Wave propagation in the Lorenz–96 model

Response to the Referees

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We would like to thank the two Referees for their thoughtful comments and constructive criticism which have helped us to improve our paper. Below we give our point-by-point response to their comments (in blue) and explain how we have incorporated their suggestions in the revised manuscript.

Response to Referee 1

1. **[...] a quibble with the use of the term "standing wave" to describe the stationary wave-like structures found for $F < 0$ and $n = 4k + 2$ [...]**

   We agree with the Referee that our use of the terminology "standing wave" is not justified, and we thank the Referee for pointing out this misnomer. In the revised paper we have followed the suggestion of the Referee and replaced "standing wave" with "stationary wave" throughout the text.

2. **[...] the authors could also make mention of other similar systems exhibiting double-Hopf bifurcation to multiple traveling wave solutions [...]**

   In the revised paper we end section 4 with a paragraph in which we give some examples of double-Hopf bifurcations in fluid dynamical models. In addition to the work of Moroz and Holmes (1984) that was suggested by the Referee we have also included the more recent works by Avila (2006), Lewis (2010), Lewis and Nagata (2003, 2005), and Marqués et al. (2002, 2003).

3. **P.2 line 15: As well as self-referencing Sterk at al. (2010), you could also mention other earlier studies that also identified Hopf bifurcations associated with the onset of low-frequency variability in atmospheres, oceans or laboratory experiments [...]**

   In the fourth paragraph of the introduction section we have included references to Simonnet et al. (2003a,b), Te Raa and Dijkstra (2002), Dijkstra et al. (2008), Frankcombe et al. (2009), Read et al. (1992), and Tian et al. (2001).

4. **P.2 line 21: word missing after "which" - "was"?**

   We have rewritten this part as "[...] which considers only the classical case [...]"

5. **P.2 line 24-27: commas recommended after "bifurcation", "that", "$F > 0$", "n", "that" and "$F < 0$" [...]**

   This has been corrected as suggested.

6. **P.3 Fig 1 Are the braided striations in the Hovmöller plots real or an artifact of the plotting?**

   In order to obtain a continuous diagram in the (j, t)-plane we have applied linear interpolation between the values x_j and x_{j+1} (see the accompanying caption). Within the red and blue bands one can see "streaks" of dark red and dark blue, which are indeed artifacts of our linear interpolation procedure. These streaks are precisely located at the j-values where x_j is a local maximum or minimum (for fixed t). At such points the linear interpolation of the x_j's is non-differentiable in j, and hence there is a large difference in gradient around either side of such points. Alternatively, one could use an interpolation method based on smooth functions, but we do not feel that this would add much to our plots.

7. **P.9 line 5 "associated with" not "to"**

   This has been corrected as suggested.

8. **P.22 line 20 Page numbers in reference Frank et al. 2014 seems to be incorrect [...]**

   The page numbers in the reference to Frank et al. (2014) were indeed incorrect. The paper is 14 pages in length, but instead of page numbers we should have included the article number 1430027 in our BibTeX file. We have corrected this, and we thank the Referee for noting the mistake.

Response to Referee 2

1. **The authors should provide a more comprehensive description of the Lorenz96 model and its geophysical interpretation.**

   The famous Lorenz-63 model is a Galerkin projection of a fluid dynamical model describing Rayleigh-Bénard convection and hence has a clear physical interpretation. In contrast, the Lorenz-96 model does not have such an interpretation. In fact, Lorenz (2006) writes:

   > "The physics of the atmosphere is present only to the extent that there are external forcing and internal dissipation, simulated by the constant and linear terms, while the quadratic terms, simulating advection, together conserve the total energy [...]"

   The interpretation of Lorenz is that the variables "[...] may be thought of as values of some atmospheric quantity in K [n in our paper] sectors of a latitude circle." This is also the interpretation we have adopted in our paper. In the paper "Regimes in simple systems" (Journal of the Atmospheric Sciences, 63, 2006, pp. 2056–2073) Lorenz writes:

> "The variations of X_n [x_j in our paper] are intended to simulate the behavior of some atmospheric quantity at N [n in our paper] equally spaced grid points about a latitude circle, but to my knowledge, the system cannot be obtained by truncating any realistic atmospheric model."

Since the Lorenz-96 model was not derived from physical principles the sign of the forcing parameter F has no physical interpretation either. The Hovmöller diagrams of the Lorenz-96 model reveal both travelling waves and stationary waves which have also been observed in physical models, such as the low-order shallow water model studied in Sterk et al. (2010). However, we do not think that it is possible to provide any stronger link with "reality" than that.

After the first paragraph in the introduction section we included a new paragraph which aims to clarify why the the Lorenz-96 model is useful despite the lack of a clear physical interpretation.

Does a larger dimension n mean a finer latitude grid or a bigger planet; what is the interpretation of the limiting wave period?

The parameter n could indeed be interpreted as the resolution of discretization with a larger n implying a finer grid. In fact, some authors interpret the Lorenz-96 model as a discretized partial differential equation; see, for example, Reich & Cotter, "Probabilistic Forecasting and Bayesian Data Assimilation", Cambridge University Press, 2015. However, our results show that spatiotemporal properties of waves do not always converge as $n \to \infty$. The lack of convergence of the dynamics with n could be caused by the fact that the coefficients of all terms in the model are 1. We consider it to be an interesting mathematical problem to investigate whether the coefficients of the model can be changed in such a way that dynamical properties do converge as $n \to \infty$. However, we also feel that this question is beyond the scope of the present manuscript.

Our results imply that the parameters n and F should be chosen carefully when using the Lorenz-96 model for testing purposes. In Sterk and Van Kekem (2017) we have shown that the predictability of extreme events in the Lorenz-96 model strongly depends on n and F. In the revised manuscript we have made this more explicit.

