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Phase-dependent dynamics of breather collisions in the compact Zakharov equation for envelope

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Abstract. We study collisions of the solitary type coherent wave structures – breathers in the nonintegrable Zakharov equation for envelope describing gravity waves on the surface of deep water. The numerical simulations of breather interactions revealed several fundamentally different effects when compared to analytical two-soliton solutions of the nonlinear Schrodinger equation. The relative phase of the breathers is shown to be the key parameter determining the dynamics of the interaction. We show that the maximum of the amplitude amplification can significantly exceed the sum of the breather amplitudes. The breathers lose up to a few percent of their energy during the collisions due to radiation of incoherent waves and in addition exchange energy with each other. The level of the energy loss increases with large values of the wave steepness, and also with certain synchronisation of breather phases. Each of the breathers can gain or lose the energy after collision resulting in the increase or decrease of the amplitude. The magnitude of the space shifts that breathers acquire after collisions depends on the relative phase and can be either positive or negative.

1 Introduction

In the complex dynamics of nonlinear waves on the surface of deep water, one type of processes is of special attention in the experimental and theoretical studies: the propagation of the coherent wave structures – solitons and breathers, and their mutual interactions. The exact model for the surface waves are the Euler equations which are rather complicated for analytical analysis and numerical simulations. At the same time approximate models for deep water surface waves demonstrate good agreement with experimental data and are widely used in fluid dynamics and geophysics.

The first-order weakly nonlinear model for the surface waves is the nonlinear Schrodinger (NLS) equation. The NLS model describes propagation of envelope of the quasi-monochromatic wave packets (Zakharov (1968)) and in the one-dimensional case is completely integrable via the inverse scattering transform (IST) method (Zakharov and Shabat (1972)).

As well known, solitons and breathers appear as a result of a balance between nonlinearity and dispersion. In the case of the NLS equation, the IST allows to find soliton solutions and describe their interactions analytically. The complete integrability leads to the remarkable properties of the interaction process: solitons of the NLS equation collide absolutely elastically. The next order nonlinear model for description of the wave envelope propagation, the so called Dysthe equation (Dysthe (1979)) – is not integrable. However, the existence of solitary type solutions in this model was also demonstrated by different analytical and numerical approaches – see Akylas (1989); Zakharov and Dyachenko (2010). In this work we will use the term "breathers" to describe solitary type solutions of the Dysthe equations and the next order equations.

The NLS and Dysthe models assume, that the wave train is quasi-monochromatic, i.e. its spectrum belongs to a narrow vicinity of some carrier wave number k_0. The compact Zakharov equation is free from this restriction and can be written not only for the envelope, but for the wave train itself (Dyachenko and Zakharov (2011, 2012)). Breather solutions of the compact Zakharov equation and the fully nonlinear Euler equations can be found numerically (Dyachenko and Zakharov (2008); Fedele and Dutykh (2012); Dyachenko et al. (2013)). Recently, the propagation and interactions of breathers have been studied in the laboratory experiments and by the numerical simulations of the fully nonlinear Euler equations (Slunyaev et al. (2013, 2017)).

The study of breather interactions in approximate high-order deep water models is an important step in understanding of the surface waves dynamics and the fundamental properties of the Euler equations. Here we focus on the compact Zakharov equation in the form suggested by Dyachenko et al. (2017a) which describes the wave train envelope without any assumptions on its spectral width. The envelope form of the compact Zakharov equation allows us to make straightforward comparison with the less precise but integrable NLS model.

The dynamics of soliton interactions in the NLS model drastically depends on their complex phases. In particular, the maximum of the amplitude amplification in the collision process is determined by sinchronization of the solitons phases. This soliton phase sinchronization plays the important role in the formation of waves of extreme amplitude (rogue waves) and and has been well studied in different contexts including not only water waves theory (Kharif et al. (2009)), but also e.g. optical waves in fibre (Antikainen et al. (2012)). Recently, analytical phase synchronization in many-soliton ensembles has been studied by Sun (2016) and by Gelash (2018). Moreover, the role of soliton phase parameters has been extensively studied in other integrable models, such as mKdV equation describing the waves on shallow water – see the paper of Slunyaev and Pelinovsky (2016).

In this work we study the dependence of the characteristics of the breathers interactions on their phases. Our studies of the amplitude amplification, energy exchange between breathers, energy loss due to emission of incoherent radiation and space shifts of the breathers after mutual collisions reveal fundamental differences when compared to the integrable NLS model.

2 Envelope equations for deep water surface gravity waves.

As was shown by Zakharov (1968), the one-dimensional free surface potential flow of an ideal incompressible deep fluid in a gravity field belongs to a class of Hamiltonian systems. In this system the shape of the fluid surface $\eta(x,t)$ and the flow potential function $\psi(x,t)$ play the role of canonically conjugated variables. The Hamiltonian approach for surface waves allows to build a perturbation theory taking the wave steepness μ as a small parameter. The compact Zakharov equation for unidirectional gravity waves corresponds to the expansion of the Hamiltonian up to second order by μ (see details in the mentioned above works of Dyachenko and Zakharov (2011, 2012)).

