one of the parameters affecting another in the statistical inference processes, the power tests of the parameters for cascade series takes a method of maintaining one parameter unchanged and performing power tests on the other one. Therefore, when keeping C_1 unchanged, α takes the values of 1.1-1.9 with a step 0.1, similarly, and when keeping α unchanged, C_1 takes 0.1-0.9 with a step 0.1. Then the probabilities of Type Ⅱ Errors are obtained by the results of the Monte Carlo estimates for such assigned parameters falling within the confidence intervals of the selected sample, and **Fig.14** and **Fig.15** show the variations of the occurrence ratios of Type Ⅱ Errors over different population parameters for the two parameters respectively. For the percentile results estimated for C_1 (**Fig.14(a)**), the probability that the null hypothesis is true is 0.86 when

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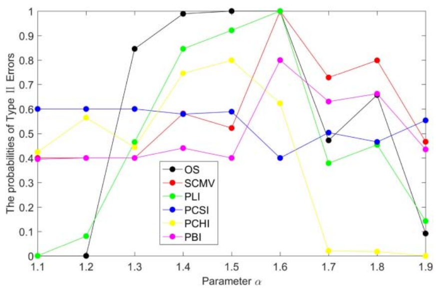
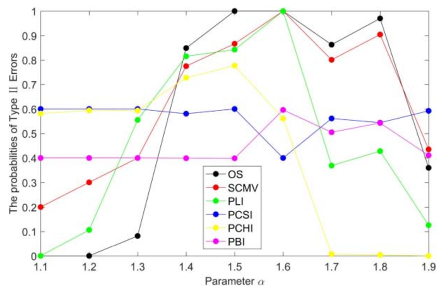
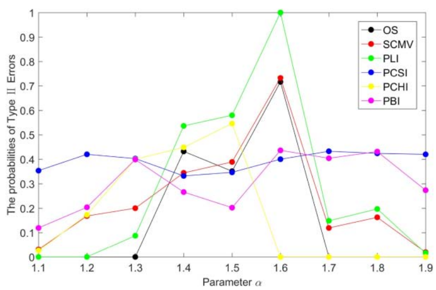
using the estimated intervals of selected sample, and the probability that $\theta = 0.3$ is misjudged is less than 0.1. We also get that the test results of original series are significantly better than that of



(a) Hypothesis testing based on the intervals estimated by percentile method



(b) Hypothesis testing based on the intervals estimated by percentile-*t* method



(c) Hypothesis testing based on the intervals estimated by percentile method at the significance level 0.5

Fig.15. Hypothesis testing of α=1.6 against the eight alternative hypotheses with the population parameters ranged within 1.1-1.9 with a step 0.1 except 1.6, where the size of each sample is set to 1000

the incomplete and interpolated series at any missing levels. On the whole, the incomplete and PLI series can provide near-original inference results, which outperform the other three interpolation methods. Among the other three interpolation methods, although some correct judgments can be made at some levels, the misjudgment probabilities of adjacent parameters cannot be ignored, even exceeding the probabilities that the null hypotheses are true. Under the test of the significance level 0.95, the percentile or percentile-*t* estimation accuracy of α for the selected series is not high, and high error ratios appear on the population parameters ranging from 1.3 to1.9 except 1.6, resulting in the neighboring parameters being misjudged and accepting the null

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hypotheses. **Fig.15 (c)** gives the percentile results at the significance level 0.5, and we can get that the PLI series at high missing levels show greater stability than the incomplete series. Although the results of the incomplete series at the low missing levels are close to the PLI series, while at the high missing levels, the probabilities of Type Ⅱ Errors of θ=1.2 and 1.3 are unable to ignore, just like the estimates of the interval endpoints, the PLI method well avoids the erroneous judgments. The high error probabilities are still present at H=1.3 and 1.4, and the main reasons for the high error ratios may be the poor stability of the estimation method and that the sample sizes are too small. Because we focus on the effects on the accuracies caused from the lack of data and interpolation methods, so these deviations have little effect on the analysis, in addition, you can also control the level of significance to control the probabilities of falling within the confidence interval.