2. **The paper seems to provide very little beyond a previous preprint of the authors (arxiv-preprint 1704.05442). In fact, only part of the manuscript considering negative forcing seems to contain unpublished material. Given that the authors' results are much less comprehensive in this situation and that this case, so far, lacks justification, the question arises whether the manuscript provides enough material to justify publication in NPG. The authors should provide clear guidance regarding what is new in this manuscript and what is not, and justify why the new material should be of interest to the general NPG readership.**

Note that there are three versions of the manuscript arXiv:1704.05442. The latest version (version 3, hereafter referred to as V3) dates from December 20, 2017 and will be published in Physica D. We will remove versions 1 and 2 in due time in order to prevent confusion.

V3 is aimed at a mathematical audience and contains mostly technical details, including proofs, normal form computations, and routes to chaos. In contrast, the present manuscript is aimed at researchers who use the Lorenz-96 model in their work. We agree with the Referee that there is some overlap with the present manuscript and V3, namely:

- the eigenvalues of the equilibrium $(F, \dots, F)$;

- theorem 1, which is a concise summary of theorems in V3;

- the bounds on the wave number for $F > 0$ and the limiting wave period.

However, we feel that this overlap is needed in order to (1) give a complete and co-herent overview of waves in the Lorenz-96 model and (2) to emphasize the difference between the cases $F > 0$ and $F < 0$. In particular, for $F < 0$ the spatiotemporal properties of waves in the Lorenz-96 model depend on the remainder of n upon division by 4. Moreover, for $F < 0$ the spatial pattern of the wave is determined by the structure of the equilibrium solution that is born through one or two pitchfork bifurcations. To our knowledge this has not been discussed in the literature before. Also note that in previous work we have not discussed traveling waves for $F < 0$.

We are not sure what the Referee means with the remark that our results are "less comprehensive" for $F < 0$. Does it mean a lack of rigorous proof? For $F > 0$ and $F < 0$ with n odd we were able to analytically compute the spatiotemporal properties of the waves. However, our example for $n = 4$ shows that this is not always possible.

The double-Hopf bifurcation for $n = 12$ also appears in V3, but the discussion in V3 is technical and focuses on detailed normal form computations. In the current manuscript we show that the number double-Hopf bifurcations grows quadratically with n. In addition, we show that for many dimensions of the 2-parameter model there is a pair of double-Hopf bifurcations close to the F-axis which causes the co-existence of *three* stable waves. This phenomenon has not been discussed in V3.

We believe that our manuscript is sufficiently interesting to the readers of NPG. Among the readers of NPG there are researchers who use the Lorenz-96 model in their work. For example, see the references Basnarkov and Kocarev (2012), Lucarini and Sarno (2011), Orrell et al. (2003), Sterk et al. (2012), and Trevisan and Palatella (2011). Table 1 shows that many researchers stick with the canonical choices $n = 36$ and $n = 40$. As argued above, these choices can influence the results of numerical experiments. We hope that by giving an coherent overview of the dynamics for various values of n and both $F > 0$ and $F < 0$ users of the Lorenz-96 model will be able to make appropriate parameter choices that suit their purposes.

3. **The paragraph starting pg.15, l.2 is pretty unclear. It seems to be important though for a major conclusion of the manuscript. The main problem seems to me that you do not define the Neimark-Sacker bifurcation. In fact, the standard definition of the Neimark-Sacker bifurcation pertains to discrete-time systems only. Please clarify. Further, the definition of "multi-stability lobe" is unclear, and what is the evidence that the two mentioned curves indeed bound such a region?**

We fully agree with the Referee that our explanation was not sufficiently detailed.

In the revised manuscript we explain what Neĭmark-Sacker bifurcations in discrete-time and continuous-time systems are. We also explain how these bifurcations lead to coexistence of periodic orbits. To that end we describe the successive bifurcations that occur while keeping $G = 0.1$ fixed and letting F increase. This scenario is not limited to the Lorenz-96 model. In fact, it occurs in any system with a double-Hopf bifurcation of type I, which follows from the normal form analysis in Kuznetsov (2004).

4. **Please decide whether you want to write "Figure (No)" or abbreviate as 'Fig. (No)'.**

The section *Manuscript preparation guidelines for authors* on the website of NPG explicitly states the following:

> The abbreviation "Fig." should be used when it appears in running text and should be followed by a number unless it comes at the beginning of a sentence, e.g.: "The results are depicted in Fig. 5. Figure 9 reveals that...".

We have checked carefully that we have followed this guideline in a consistent manner.

5. **pg.2, l.21, replace "which restricted to" with "which considers only" or similar.**

We have rewritten this part as "[...] which considers only the classical case [...]"

6. **pg.2, l.21, replace and "thereby we" with "; these two manuscripts together" or similar.**

We have rewritten this part as "; these two manuscripts together give a comprehensive picture of wave propagation in the Lorenz-96 model."

7. **I think the appropriate technical term is Jacobi matrix.**

In the languages like German or Dutch the $m \times n$ matrix with first-order partial derivatives of a map $F : \mathbb{R}^n \to \mathbb{R}^m$ is referred to as "Jacobi matrix". However, we have consulted various English text books on mathematics and in those texts the term "Jacobian matrix" is used. Hence, we have not changed the terminology.

8. **pg.4, l.2 mention that "equilibrium solution" and "steady flow" mean time-independent solutions.**

We have rewritten this sentence as follows:

> "Assume that for the parameter value μ_0 the system has an equilibrium $\vec{x}_0$; this means that $\vec{f}(\vec{x}_0, \mu_0) = \vec{0}$ and hence $\vec{x}_0$ is a time-independent solution of eq. (2)."

9. **pg.4, l.8 should be "genericity conditions".**

The word "nongenericity conditions" was indeed a slip of the pen. We have changed it into "non-degeneracy conditions" which is the appropriate technical term in the literature on bifurcation theory.

10. **pg.4, l.9 you never actually define what a Hopf bifurcation is.**

We agree with the Referee that the flow of this paragraph is not completely logical. This indeed leads to using the word "Hopf bifurcation" without explaining it.