Recently Dyachenko et al. (2016a, 2017b) suggested the canonical transformation from physical real-valued Hamiltonian variables $\eta(x,t)$ and $\psi(x,t)$ to the complex normal variable $c(x,t)$, such that the compact Zakharov equation can be written in

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x-space in the following "super" compact form:

$$\frac{\partial c}{\partial t} + i\hat{\omega}_k c - i\partial_x^+ \left(|c|^2 \frac{\partial c}{\partial x} \right) = \partial_x^+ \left(\hat{k}(|c|^2) c \right). \tag{1}$$

The operators $\hat{k}$ and $\hat{\omega}$ act in the Fourier space as multiplication by $|k|$ and $\sqrt{gk}$ respectively. Here and below g means the gravity acceleration, so that the term $\sqrt{gk}$ corresponds to the dispersion law of the gravity waves. The operator ∂_x^+ in the Fourier space is $ik\theta(k)$, where $\theta(k)$ is the Heaviside step function. Physical variables – the free surface profile $\eta(x,t)$ and flow potential $\psi(x,t)$ can be recover by the known $c(x,t)$ using the canonical transformation. We present below only the transformation of the free surface up to the second order by $c(x,t)$:

$$\eta(x,t) = \frac{1}{\sqrt{2} g^{\frac{1}{4}}} (\hat{k}^{-\frac{1}{4}} c(x,t) + \hat{k}^{-\frac{1}{4}} c(x,t)^*) + \frac{\hat{k}}{4\sqrt{g}} \left[\hat{k}^{-\frac{1}{4}} c(x,t) - \hat{k}^{-\frac{1}{4}} c(x,t)^* \right]^2. \tag{2}$$

Here operator $\hat{k}^\alpha$ is the multiplication by $|k|^\alpha$ in the Fourier space and * means the complex conjugation.

The equation (1) has the breather solution:

$$c(x,t) = c_{br}(x - Vt) e^{i(\tilde{k}x - \tilde{\omega}t)}, \tag{3}$$

where $\tilde{k}$ is a carrier wavenumber, $V = \frac{1}{2}\sqrt{\frac{g}{\tilde{k}}}$ is the group velocity and $\tilde{\omega}$ is frequency close to $\sqrt{g\tilde{k}}$. In the Fourier space this solution has the following form:

$$c_k(t) = \frac{1}{\sqrt{2\pi}} \int c_{br}(x - Vt) e^{i(\tilde{k}-k)x} e^{-i\tilde{\omega}t} dx = \frac{1}{\sqrt{2\pi}} \int c_{br}(\xi) e^{i(\tilde{k}-k)\xi} e^{-i(\tilde{\omega} - \tilde{k}V + kV)t} d\xi = \varphi_k e^{-i(\Omega + Vk)t}, \tag{4}$$

where

$$\varphi_k = \frac{1}{\sqrt{2\pi}} \int c_{br}(\xi) e^{i(\tilde{k}-k)\xi} d\xi \tag{5}$$

and

$$\Omega = \tilde{\omega} - \tilde{k}V. \tag{6}$$

The breather solution can be found numerically by Petviashvili method (Petviashvili (1976)). Details of application of the Petviashvili method to the super compact equation are given in (Dyachenko et al. (2017b)). The solution φ_k (5) can be found numerically by iterations:

$$\varphi_k^{n+1} \quad = \quad \frac{NL_k^n}{M_k} \left[\frac{\sum_{k'} (\varphi_{k'}^n \, NL_{k'}^n)}{\sum_{k'} (\varphi_{k'}^n \, M_{k'} \varphi_{k'}^n)} \right]^{-\frac{3}{2}} \tag{7}$$

Here φ_k^n is the breather solution φ_k on n-th iteration,

$$M_k = \Omega + Vk - \omega_k. \tag{8}$$

The notation NL^n corresponds to the nonlinear part of the equation (1) on n-th iteration in the x-space:

$$NL^n \quad = \quad -\frac{\partial^+}{\partial x} \left(|\varphi^n|^2 \frac{\partial \varphi^n}{\partial x} \right) + i \frac{\partial^+}{\partial x} \left(\hat{k} \left(|\varphi^n|^2 \right) \varphi^n \right). \tag{9}$$

In the equation (7) NL^n_k means the Fourier transform of NL^n. The breather solution is defined by two parameters: the group velocity V and the frequency Ω: The breather solutions found using the iteration procedure (7) for certain V and Ω may have an arbitrary phase ϕ, i.e. differ by factor $e^{i\phi}$.

