Parametric covariance dynamics for the nonlinear diffusive Burgers equation

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Referee report by S. E. Cohn

This interesting paper considers the tangent linear covariance dynamics of the (nonlinear, diffusive) Burgers equation under stochastic initial conditions. The tangent linear covariance dynamics can be expressed as a series expansion in the distance between any two points in the spatial domain. To terminate the expansion at finite order requires a closure approximation due to the diffusion term in Burgers' equation. The authors develop such an approximation through a careful examination of properties of the two-point correlation function. Their proposed approximation, appropriately termed a *locally homogeneous Gaussian closure*, terminates the series expansion at second order by approximating the fourth-order term in terms of the second-order one. This results in a system of three nonlinear, nonlinearly coupled, time-dependent PDEs to be solved on the spatial domain, for the mean, the variance, and the second-order term of the correlation function expansion. The authors carry out numerical experiments to test their approach against a Monte Carlo approach used as a basis of comparison.

The appeal of the approach considered in this paper, beyond increased scientific understanding of covariance evolution, is to dramatically reduce computational costs: one needs to solve only three PDEs instead of carrying out, for instance, the tens, hundreds, thousands or more integrations of the original (Burgers equation) dynamics with Monte Carlo approaches. The authors discuss the potential of their approach for the important applications of data assimilation and probabilistic forecasting.

The experimental results shown in the paper are convincing but also somewhat unsettling. They show that, for small initial variance (1% standard deviation), the closure approach accurately reproduces the Monte Carlo results, thus validating in this case both the closure approximation and the tangent linear covariance dynamics. For much larger initial variance (10% standard deviation), however, the closure approach reproduces the Monte Carlo results only away from the region where the gradient of the mean state steepens. Through further numerical experiments, the authors demonstrate that it is likely that it is the use of the tangent linear covariance dynamics themselves, to which the closure approximation is applied, rather than the closure approximation per se, that causes the fairly substantial difference between results of their approach and the Monte Carlo simulations in the vicinity of sharp gradients. An obvious conclusion is that further research will be needed to investigate the limitations of tangent linear covariance dynamics vis-à-vis fully nonlinear covariance dynamics, such as those obtained from Monte Carlo methods.

I did carefully check all the mathematical derivations in the paper, including those in the Appendix, and I found two, both of which could affect the experimental results. First, there is an error passing from Eq. (27b) to Eq. (28b): a coefficient 2 has crept into the third-from-last term on the right side of Eq. (28b) which does not belong there, and it is repeated in the final Eq. (29c). If this is just a typo, it simply needs to be corrected. But if this error has also made its way into the computer code, then the numerical experiments will need to be re-run.

Second, the Gaussian initial covariance function, Eq. (30), is not appropriate for the geometry of the numerical experiments. Since the domain is periodic, the distance |x-y| should be replaced by a distance function that reflects this periodicity, such as the great-circle distance or chordal distance. As it stands, the covariance function has a slight first-derivative discontinuity at |x-y| = D/2, and this introduces spurious odd-order terms in the series expansion of the correlation function that have been neglected. Although this might be a small effect initially, since the initial correlation length L was taken to be small, the numerical experiments showed that the correlation length grows by nearly an order of magnitude. The numerical experiments will need to be repeated with an appropriate initial covariance model.

Here are a few smaller issues, including typos:

1. It is mentioned a few times that "operator splitting" (p.2 l.25) or "time splitting" (p.6 l.26) is used in the derivation. Actually, the authors are simply carrying out the derivation term-by-term without any approximation introduced by doing so. The authors' use of this terminology is not at all standard; usually it means that a time discretization error is introduced in a numerical method. I suggest removing the terminology altogether.
2. P.7 l.13: I would change "Hilbert space" to the more general "function space" since no Hilbert space apparatus has been introduced in the paper.
3. In the title itself, the apostrophe after Burgers should be removed: the possessive is not correct here.
4. P.4 l.16: recipes → recipe
5. P.6 l.12: te → the
6. Eq. (25): One appearance of δx^2 in the second term, and one in the third term, should be removed.
7. P.10 l.8: express → expressed
8. Eq. (29): The subscript x on the symbol V should be used consistently.
9. P.13 l.33: 8.8 of → 8.8 times
10. P.14 l.9: 5.5 of → 5.5 times
11. P.14 l.12: 7.5 of → 7.5 times
12. P.18 l.13: variance field → normalized variance field
13. P.18 l.21: third term → third order term
14. P.19 l.1: fourth term → fourth order term
15. P.19 l.8: third order term → fourth order term