3.3 Validation of actual data with missing values

In addition to using the simulation data for the examination, this study also uses empirical data $PM_{2.5}$ series collected in Beijing from January 2016 to December 2016 for one year, with a total of 8764 data points, and the parameters C_1 of α are estimated to be 0.1343 and 1.992 respectively, which indicate the distribution of the series is close to the lognormal distribution. The examined data are formed according to the preceding procedures in advance, **Table 3** gives the comparative RMS errors of the two parameters for various time series at all cases. It can be seen that as the missing level increases, the errors are increased and the accuracy of the estimates decrease. The accuracy of PLI and PBI methods is slightly better than that of the incomplete data, while the familiar performances of the estimates as in the simulations examination occur on the PCSI and PCHI methods.

As can be seen from **Fig.16** and **Fig.17**, the accuracy of the left and right endpoint estimates of confidence intervals are decreased as the amounts of missing data increase for various series. For the estimates of C_1 (**Fig.16**), the accuracy of PBI and PLI are the highest, and the errors of the two endpoints for the original series are evidently smaller than the rest of the series, while the incomplete series will not be correctly estimated. Combined with the performances of the simulation data, it can be inferred that the parameter C_1 is sensitive to the nature of the data, because C_1 reflects the sparsity of the data, therefore, when a data is somewhat or more sparse, using the incomplete data directly may yield more reasonable result. Compared with the right endpoint estimates, the accuracy of left endpoint estimates of the two interpolation methods are higher, except PCSI and PCHI methods. For the estimates of α (**Fig.17**), the incomplete, PLI and PBI series are all better, and the other two interpolation methods are less accurate. Compared with the percentile method, the percentile-*t* interval of the original series is narrowed, and we can get that the percentile-*t* method is more accurate by comparing with the endpoints of confidence intervals estimated for various series.

Table 3 RMS errors of parameter C_1 and α calculated for Beijing PM2.5 time series containing missing values (SCMV) and four kinds of interpolated series at five missing levels

Data	C_1					α				
	Level1	Level2	Level3	Level4	Level5	Level1	Level2	Level3	Level4	Level5
SCM	0.0006	0.0016	0.0029	0.0044	0.0061	0.0083	0.0096	0.0092	0.0107	0.0134
PLI	0	0.0001	0.0001	0.0001	0.0001	0.004	0.008	0.0116	0.0125	0.0126
PCSI	0.0499	0.0931	0.1342	0.1526	0.1801	0.1049	0.1971	0.2484	0.3004	0.2998
PCHI	0.0191	0.0444	0.0656	0.0882	0.1032	0.0757	0.1305	0.1767	0.1678	0.1894
PBI	0	0	0.0001	0.0001	0.0002	0.0027	0.0054	0.0086	0.0116	0.0133

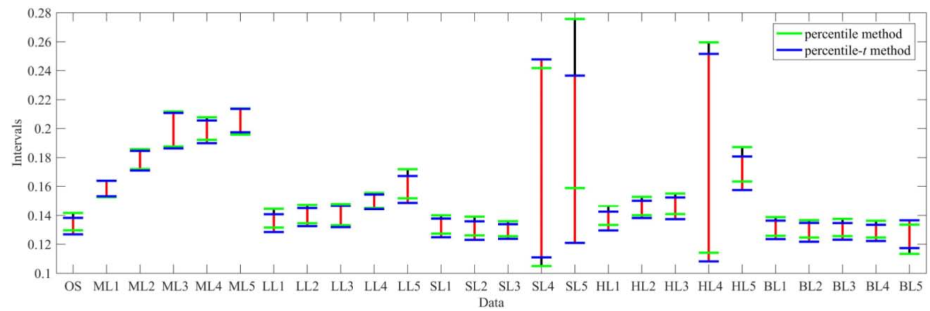

Fig.16. The intervals of C_1 estimated for the six kinds of time series constructed from Beijing PM2.5 time series at five levels by using percentile and percentile-*t* methods