Recently Dyachenko et al. (2017a) derived the envelope version of the super compact equation (1) using the envelope function $C(x,t)$:

$$c(x,t) = C(x,t)\mathrm{e}^{i(k_0 x - \omega_{k_0} t)}. \tag{10}$$

The envelope equation written in the framework moving with the group velocity $V_0 = \frac{\partial \omega_{k_0}}{\partial k_0}$ (that corresponds to the carrier wave number k_0) has the following form:

$$\begin{aligned}\frac{\partial C}{\partial t} \quad &+ \quad i\left[\omega_{k_0+k} - \omega_{k_0} - \frac{\partial \omega_{k_0}}{\partial k_0}\hat{k}\right]\hat{\theta}_{k_0+k}C + ik_0^2\hat{\theta}_{k_0+k}\left[|C|^2 C\right] + \\ &+ \quad k_0\hat{\theta}_{k_0+k}\left[C\frac{\partial}{\partial x}|C|^2 + 2|C|^2\frac{\partial C}{\partial x} - i\hat{k}(|C|^2)C\right] - \\ &- \quad \hat{\theta}_{k_0+k}\frac{\partial}{\partial x}\left[\hat{k}(|C|^2)C + i|C|^2\frac{\partial C}{\partial x}\right] = 0.\end{aligned} \tag{11}$$

Note, that this equation was derived without any assumptions about spectral width of the wave field and has the same range of applicability as the equation (1). The breather solutions (3), written in terms of the envelope function $C(x,t)$ has the form:

$$C_{br}(x,t) = c_{br}(x - Vt)e^{i(\tilde{k}-k_0)x - i(\tilde{\omega}-\omega_{k_0})t}. \tag{12}$$

In the assumption of narrow spectral bandwidth and small wave steepness the envelope equation (11) transforms into the well known models for envelopes of gravitational waves on deep water – the Dyste equation and the NLS equation. In this work we study only the model (11) itself and the NLS equation which can be extracted from (11) as:

$$\frac{\partial C}{\partial t} + \frac{i\omega_{k_0}}{8k_0^2}\frac{\partial^2 C}{\partial x^2} + ik_0^2\left[|C|^2 C\right] = 0. \tag{13}$$

Usually the NLS equation is written in the dimensionless units (see e.g. the monograph Novikov et al. (1984)). Here we keep the dimensional constants ω_{k_0} and k_0 to compare dynamics of soliton interactions in both studied models straightforwardly.

3 Soliton and breather solutions

We study soliton dynamics in the frame moving with the velocity $V_0 = \frac{\partial \omega_{k_0}}{\partial k_0} = 0.05$, that corresponds to the carrier wavenumber $k_0 = 100$ (here and further $g = 1$). The envelope one-soliton solution of the NLS equation (13) moving in this frame with relative velocity U can be written as:

$$C_s(x,t) = C_0 sech\left[\frac{2C_0k_0^2}{\sqrt[4]{gk_0}}\left((x-x_0) - Ut\right)\right] exp\left[-i\frac{4k_0^2}{\sqrt{gk_0}}U(x-x_0) + i\frac{2U^2k_0^2}{\sqrt{gk_0}}t - i\frac{C_0^2k_0^2}{2}t + i\phi_0\right], \tag{14}$$

where C_0 is the soliton amplitude and ϕ_0 is an arbitrary phase.

As one can see from the equation (14), the modulus of the NLS soliton envelope function $|C_s(x)|$ for a fixed wavenumber k_0 (and velocity V_0) is defined by the amplitude parameter C_0. The relative velocity U in (14) only affects the carrier wave and

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shifts its wavenumber by the value $\Delta k = \frac{4k_0^2}{\sqrt{gk_0}}U$. In this work we focus on the interactions of solitons and breathers of equal amplitude C_0 and different velocities $V = V_0 + U$. To be able to describe soliton collisions using the NLS model analytically we should hold the carrier wave number k_0 and vary relative velocity U. Thus in our studies all solitons have the same shape of the modulus $|C_s(x)|$.

The dynamics of breather collisions in the supercompact envelope Zakharov equation can be investigated only by numerical simulations. We study interactions of breathers of the same amplitudes C_0 and different velocities $V = V_0 + U$, like in the case of solitons. The amplitude of the breather C_0 is not an independent parameter of the solution. To find the breather having the given velocity V and amplitude C_0 we vary parameter Ω in the Petviashvili method. The envelopes of the breathers solutions found in this way differ from each other, or more exactly, have different characteristic width (see the curves 1,2 and 3 in figure 1). Only when $U = 0$ the breather solution (12) of the exact envelope equation almost coincides with the NLS soliton (14) (see the curves 3 and 4 in figure 1). To obtain solitons of almost the same characteristic width as breathers we should vary the carrier wave number k_0 instead of the relative velocity U in the solution (14). But this procedure is not appropriate for our work since in this case we are not able to use the NLS equation to study soliton interactions.

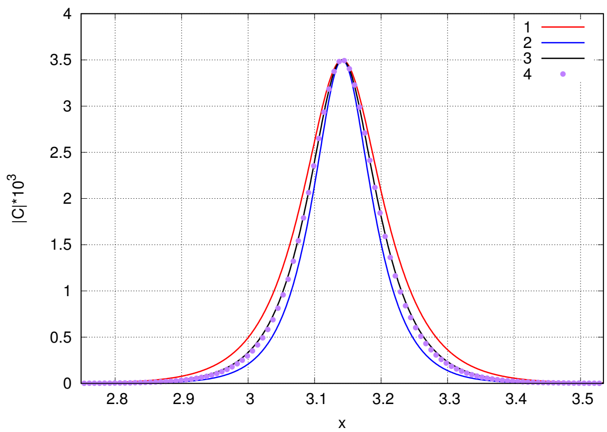

Figure 1. Comparison of breathers and solitons with the same amplitude $C_0 = 3.5 \cdot 10^{-3}$. The red curve (1) shows the absolute value of envelope function $|C(x)|$ of the breather with $U = 0.002$, $\Omega_1 = 4.85735$; the blue curve (2) corresponds to the breather with $U = -0.002$, $\Omega_2 = 5.27646$ and the black curve (3) – to the breather with $U = 0$, $\Omega_0 = 5.05798$. The purple dots (4) show the soliton solution with $U = 0$.

