Comparison of stochastic parameterizations in the framework of a coupled ocean-atmosphere model – Response to the 3rd reviewer

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We thank the reviewer for his/her comment. We have taken the liberty of breaking down the few points he/she adresses as follow:

1.1 At the beginning of Section 3.1.2 the authors state that "These methods are applicable for parameterization purposes if the problem can be cast into a backward Kolmogorov equation." This is not a correct statement: Every dynamical system (deterministic or stochastic) can be cast into a backward Kolmogorov equation. Casting a nonlinear dynamical system into the linear backward Kolmogorov equation (or alternatively the linear Fokker-Planck equation) allows for the machinery of perturbation theory of linear systems to derive explicit formulae for the homogenized equations. The authors may want to reference here the monograph "Multiscale Methods: Averaging and Homogenization" by Pavliotis and Stuart.

Answer: We agree with the referee and have modified the statement as:

> *These methods are applicable for parameterization purposes if the problem can be cast into a linear backward Kolmogorov equation (Pavliotis and Stuart, 2008).*

1.2 Furthermore, the authors provide references for the mathematical justification of MTV (including Papanicolaou). These are references if the underlying dynamical system is stochastic; for the applicability of the MTV procedure for deterministic systems, the references are Melbourne and Stuart (2011), Nonlinearity 24, pp 1361, Gottwald and Melbourne (2013), Proc. Roy. Soc. A 469, pp 2013020 and Kelly and Melbourne (2017), Journal of Functional Analysis 10, pp 4063.

Answer: We agree and have added the references:

> *This approach is based on the singular perturbation methods that were developed for the analysis of the linear Boltzmann equation in an asymptotic limit (Grad, 1969; Ellis and Pinsky, 1975; Papanicolaou, 1976; Majda et al., 2001) and it has been applied to deterministic systems as well (Melbourne and Stuart, 2011; Kelly and Melbourne, 2017).*

1.3 The choice of notation with ρ for the backward Kolmogorov equation which propagates expectation values, is odd. ρ is usually reserved for densities. I also found the backward in time integration with the expectation value being defined at the final time cumbersome; why not have a positive sign on the left hand side of (25) and (26), and have the expectation value equal to $f(X)$ at $t = 0$? Also, the generator $\mathcal{L}$ is not defined when it is introduced.

Answer: Please see our answer to the 1st referee. For the choice of the backward integration, we align with the MTV derivation in Majda et al. (2001). We have modified the first sentence where the operators appear as:

In this setting, the backward Kolmogorov equation reads (Majda et al., 2001):

$$-\frac{\partial \rho^\delta}{\partial s} = \left[\frac{1}{\delta^2}\mathcal{L}_1 + \frac{1}{\delta}\mathcal{L}_2 + \mathcal{L}_3\right]\rho^\delta$$

where the probability density $\rho^\delta(s, \boldsymbol{X}, \boldsymbol{Y}|t)$ is defined with the final value problem $f(\boldsymbol{X})$: $\rho^\delta(t, \boldsymbol{X}, \boldsymbol{Y}|t) = f(\boldsymbol{X})$. The density ρ^δ can be expanded in term of δ and inserted in Eq. (25). The zeroth order of this equation ρ^0 can be shown to be independent of $\boldsymbol{Y}$ and its evolution given by a closed, averaged backward Kolmogorov equation (Kurtz, 1973):

$$-\frac{\partial \rho^0}{\partial s} = \bar{\mathcal{L}}\rho^0 \, .$$

The precise definition of the operators $\mathcal{L}_i$ and $\bar{\mathcal{L}}$ acting on the densities is given in Appendix A.

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2. In the case when the unresolved scales consist of the wavenumber 2 atmospheric variables, the authors found that the WL approach leads to unstable dynamics. I am surprised by that. The authors relate this to a cubic form. Can the authors comment on which assumption of WL is being violated by this choice of unresolved variables? Are higher-order corrections needed?

Answer:

The weak-coupling assumption is violated in the atmosphere when ε is set to 1 as we have done for every experiment in the article. This might be a reason and a simple way (but computationnaly expensive) to check this is to repeat the experiment with smaller values of ε until the instability disappears. Higher-order corrections might solve the problem but are quite tedious to compute. What we find the most intriguing is the fact that the MTV method is stable, but not the WL one, leading us to suspect that the method is sensitive to the accuracy of the memory term M_3 integration.

3. There are a few typos in the text:

Eqn (24): missing full stop.

Above (42): let -> left

Answer:

Thanks.

We thank again the reviewer for his/her suggestions.

References

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