From the coverage results of the confidence intervals estimated by using bootstrap method, the percentile coverage ratios of PLI at all missing cases exceed 40%, and this indicates that its accuracy is the highest (**Fig.18** and **Fig.19**). The sensitivity of C_1 to the missing values lead to all the estimates exceeding the interval of the original series from percentile or percentile-*t* results for the series with missing values, and for all series, the accuracy of the α estimates is better than that of C_1. For the estimates of C_1, as for the slight differences of the coverage between the percentile method and the percentile-*t* method, the reason is that the interval estimates using percentile-*t* method of the original series is narrowed. Since there are no repeated observations for a fixed parameter, so the empirical series is no longer to perform the power test here.

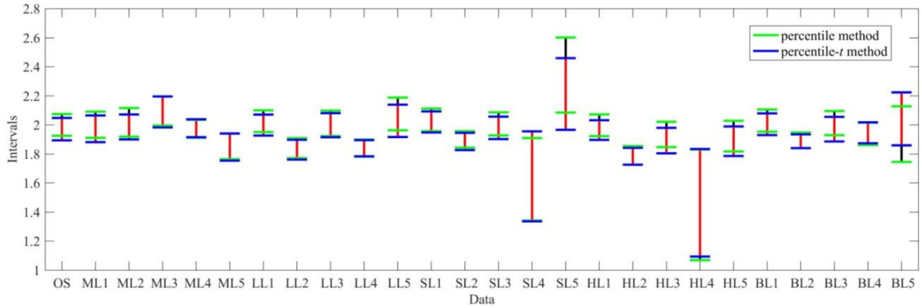

Fig.17. The intervals of α estimated for the six kinds of time series constructed from Beijing PM2.5 time series at five levels by using percentile and percentile-*t* methods

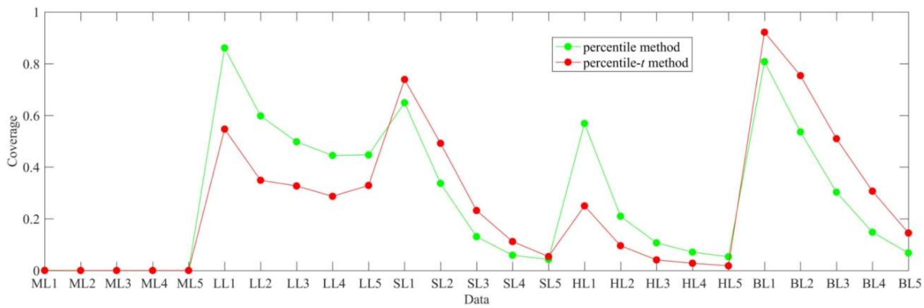

Fig.18 The coverage ratios of C_1 estimated for the six kinds of time series constructed from Beijing $PM_{2.5}$ at five levels by using percentile and percentile-*t* methods

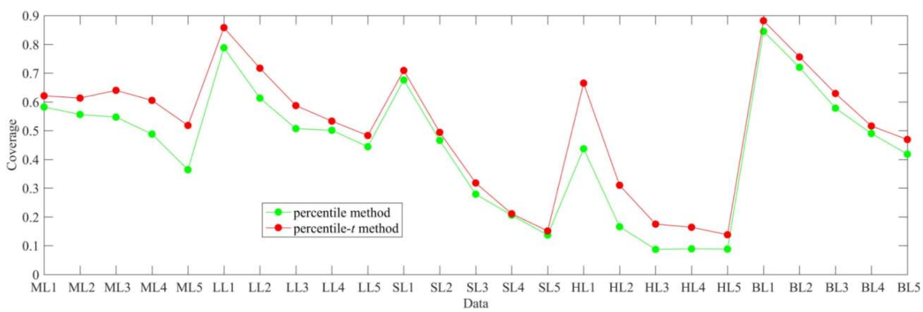

Fig.19. The coverage ratios of α estimated for the six of time series constructed from Beijing $PM_{2.5}$ at five levels by using percentile and percentile-*t* methods