4 Interactions of solitons and interactions of breathers.

We compare three different cases of two-soliton and two-breather interaction corresponding to the following three different values of the maximum wave steepness (and amplitudes correspondingly): $\mu \sim 0.1$ (amplitude $C_0 = 2.25 \cdot 10^{-3}$), $\mu \sim 0.15$ ($C_0 = 2.8 \cdot 10^{-3}$) and $\mu \sim 0.2$ ($C_0 = 3.5 \cdot 10^{-3}$). To find the maximum wave steepness we calculate the maximum of the derivative of the surface elevation: $\mu = max(\eta'(x))$, where $\eta(x)$ is recovered via the transformation (2). In each case the

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velocity of colliding solitons (as well as breathers) are equal to $V = V_0 \pm U_0$, where $V_0 = 0.05$ (i.e. $k_0 = 100$ as in the previous paragraph) and $U_0 = 0.002$. At the initial moment of time $t = 0$ solitons (or breathers) are located at $x = \pi/2$ and $x = 3\pi/2$. We study the dynamics of interaction up to the time $t = \pi/U_0$.

As we mentioned above the NLS equation is completely integrable model and exact N-soliton solution (N means the number of solitons) is known starting from the work of Zakharov and Shabat (1972). Here we apply analytical two-soliton solution formulas to study collisions of solitons. In figure (2) we present an example of interacting solitons (with equal amplitudes C_0) and demonstrate that their collision leads to space shifts in soliton positions and formation of nonlinear wave profile with maximum amplitude amplification equal to $2C_0$. The latter is possible only at certain phase synchronization – see the text below. In the NLS model the space shift δx are determined by soliton amplitudes and velocities and does not depend from their phases. In our case each of solitons acquires a positive shift in the direction of soliton propagation, that can be calculated via the following formula (see e.g. the monograph of Novikov et al. (1984)):

$$\delta x = \frac{\sqrt{gk_0}}{2C_0 k_0^2} \lg\left(1 + \frac{C_0}{U_0}\frac{\sqrt{gk_0}}{4}\right). \tag{15}$$

For parameters corresponding to the figure (2), the value of δx is equal to 0.0975. In addition to the space shift (15) solitons acquire a phase shift $\delta\phi$ that can be calculated using similar expression and also depends only from soliton amplitudes and velocities. As one can see from (14) the dependence of soliton phase (at the maximum of the modulus of the envelope) on time is determined by:

$$\phi(t) = \phi_0 + \left(-\frac{2U^2 k_0^2}{\sqrt{gk_0}} - \frac{C_0^2 k_0^2}{2}\right)t. \tag{16}$$

Thus for a couple of solitons with equal amplitudes C_0 and relative velocities $\pm U_0$ the phase difference is time invariant: $\Delta\phi(t) \equiv \phi_{02} - \phi_{01}$. Here we neglected the additional phase shifts $\delta\phi_1$ and $\delta\phi_2$ that solitons acquire as a result of interaction since in our case they compensate each other in $\Delta\phi$.

The maximum amplitude $2C_0$ is achieved when the phase difference between colliding solitons is equal to zero: $\Delta\phi = 0$ (see e.g. Antikainen et al. (2012)). The value of the maximum amplification depends from the relative phase of the interacting solitons $\Delta\phi$. In this work we use the normalised definition for the maximum amplification A of the wave field amplitude:

$$A(\Delta\phi) = \frac{max(|C(\Delta\phi)|)}{2C_0}, \tag{17}$$

In the NLS model the amplitude amplification decrease with increasing of $\Delta\phi$ – see the figure (3). Note that in the general case of different solitons the phase difference $\Delta\phi$ is not time invariant. However, the dependence of maximum amplification on $\Delta\phi$ calculated at the moment of time when solitons located at the same point is similar to the case presented in the figure (3) with maximum of the function $A(\Delta\phi)$ shifted from zero, that is caused by different additional phase shifts $\delta\phi_1$ and $\delta\phi_2$.

The numerical simulation of breather interactions in compact Zakharov equation were carried out in the periodic domain $x \in [0, 2\pi]$. To study the influence of the relative phase of the breathers at the moment of the collision on the value of the recorded maximum amplification $A(\Delta\phi)$ (17), a series of numerical experiments was performed with different value of the initial phase ϕ_{01} for the left breather. Using (12) and taking into account the definition (6), it is easy to calculate the dependence