In the revised manuscript we have rewritten the starting sentences of this paragraph as follows:

> Assume that $D\vec{f}(\vec{x}_0, \mu_0)$ has two eigenvalues $\pm\omega i$ on the imaginary axis. This indicates the occurrence of a Hopf bifurcation, i.e. the birth of a periodic solution from an equilibrium that becomes unstable.

Wave propagation in the Lorenz-96 model

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Abstract. In this paper we study the spatiotemporal properties of waves in the Lorenz-96 model and their dependence on the dimension parameter n and the forcing parameter F. For $F > 0$ the first bifurcation is either a supercritical Hopf or a double-Hopf bifurcation and the periodic attractor born at these bifurcations represents a traveling wave. Its spatial wave number increases linearly with n, but its period tends to a finite limit as $n \to \infty$. For $F < 0$ and odd n the first bifurcation is again a supercritical Hopf bifurcation, but in this case the period of the traveling wave also grows linearly with n. For $F < 0$ and even n, however, a Hopf bifurcation is preceded by either one or two pitchfork bifurcations, where the number of the latter bifurcations depends on whether n has remainder 2 or 0 upon division by 4. This bifurcation sequence leads to ~~standing~~ stationary waves and their spatiotemporal properties also depend on the remainder after dividing n by 4. Finally, we explain how the double-Hopf bifurcation can generate two or more stable waves with different spatiotemporal properties that coexist for the same parameter values n and F.

Copyright statement.

1 Introduction

In this paper we study the Lorenz-96 model which is defined by the equations

$$\frac{dx_j}{dt} = x_{j-1}(x_{j+1} - x_{j-2}) - x_j + F, \quad j = 0, \dots, n-1, \tag{1}$$

together with the periodic "boundary condition" implied by taking the indices j modulo n. The dimension $n \in \mathbb{N}$ and the forcing parameter $F \in \mathbb{R}$ are free parameters. ~~The model was introduced by~~ Lorenz (2006) ~~for numerical experiments in predictability studies. In his paper, Lorenz~~ interpreted the variables x_j as values of some atmospheric quantity in n equispaced sectors of a latitude circle, where the index j plays the role of "longitude". Hence, a larger value of n can be interpreted as a finer latitude grid. Lorenz also remarked that the vectors $(x_0, \dots, x_{n-1})$ can be interpreted as wave profiles, and he observed that for $F > 0$ sufficiently large these waves slowly propagate "westward", i.e. in the direction of decreasing j. Figure 1 shows a Hovmöller diagram illustrating two traveling waves with wave number 5 for dimension $n = 24$ and the parameter values $F = 2.75$ (in the periodic regime) and $F = 3.85$ (in the chaotic regime).

The Lorenz-96 model was introduced as a tool for numerical experiments in predictability studies, rather than as a physically realistic model. Indeed, Lorenz (2006) wrote that "The physics of the atmosphere is present only to the extent that there are external forcing and internal dissipation, simulated by the constant and linear terms, while the quadratic terms, simulating advection, together conserve the total energy [...]" The value of the Lorenz-96 model primarily lies in the fact that it has a very

5 simple implementation in numerical codes while at the same time it can exhibit very complex dynamics for suitable choices of the parameters n and F. The famous Lorenz-63 model (Lorenz, 1963), which *does* have a clear physical interpretation (namely, as the Galerkin projection of a fluid dynamical model describing Rayleigh-Bénard convection) has two disadvantages. Firstly, it consists of only three ordinary differential equations. Secondly, for the classical parameter values the model has Lyapunov spectrum $(0.91, 0, -14.57)$, which makes the model very dissipative. Such properties are not typical for atmospheric models.

10 In contrast, the dimension of the Lorenz-96 model can be chosen arbitrarily large, and for suitable values of the parameters the Lyapunov spectrum is similar to those observed in models obtained from discretizing partial differential equations. For those reasons the Lorenz-96 model has become a test model for a wide range of geophysical applications.

Table 1 lists some recent papers with applications of the Lorenz-96 model. In most studies the dimension n is chosen ad hoc, but $n = 36$ and $n = 40$ appear to be popular choices. Many applications are related to geophysical problems, but the model

15 has also attracted the attention of mathematicians working in the area of dynamical systems for phenomenological studies in high-dimensional chaos. Note that Eq. (1) is in fact a family of models parameterized by means of the discrete parameter n. An important question is to what extent both the qualitative and quantitative dynamical properties of Eq. (1) depend on n. ~~Answers to these questions can be used in selecting the most appropriate values of~~ For example, the dimension n ~~and F in specific applications. For example~~ has a strong effect on the predictability of large amplitudes of traveling waves in weakly chaotic

20 regimes of the Lorenz-96 model (Sterk and Van Kekem, 2017). In general, the statistics ~~and predictability~~ of extreme events ~~can depend very much on the dynamical regime of a model~~ in dynamical systems strongly depend on topological properties and recurrence properties of the system (Holland et al., 2012, 2016). Therefore, a coherent overview of the dependence of spatiotemporal properties on the parameters n and F is useful to assess the robustness of results when using the Lorenz-96 model in predictability studies.

25 In this paper we address the question how the spatiotemporal properties of waves, such as their period and wave number, in the Lorenz-96 model depend on the dimension n and whether these properties tend to a finite limit as $n \to \infty$. We will approach this question by studying waves represented by periodic attractors that arise through a Hopf bifurcation of a stable equilibrium. Along various routes to chaos these periodic attractors can bifurcate into chaotic attractors representing irregular waves which "inherit" their spatiotemporal properties from the periodic attractor. For example, the wave shown in the left

30 panel of Fig. 1 bifurcates into a 3-torus attractor which breaks down and gives rise to the wave in the right panel. Note that both waves have the same wave number. Figure 2 shows power spectra of these waves, and clearly their dominant peaks are located at roughly the same period. Inheritance of spatiotemporal properties also manifests itself in a shallow water model studied by Sterk et al. (2010) in which a Hopf bifurcation (related to baroclinic instability) explains the observed time scales of atmospheric low-frequency variability. The Hopf bifurcation plays a key role in explaining the physics of low-frequency variability

35 in many geophysical contexts. Examples are the Atlantic Multidecadal Oscillation (Te Raa and Dijkstra, 2002; Dijkstra et al.,

2008; Frankcombe et al., 2009), the wind-driven ocean circulation (Simonnet et al., 2003a, b), and laboratory experiments (Read et al., 1992; Tian et al., 2001).