4 Conclusions

An observation series obtained in the natural world is often incomplete and contains many missing values, therefore, the fractal modeling of a time series with missing values has certain uncertainties, or the problem is whether it is essential to perfect the data with missing values prior to the multifractal analysis. As we all know, a fractal series is irregular and scalar, but these interpolation methods do not consider the scaling characteristics of the series, such as the four interpolation methods adopted in the paper, while they make the data complete by using the relationships between the missing points and its adjacent sample values, therefore these interpolation methods can not completely replace the losses caused by missing data, and then how about the accuracy of the estimates of parameters for multifractal data, or the maintenance of the multifractal features for various interpolation methods?

From the results of the RMS errors, PLI method is reliable, which can provide reasonable estimation accuracy not only for the simulation data such as monofractal or multifractal data, but also for the experiment data, and the results of the direct use of the incomplete data are proved to be effective at the low missing levels. Both PCSI and PCHI methods behave badly, and it is not difficult to find that the result is caused by the smoothness yielded from the interpolation, it is also concluded that with the increase of the missing values, the influence of PCSI and PCHI methods on the estimation accuracy gradually become greater than the other three methods.

In order to study the distributions of the estimates of the parameters, a new resampling method, which focused on fully using the information of the series and is only controlled by the random numbers, is used to estimate the confidence intervals of the series containing missing values and various interpolated series. The resampling method was proved effective based on the fact that the bootstrap intervals with quantile 0.95 can well cover the estimates of the Monte Carlo simulations. For the estimates of Hurst index h, it can be concluded that when the amounts of missing data are large, the direct use of the series containing missing values cannot be correctly estimated, but the more accurate estimates can be obtained through PLI and PBI preprocessing. With some differences between percentile method and percentile-t method, the intervals estimated for the former are shifted to the left compared to the latter. For C_1 and α, compared to the confidence intervals of the original data, the left endpoints of incomplete data are estimated to have a significant left deviation at the high missing levels, and the right endpoint errors are smaller. The left endpoint estimates of PLI are close to the original series, and the accuracy improved significantly compared with the incomplete data. Through the investigation of the experimental

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data, it is found that the estimates of C_1 are more sensitive to the properties of the data, and it can not be accurately estimated by directly using the series containing missing values. For the estimates of α, we can see that the incomplete, PLI and PBI series have better performances. Compared with the percentile method, the percentile-*t* intervals of the original series are narrowed, and the accuracy of the percentile-*t* method is higher by comparing the endpoint estimates of various series.

The fBm and cascade simulations with known values for parameters were used to study the probabilities of Type Ⅱ errors, i.e., the probabilities of falling within the confidence intervals of the selected sample estimated by bootstrap method for such fixed population parameters. For fBm, eight alternative hypotheses were assigned, such as 0.1, 0.2, . . . 0.9, except 0.6, while for cascade, keep one parameter unchanged, and let another take a value in a range 1.1-1.9 or 0.1-0.9 respectively. For Hurst index *h*, it was analyzed that the probability of Type Ⅱ error for each alternative hypothesis is the closest to the original series for directly using the series containing missing values and PLI method at the low missing levels, which is more effective than the other three interpolation methods, while at the high missing levels the error ratios of the population parameters around 0.6 keep rising. For the multifractal series, the similar conclusion can be drawn, but when faced with low missing levels, the performance of the PLI is even better than using the series containing missing values directly. Moreover, when the significance level is set to 0.5, it can be concluded that the PLI method has more stability than directly using the series containing missing values.

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