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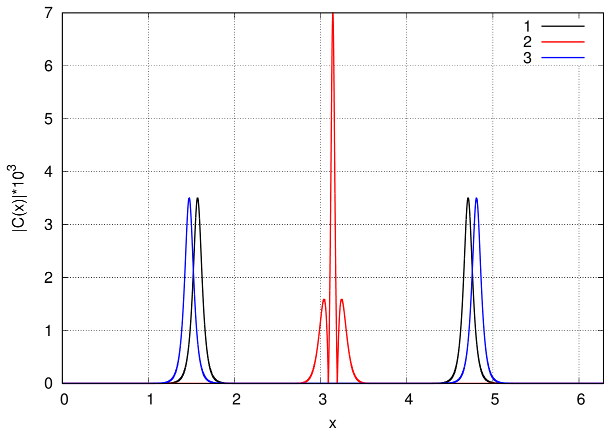

Figure 2. Collision of solitons with amplitudes $C_0 = 3.5 \cdot 10^{-3}$ and velocities $U = \pm U_0 = \pm 0.002$. The black curve (1) shows the absolute value of envelope function $|C(x)|$ at $t = 0$, the red curve (2) corresponds to the moment of maximum amplitude amplification during collision process and the blue curve (3) corresponds to solitons at $t = \pi/U_0$.

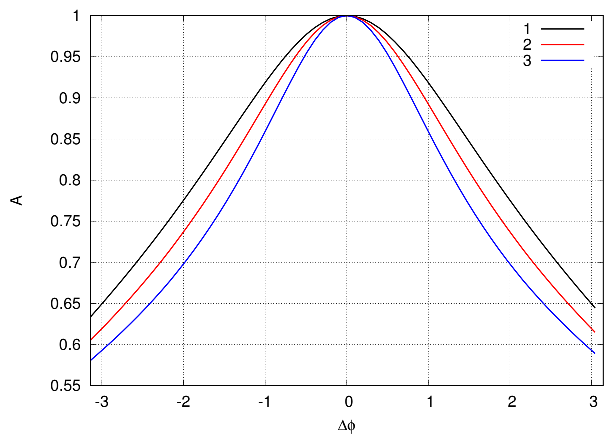

Figure 3. Dependence of the maximum amplitude amplification A from the relative phase $\Delta\phi$ of colliding solitons. The black curve (1), the red curve (2) and the blue curve (3) correspond to the soliton wave steepnesses $\mu \sim 0.1$ ($C_0 = 2.25 \cdot 10^{-3}$), $\mu \sim 0.15$ ($C_0 = 2.8 \cdot 10^{-3}$) and $\mu \sim 0.2$ ($C_0 = 3.5 \cdot 10^{-3}$) respectively.

of the phase $\phi(t)$ of the breather (at the maximum of the modulus of the envelope) on t with the parameters Ω, V and the initial phase ϕ_0:

$$\phi(t) = \phi_0 - (\Omega + k_0 V - \omega_{k_0})t. \tag{18}$$

The colliding breathers have slightly different width, so their relative phase is not time invariant. In addition breathers acquire slightly different phase shifts $\delta\phi_1$ and $\delta\phi_2$ which cannot be simply calculated and accounted in $\Delta\phi$. Thus, we define the phase

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difference of the breathers at $t = \frac{\pi}{2U_0}$ (i.e. at the moment of time when breathers located at the same point):

$$\Delta\phi = (\phi_{02} - \phi_{01}) - \frac{\Omega_2 - \Omega_1}{2U_0}\pi + k_0\pi. \tag{19}$$

4.1 Breather collisions: amplitude amplification and energy loss.

Our numerical simulations show that the dependences $A(\Delta\phi)$ for the colliding breathers of small amplitude (and small $\mu \approx 0.1$) and for the colliding solitons are similar. The maximum of the amplitude amplification function is observed at $\Delta\phi \approx 0$ (similar to the NLS case), when we choose the phase difference in the form (19) – compare the black curves in figures 3 and 4. At larger values of the wave steepness the position of maximum of $A(\Delta\phi)$ is shifted from $\Delta\phi = 0$ more significantly – see the figure 4.

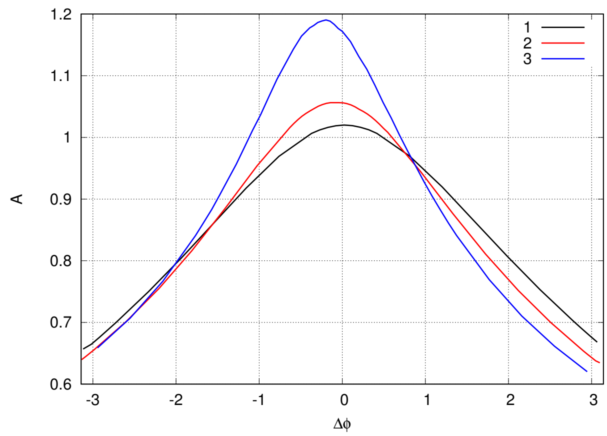

Figure 4. Dependence of the maximum amplitude amplification A from the relative phase $\Delta\phi$ of colliding breathers. The black curve (1), the red curve (2) and the blue curve (3) correspond to the soliton wave steepnesses $\mu \sim 0.1$ ($C_0 = 2.25 \cdot 10^{-3}$), $\mu \sim 0.15$ ($C_0 = 2.8 \cdot 10^{-3}$) and $\mu \sim 0.2$ ($C_0 = 3.5 \cdot 10^{-3}$) respectively.