In addition to *traveling* waves, such as illustrated in Fig. 1, we will also show the existence of *stationary* waves. In a recent paper by Frank et al. (2014) stationary waves have also been discovered in specific regions of the *multi-scale* Lorenz-96 model. Their paper uses dynamical indicators such as the Lyapunov dimension to identify the parameter regimes with stationary waves. Moreover, we will explain two bifurcation scenarios by which waves with different spatiotemporal properties coexist. This paper complements the results of our previous work (Van Kekem and Sterk, 2018) which considers only the case $F > 0$; these two manuscripts together give a comprehensive picture of wave propagation in the Lorenz-96 model.

The remainder of this paper is organized as follows. In Sect. 2 we explain how to obtain an approximation of the periodic attractor born at a Hopf bifurcation, which enables us to derive spatiotemporal properties of the waves in the Lorenz-96 model. In Sect. 3.1 we show that, for $F > 0$, periodic attractors indeed represent traveling waves as suggested by Lorenz. Also for $F < 0$ and odd values of n, periodic attractors represent traveling waves, as is demonstrated in Sect. 3.2. In Sect. 3.3, however, we show analytically that, for $n = 6$ and $F < 0$, *stationary* waves occur. By means of numerical experiments we show in Sect. 3.4 that stationary waves occur in general for even n and $F < 0$. In Sect. 4 we discuss the bifurcation scenarios by which stable waves with different spatiotemporal properties can coexist for the same values of the parameters n and F.

2 Hopf bifurcations

In this section we consider a general geophysical model in the form of a system of ordinary differential equations:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}, \mu), \qquad \boldsymbol{x} \in \mathbb{R}^n. \tag{2}$$

In this equation, $\mu \in \mathbb{R}$ is a parameter modeling external circumstances such as forcing. Assume that for the parameter value μ_0 the system has an equilibrium $\boldsymbol{x}_0$; this means that $\boldsymbol{f}(\boldsymbol{x}_0, \mu_0) = \boldsymbol{0}$ and hence $\boldsymbol{x}_0$ is a time-independent solution of eq. (2). In the context of geophysics $\boldsymbol{x}_0$ represents a steady flow, and its linear stability is determined by the eigenvalues of the Jacobian matrix $D\boldsymbol{f}(\boldsymbol{x}_0, \mu_0)$. An equilibrium becomes unstable when eigenvalues of the Jacobian matrix cross the imaginary axis upon variation of the parameter μ. Dijkstra (2005) provides an extensive discussion of the physical interpretation of bifurcation behaviour.