We found that the maximum value of amplitude amplification $A(\Delta\phi)$ increases with the initial amplitude (and steepness) of the breathers and exceeds 1 by almost 20% for maximum value of the wave steepness studied in this work ($\mu \approx 0.2$) – see again the figure 4. The envelope profiles of colliding breathers are shown in the figures 5 and 6 for the values of the relative phase $\Delta\phi$ corresponding to the minimum and maximum amplification of $A(\Delta\phi)$.

The interactions of breathers are inelastic (Dyachenko et al. (2013)), which is manifested by radiation of incoherent waves as can be seen from the figures 5 and 6. In this work we have observed that level of the radiation is strongly dependent on the relative phase – compare the lower pictures in the figures 5 and 6.

In this work we quantitatively study the dependence of breather energy losses ΔE_{loss} on the relative phase $\Delta\phi$. We estimate the values of energy changes of each of the breathers $\delta E^{(1)}$ and $\delta E^{(2)}$ after collision. Here we denote the energy change for the breather initially located at $x = \pi/2$ as $\delta E^{(1)}$, while $\delta E^{(2)}$ corresponds to the energy change of the breather initially located at $x = 3\pi/2$. To estimate $\delta E^{(1)}$ and $\delta E^{(2)}$ we cut out each breather after collision by a window function centered at the maximum of the envelope module. The window function was chosen so that being applied to a single (i.e. non-interacting) breather gives

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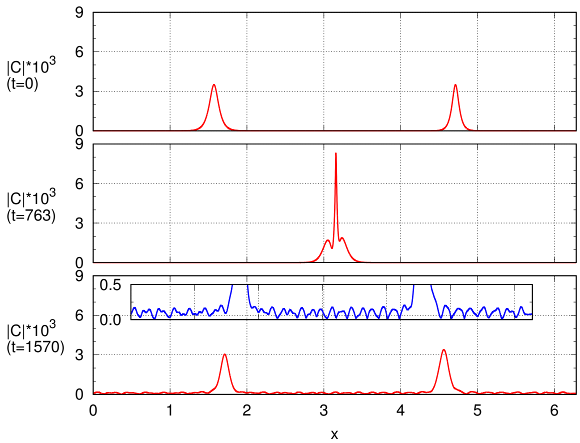

Figure 5. Snapshots of the envelope of colliding breathers for the phase difference $\Delta\phi \approx -0.2$ and the wave steepness $\mu \sim 0.2$. The upper picture shows the absolute value of the envelope $|C(x)|$ at the initial moment of time ($t = 0$); the middle picture corresponds to the moment of maximum amplitude amplification (at $t = 763$) and the lower picture shows the final state at $t = 1570$. Zoom of the final profile is shown by the blue curve.

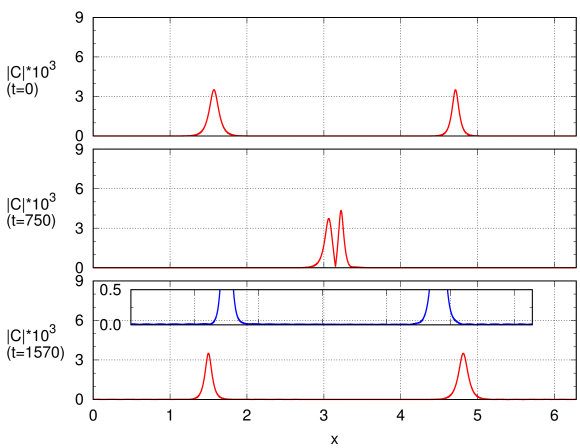

Figure 6. Snapshots of the envelope of colliding breathers for the phase difference $\Delta\phi \approx \pi$ and the wave steepness $\mu \sim 0.2$. The upper picture shows the absolute value of the envelope $|C(x)|$ at the initial moment of time ($t = 0$); the middle picture corresponds to the moment of maximum amplitude amplification (at $t = 750$) and the lower picture shows the final state at $t = 1570$. Zoom of the final profile is shown by the blue curve.

the value of the estimated energy which is different from the breather complete energy not more than 0.01%. We define the total energy losses as

$$\Delta E_{loss} = -(\delta E^{(1)} + \delta E^{(2)}). \tag{20}$$

The figure 7 shows the energy loss as a function of the relative phase $\Delta\phi$ for the steepness of the breathers $\mu \approx 0.2$. We have found that the value of the energy losses can reach $\approx 3\%$. As one can see from the figure 7, the positions of maximum amplitude amplification and maximum of energy losses are correlated.

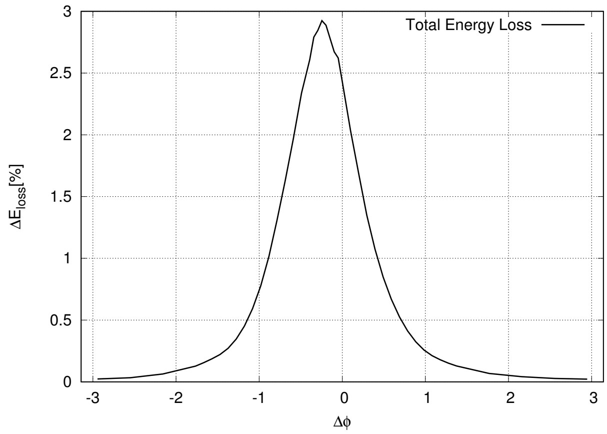

Figure 7. Energy loss ΔE_{loss} (in percent) in dependence on the phase difference $\Delta\phi$ for $\mu \approx 0.2$.

4.2 Breather collisions: space shifts and energy interchange.

In this paragraph we describe the individual changes of each breather after collision. First of all, we found that breathers exchange energy with each other. Each of breathers can gain or lose the energy after collision in dependence of the relative phase $\Delta\phi$ – see the figure 8. As one can see the maximum energy gain of the first breather is achieved at $\Delta\phi = -0.98$, while the maximum energy gain of the second breather is achieved at $\Delta\phi = 0.58$.

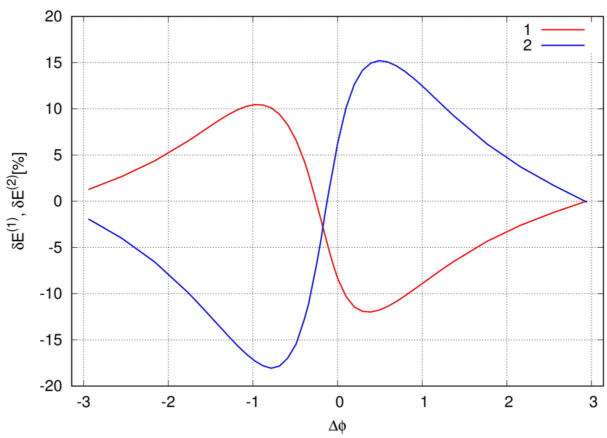

Figure 8. Energy change of each breather $\delta E^{(1)}$ (red curve 1) and $\delta E^{(2)}$ (blue curve 2) in dependence on the relative phase $\Delta\phi$ for $\mu \approx 0.2$.

The energy exchange results in the increase or decrease of the breather amplitudes, that is demonstrated by the figures 10 and 9. In these figures we show the envelope profiles of the breathers after the collision (at $t = 1570$) in comparison with non-interacting breathers at the same time. The figure 10 corresponds to the relative phase $\Delta\phi = -0.98$ and the figure 9 – to the relative phase $\Delta\phi = 0.58$.

In addition, the figures 9 and 10 demonstrate that the space positions of breathers after the interaction also depend on the relative phase $\Delta\phi$. We calculate the space shifts of the breathers $\delta x^{(1)}$ and $\delta x^{(2)}$ by the difference between the positions of interacting and the corresponding single breathers at the same time $t = 1570$. We demonstrate the dependence of $\delta x^{(1)}$ and $\delta x^{(2)}$ on the relative phase $\Delta\phi$ in figure 11. In contrast to the NLS model the space shifts of breathers can be either positive or negative.

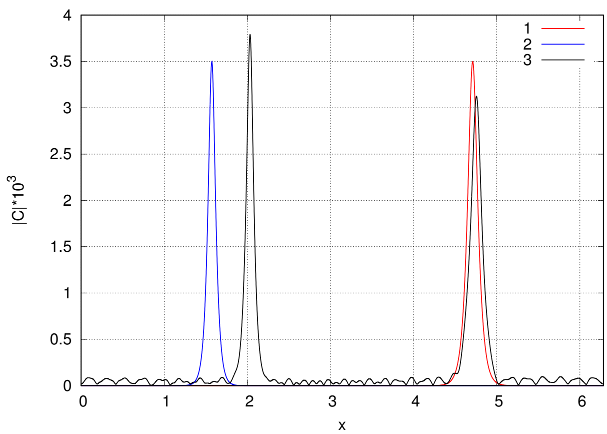

Figure 9. Modulus of the breather envelopes $|C(x)|$. The red curve (1) corresponds to the single breather with $U = 0.002$ at $t = 1570$ in the absence of collision. The blue curve (2) corresponds to the single breather with $U = -0.002$ at $t = 1570$ in the absence of collision. The black curve (3) shows the modulus of envelope $|C(x)|$ of breathers after collision at $t = 1570$ with the relative phase $\Delta\phi = 0.58$.

5 Conclusions

In this work we have studied how the relative phase of breathers affects the key properties of their interaction. Previously, Dyachenko et al. (2013) shown, that the collision of breathers in the compact Zakharov equation leads to the minor radiation of incoherent waves (inelasticity of the interaction). The findings demonstrate nonintegrability of this model (additionally Dyachenko et al. (2013) presented an analytical proof of the nonintegrability of Zakharov equation). Here we have studied the influence of the relative phase of the colliding breathers on the level of the radiation. We have found that the total energy loss due to the radiation is enhanced at large values of the wave steepness, that was expected for a nonintegrable model, and also at a certain synchronisation of the relative phase between breathers: $\Delta\phi \approx 0$. We explain the latter in the following way. The maximum amplitude amplification at $\Delta\phi \approx 0$ is accompanied also by the formation of the wave profile of high steepness. We