Assume that $D\boldsymbol{f}(\boldsymbol{x}_0, \mu_0)$ has two eigenvalues $\pm \omega i$ on the imaginary axis. This indicates the occurrence of a Hopf bifurcation, i.e. the birth of a periodic solution from an equilibrium that changes stability. If the equilibrium $\boldsymbol{x}_0$ is stable for $\mu < \mu_0$ and unstable for $\mu > \mu_0$, then under suitable non-degeneracy conditions the Hopf bifurcation is supercritical, which means that a stable periodic orbit exists for $\mu > \mu_0$ (Kuznetsov, 2004). For small values of $\varepsilon = \sqrt{\mu - \mu_0}$ the periodic orbit that
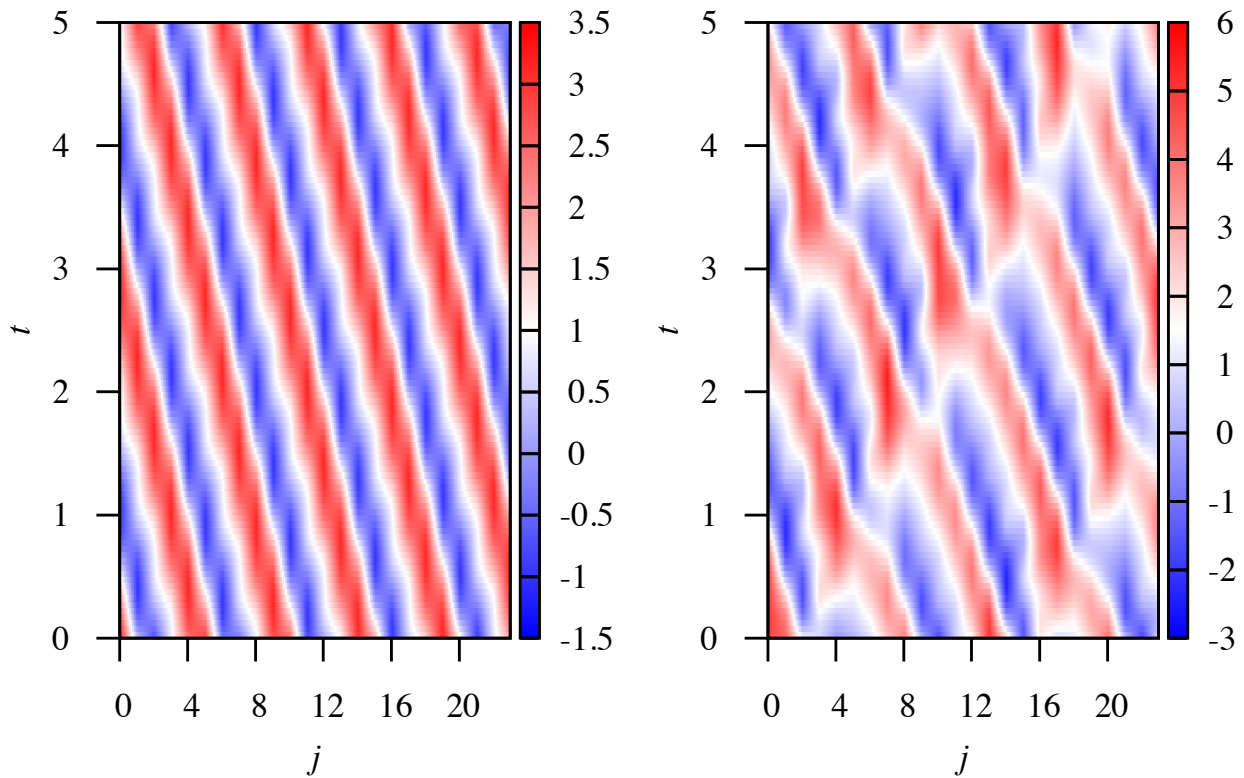
Figure 1. Hovmöller diagrams of a periodic attractor (left, $F = 2.75$) and a chaotic attractor (right, $F = 3.85$) in the Lorenz-96 model for $n = 24$. The value of $x_j(t)$ is plotted as a function of t and j. For visualization purposes linear interpolation between x_j and x_{j+1} has been applied in order to make the diagram continuous in the variable j.
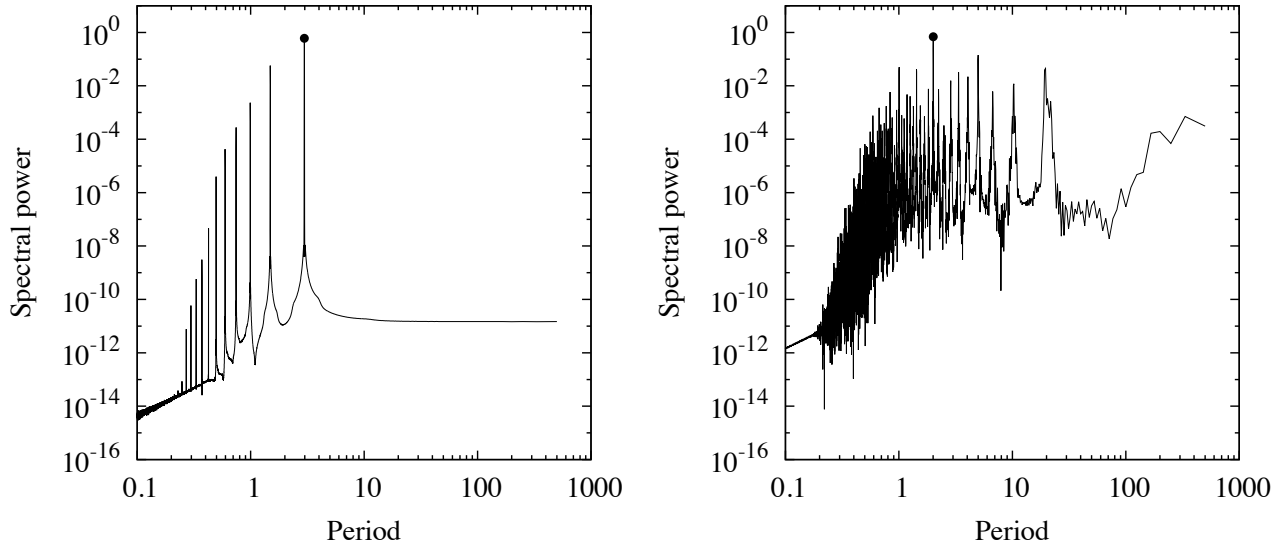
Figure 2. Power spectra of the attractors of Fig. 1. Note that the maximum spectral power (indicated by a circle) is attained at nearly the same period.

is born at the Hopf bifurcation can be approximated by

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \varepsilon \operatorname{Re}\left[(\boldsymbol{u} + i\boldsymbol{v})e^{i\omega t}\right] + O(\varepsilon^2), \tag{3}$$

see Beyn et al. (2002). Without loss of generality we may assume that the corresponding complex eigenvectors $\boldsymbol{u} \pm i\boldsymbol{v}$ of the matrix $D\boldsymbol{f}(\boldsymbol{x}_0, \mu_0)$ have unit length. In the context of geophysical applications this first-order approximation of the periodic

5 orbit can be interpreted as a wave-like perturbation imposed on a steady mean flow. The spatiotemporal properties of this wave can now be determined by the vectors $\boldsymbol{x}_0, \boldsymbol{u}, \boldsymbol{v}$ and the frequency ω.

3 Waves in the Lorenz-96 model

In this section we study waves in the Lorenz-96 model and how their spatiotemporal characteristics depend on the parameters n and F.

10 ### 3.1 Traveling waves for $n \geq 4$ and $F > 0$

For all $n \in \mathbb{N}$ and $F \in \mathbb{R}$ the point $\boldsymbol{x}_F = (F, \ldots, F)$ is an equilibrium solution of Eq. (1). This equilibrium represents a steady flow, and since all components are equal the flow is spatially uniform. The stability of $\boldsymbol{x}_F$ is determined by the eigenvalues of the Jacobian matrix of Eq. (1). Note that the Lorenz-96 model is invariant under the symmetry $x_i \to x_{i+1}$ while taking into