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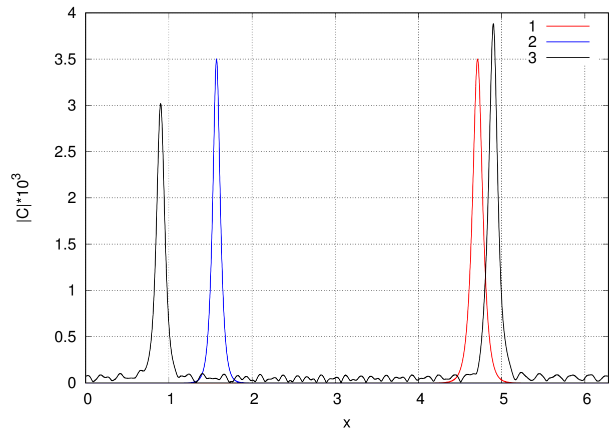

Figure 10. Modulus of the breather envelopes $|C(x)|$. The red curve (1) corresponds to the single breather with $U = 0.002$ at $t = 1570$ in the absence of collision. The blue curve (2) corresponds to the single breather with $U = -0.002$ at $t = 1570$ in the absence of collision. The black curve (3) shows the modulus of envelope $|C(x)|$ of breathers after collision at $t = 1570$ with the relative phase $\Delta\phi = -0.98$.

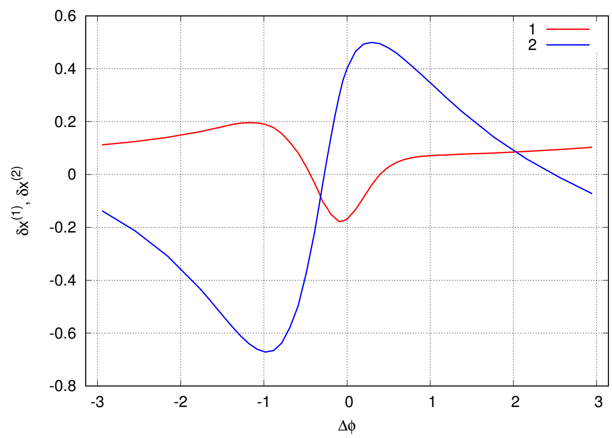

Figure 11. The dependence of the breather space shifts on the relative phase $\Delta\phi$ for $\mu \approx 0.2$. The red curve (1) corresponds to the breather with $U = 0.002$; the blue curve (2) corresponds to the breather with $U = -0.002$.

have found that the maximum steepness reaches the value $\mu \approx 0.7$ during the collision process and thus the deviation of wave dynamics from the integrable model becomes to be significant.

Interactions of the breathers in the compact Zakharov equation at a certain phase synchronization can lead to the formation of extreme amplitude waves. It is well known, that the maximum value of the amplitude amplification as a result of soliton interactions in the NLS model is equal to the sum of the soliton amplitudes. In this work we have found, that in the compact Zakharov equation the maximum amplification can be higher than the sum of amplitudes of the breathers. Interestingly, at large values of the wave field steepness this effect is enhanced, that can be a valuable complement in extreme amplitude waves studies.

We also have studied the phenomena of the energy exchange between the colliding breathers. This energy exchange is caused by inelasticity of the breathers interaction. The universal long term consequences of this process was studied in different nonintegrable models (Krylov and Iankov (1980); Dyachenko et al. (1989)). It was shown that the numerous collisions and interactions with waves of radiation leads to formation of the powerful single solitary type wave (see the review by Zakharov and Kuznetsov (2012)). Here we have found that dynamics of a single collision is not universal: the direction of energy swap is determined by the breathers phases.

Furthermore, we have studied space shifts that breathers acquire after the collision. Soliton of the NLS equation always acquire a positive constant shift δx to its space position after interaction with other soliton moving with different velocity. The value of δx is defined only by the amplitudes and velocities of the colliding solitons. The interaction of breathers in the compact Zakharov equation also leads to the appearance of the space shifts. We show that the character of this effect is not universal (δx can be positive or negative) and is determined by the breather phases.

The interactions of solitary-type wave structures in the nonintegrable models are needing further research. The complexity of the studies caused by the absence of exact N-soliton solution formulas, and also the inelasticity of the interaction that is able to destroy the initially coherent wave groups. However, as we have demonstrated here the total energy loss for interactions describing by the equation (1) does not exceed a few percent of energy of the breathers and we expect that observation of several subsequent breather collisions is possible. The study of the influence of the relative phase of the colliding breathers in the fully nonlinear model is of fundamental interest. As was shown by Dyachenko et al. (2016b), the compact Zakharov equation quantitatively describes strongly nonlinear phenomena at the surface of deep fluid. Thus we believe that the effects reported here for the breathers of the compact Zakharov equation can be also observed for the fully nonlinear Euler equations.

Pairwise collisions of breathers is an important elementary process that can be observed in the wave dynamics of arbitrary disturbed fluid surface. For example, the recent numerical simulations of the compact Zakharov equation demonstrate that an ensemble of interacting breathers can appear as a result of modulation instability driven by random perturbations of an unstable plane wave (Dyachenko et al. (2017a)). We believe that results presented here can serve as a starting point in analytical description of such processes. Moreover, the reported dependence of breather interaction dynamics on the relative phase is to be verified in laboratory experiments.

Competing interests. We declare that no competing interests are present.

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