Reference	Application	n
Basnarkov and Kocarev (2012)	Forecast improvement	960
Danforth and Yorke (2006)	Forecasting in chaotic systems	40
Dieci et al. (2011)	Computing Lyapunov exponents	40
Gallavotti and Lucarini (2014)	Non-equilibrium ensembles	32
Hallerberg et al. (2010)	Bred vectors	1024
Hansen and Smith (2000)	Operational constraints	40
Haven et al. (2005)	Predictability	40
Karimi and Paul (2010)	Chaos	$4, \ldots, 50$
De Leeuw et al. (2017)	Data assimilation	36
Lorenz (2006)	Predictability	$4, 36$
Lorenz (2005)	Designing chaotic models	30
Lorenz and Emanuel (1998)	Data assimilation	40
Lucarini and Sarno (2011)	Ruelle linear response theory	40
Orrell et al. (2001)	Model error	8
Orrell (2002)	Metric in forecast error growth	8
Orrell and Smith (2003)	Spectral bifurcation diagrams	$4, 8, 40$
Ott et al. (2004)	Data assimilation	$40, 80, 120$
Pazó et al. (2008)	Spatiotemporal chaos	128
Stappers and Barkmeijer (2012)	Adjoint modelling	40
Sterk and Van Kekem (2017)	Predictability of extremes	$4, 7, 24$
Sterk et al. (2012)	Predictability of extremes	36
Trevisan and Palatella (2011)	Data assimilation	$40, 60, 80$

Table 1. Recent papers with applications of the Lorenz-96 model and the values of n that were used.

account the periodic boundary condition. As a consequence the Jacobian matrix evaluated at $\boldsymbol{x}_F$ is circulant which means that
each row is a right cyclic shift of the previous row, and so the matrix is completely determined by its first row. If we denote
this row by

$$(c_0, c_1, \ldots, c_{n-1}),$$

then it follows from Gray (2006) that the eigenvalues of the circulant matrix can be expressed in terms of roots of unity
$\rho_j = \exp(-2\pi i j / n)$ as follows:

$$\lambda_j = \sum_{k=0}^{n-1} c_k \rho_j^k, \qquad j = 0, \ldots, n-1. \tag{4}$$

An eigenvector corresponding to λ_j is given by

$$\boldsymbol{v}_j = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & \rho_j & \rho_j^2 & \cdots & \rho_j^{n-1} \end{pmatrix}^\top.$$

In particular, for the ~~Lorenz–96 model~~ Lorenz-96 model in Eq. (1) ~~we have that~~ the Jacobian matrix at x_F has only three nonzero elements on its first row, viz. $c_0 = -1$, $c_1 = F$, $c_{n-2} = -F$. Hence, the eigenvalues λ_j can be expressed in terms of n and F as follows:

$$\lambda_j = -1 + Ff(2\pi j/n) + Fg(2\pi j/n)i,$$

where the functions f and g are defined as

$$f(x) = \cos(x) - \cos(2x),$$
$$g(x) = -\sin(x) - \sin(2x). \tag{5}$$

For $F = 0$ the equilibrium x_F is stable as $\operatorname{Re}\lambda_j = -1$ for all $j = 0, \ldots, n-1$. The real part of the eigenvalue λ_j changes sign if the equation

$$F = \frac{1}{f(2\pi j/n)} \tag{6}$$

is satisfied. The graph of f in Fig. 3 shows that for $F > 0$ Eq. (6) can have at most four solutions. Since f is symmetric around $x = \pi$ it follows that if j is a solution of Eq. (6) then so is $n - j$. This means that the equilibrium x_F becomes unstable for $F > 0$ when either a pair or a double pair of eigenvalues becomes purely imaginary. The main result is summarized in the following theorem.

Theorem 1. *Assume that $n \geq 4$ and $l \in \mathbb{N}$ satisfies $0 < l < \frac{n}{2}, l \neq \frac{n}{3}$. Then the l-th eigenvalue pair $(\lambda_l, \lambda_{n-l})$ of the trivial equilibrium x_F crosses the imaginary axis at the parameter value $F_H(l, n) := 1/f(2\pi l/n)$ and thus x_F bifurcates through either a Hopf or a double-Hopf bifurcation. A double-Hopf bifurcation, with* two *pairs of eigenvalues crossing the imaginary axis, occurs if and only if there exist $l_1, l_2 \in \mathbb{N}$ such that*

$$\cos\left(\frac{2\pi l_1}{n}\right) + \cos\left(\frac{2\pi l_2}{n}\right) = \frac{1}{2}. \tag{7}$$

Otherwise, a Hopf bifurcation occurs. Moreover, the first Hopf bifurcation of x_F is always supercritical.

The proof of Theorem 1 can be found in Van Kekem and Sterk (2018) in which also an expression for the first Lyapunov coefficient is derived which determines for which l the Hopf bifurcation is sub- or supercritical. Observe that Theorem 1 implies that a double-Hopf bifurcation occurs for $n = 10m$ (with $l_1 = m, l_2 = 3m, F = 2$) and $n = 12m$ (with $l_1 = 2m$, $l_2 = 3m$, $F = 1$). In Sect. 4 we will explain how double-Hopf bifurcations lead to the coexistence of two or more stable traveling waves with different wave numbers.

From the eigenvalues that cross the imaginary axis and the corresponding eigenvectors we can deduce the physical characteristics of the periodic orbit that arises after a Hopf bifurcation. When the l-th eigenvalue pair $(\lambda_l, \lambda_{n-l})$ crosses the imaginary axis we can write

$$\lambda_l = \frac{g(2\pi l/n)}{f(2\pi l/n)}i = -\cot(\pi l/n)i, \quad \lambda_{n-l} = \bar{\lambda}_l = \cot(\pi l/n)i.$$
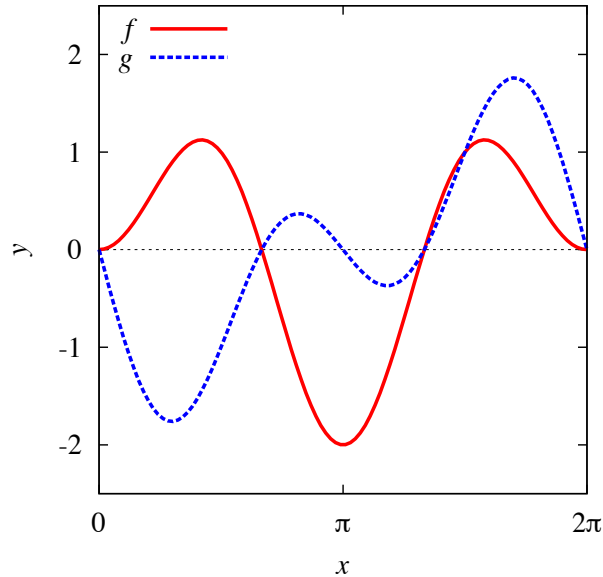
Figure 3. Graphs of the functions f and g as defined in Eq. (5). The eigenvalues of the equilibrium $x_F = (F, \ldots, F)$ are given by $\lambda_j = -1 + F f(2\pi j/n) + F g(2\pi j/n)i$ for $j = 0, \ldots, n-1$. The shapes of the graphs of f and g imply that the equilibrium x_F can only lose stability through either a Hopf or a double-Hopf bifurcation for $F > 0$, see main text.

If we set $\omega = \cot(\pi l/n)$, then according to Eq. (3) an approximation of the periodic orbit is given by

$$
\begin{aligned}
x_j(t) &= F + \varepsilon \operatorname{Re} \frac{e^{i(\omega t - 2\pi i j l/n)}}{\sqrt{n}} + O(\varepsilon^2) \\
&= F + \frac{\varepsilon}{\sqrt{n}} \cos(\omega t - 2\pi j l/n) + O(\varepsilon^2).
\end{aligned}
$$

This is indeed the expression for a *traveling* wave in which the spatial wave number and the period are given by respectively l and $T = 2\pi/\omega = 2\pi \tan(\pi l/n)$. Thus the index of the eigenpair that crosses the imaginary axis determines the propagation characteristics of the wave.

Note that Hopf bifurcations of an *unstable* equilibrium will result in an unstable periodic orbit. Therefore, not all waves that are guaranteed to exist by Theorem 1 will be visible in numerical experiments. Equation (6) implies that for $F > 0$ the *first* Hopf bifurcation occurs for the eigenpair $(\lambda_l, \lambda_{n-l})$ with index

$$
l_1^+(n) = \underset{0 < j < n/3}{\arg\max} f(2\pi j/n). \tag{8}
$$

In Appendix A1 it is shown that, except for $n = 7$, the integer $l_1^+(n)$ satisfies the bounds

$$
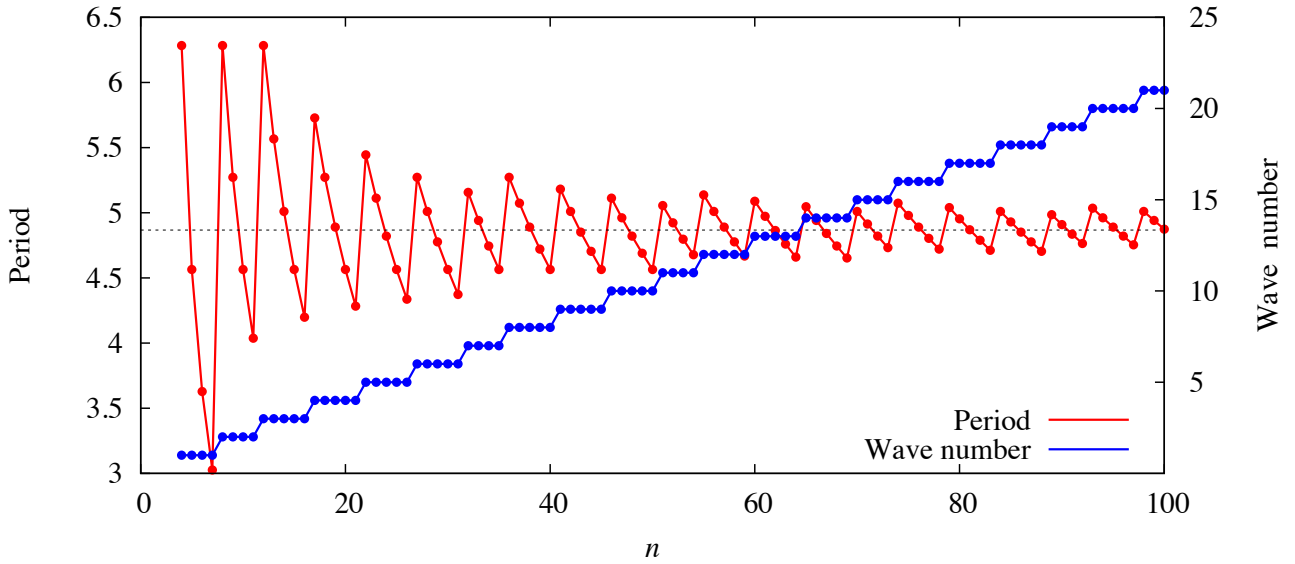\frac{n}{6} \le l_1^+(n) \le \frac{n}{4}, \tag{9}
$$

8

Figure 4. As the equilibrium $x_F = (F, \ldots, F)$ loses stability through a (double-)Hopf bifurcation for $F > 0$ a periodic attractor is born which represents a traveling wave. The spatial wave number increases linearly with n, whereas the period tends to a finite limit.

which means that the wave number increases linearly with the dimension n. Since the function f has a maximum at $x = \arccos(\frac{1}{4})$ we have

$$\lim_{n \to \infty} \frac{2\pi l_1^+(n)}{n} = \arccos(\tfrac{1}{4}),$$

which is consistent with Eq. (9). As a corollary we find that the period of this wave tends to a finite limit as $n \to \infty$:

$$T_\infty = \lim_{n \to \infty} 2\pi \tan\left(\frac{\pi l_1^+(n)}{n}\right) = 2\pi \tan(\tfrac{1}{2}\arccos(\tfrac{1}{4})) \approx 4.867.$$

Figure 4 shows a graph of the period and the wave number as a function of n. Note that the period settles down on the value T_∞.

3.2 Traveling waves for odd $n \geq 4$ and $F < 0$

Now assume that n is odd. For $F < 0$ Eq. (6) has precisely two solutions which implies that the first bifurcation of x_F is a supercritical Hopf bifurcation. The index of the first bifurcating eigenpair $(\lambda_l, \lambda_{n-l})$ follows by minimizing the value of the function f in Eq. 5:

$$l_1^-(n) = \frac{n-1}{2}.$$

9

Again, the wave number increases linearly with n, but at a faster rate than in the case $F > 0$. Now the period of the wave is given by

$$10 \quad T = 2\pi \tan\left(\frac{\pi(n-1)}{2n}\right) = O(4n),$$

where the last equality ~~sign~~ follows from the computations in Appendix A2. This implies that contrary to the case $F > 0$ the period increases monotonically with n and does not tend to a limiting value as $n \to \infty$.

Note that for even n and $F < 0$ the first bifurcation is *not* a Hopf bifurcation since $\lambda_{n/2} = -1 - 2F$ is a *real* eigenvalue that changes sign at $F = \frac{1}{2}$. Surprisingly, the case $n = 4$ is not analytically tractable. The case $n = 6$ will be studied analytically in Sect. 3.3. In Sect. 3.4 we will numerically study the bifurcations for other values of n and $F < 0$.

3.3 ~~Standing~~ Stationary waves for $n = 6$ and $F < 0$

We now consider the dimension $n = 6$. At $F = -\frac{1}{2}$ the eigenvalue λ_3 changes sign. Note that the equilibrium $\boldsymbol{x}_F$ *cannot*
5 exhibit a saddle-node bifurcation since $\boldsymbol{x}_F$ continues to exist for $F < -\frac{1}{2}$. Instead, at $F = -\frac{1}{2}$ there must be a branching point which is ~~associated to~~ either a pitchfork or a transcritical bifurcation. If we try for $F < -\frac{1}{2}$ an equilibrium solution of the form $x_P = (a,b,a,b,a,b)$ then it follows that a and b must satisfy the equations

$$b(b-a) - a + F = 0,$$
$$a(a-b) - b + F = 0.$$

Of course $a = b = F$ is a solution to these equations, but this would lead to the already known equilibrium $\boldsymbol{x}_F = (F, F, F, F, F, F)$.
10 There is an additional pair of solutions which is given by

$$a = \frac{-1 + \sqrt{-1 - 2F}}{2},$$
$$b = \frac{-1 - \sqrt{-1 - 2F}}{2}. \tag{10}$$

With these values of a and b we obtain two new equilibria $x_{P,1} = (a,b,a,b,a,b)$ and $x_{P,2} = (b,a,b,a,b,a)$ that exist for $F < -\frac{1}{2}$ in addition to the equilibrium $\boldsymbol{x}_F$. This means that a pitchfork bifurcation occurs at $F = -\frac{1}{2}$.

As F decreases, each of the new equilibria ~~$x_P^{1,2}$~~ $x_{P,1,2}$ may bifurcate again. We first consider the equilibrium $x_{P,1}$ for which
15 the Jacobian matrix is given by

$$J = \begin{pmatrix} -1 & b & 0 & 0 & -b & b-a \\ a-b & -1 & a & 0 & 0 & -a \\ -b & b-a & -1 & b & 0 & 0 \\ 0 & -a & a-b & -1 & a & 0 \\ 0 & 0 & -b & b-a & -1 & b \\ a & 0 & 0 & -a & a-b & -1 \end{pmatrix}.$$

Note that J is no longer circulant: in addition to shifting each row in a cyclic manner, the values of a and b also need to be interchanged. In particular, this means that the eigenvalues can no longer be determined by means of Eq. (4). Symbolic manipulations with the computer algebra package Mathematica (Wolfram Research, Inc., 2016) show that an eigenvalue crossing occurs for $F = -\frac{7}{2}$ in which case $a = \frac{1}{2}(-1+\sqrt{6})$ and $b = \frac{1}{2}(-1-\sqrt{6})$ ~~and~~ so that the characteristic polynomial of J is given by

$$\det(J - \lambda I) = 468 + 219\lambda + 246\lambda^2 + 91\lambda^3 + 33\lambda^4 + 6\lambda^5 + \lambda^6$$
$$= (3 + \lambda^2)(12 + \lambda + \lambda^2)(13 + 5\lambda + \lambda^2).$$

This expression shows that J has two purely imaginary eigenvalues $\pm i\sqrt{3}$ and the remaining four complex eigenvalues have negative real part. Therefore the equilbrium $x_{P,1}$ undergoes a Hopf bifurcation at $F = -\frac{7}{2}$. Numerical experiments with Mathematica show that the matrix $J - i\sqrt{3}I$ has a null vector of the form

$$v = \begin{pmatrix} v_0 & v_1 & v_0 e^{2\pi i/3} & v_1 e^{2\pi i/3} & v_0 e^{-2\pi i/3} & v_1 e^{-2\pi i/3} \end{pmatrix}^\top,$$

where we can take

5   $v_0 = 6\sqrt{2} + 2i$ and $v_1 = 3\sqrt{2} + 5\sqrt{3} - (5 + \sqrt{6})i.$

Hence, using Eq. (3) the periodic orbit can be approximated as

$$x_0(t) = \frac{-1+\sqrt{6}}{2} + \frac{\varepsilon}{\|v\|} \operatorname{Re} v_0 e^{i\sqrt{3}t} + O(\varepsilon^2),$$

$$x_1(t) = \frac{-1-\sqrt{6}}{2} + \frac{\varepsilon}{\|v\|} \operatorname{Re} v_1 e^{i\sqrt{3}t} + O(\varepsilon^2),$$

$$x_2(t) = \frac{-1+\sqrt{6}}{2} + \frac{\varepsilon}{\|v\|} \operatorname{Re} v_0 e^{i(\sqrt{3}t+2\pi/3)} + O(\varepsilon^2),$$

$$x_3(t) = \frac{-1-\sqrt{6}}{2} + \frac{\varepsilon}{\|v\|} \operatorname{Re} v_1 e^{i(\sqrt{3}t+2\pi/3)} + O(\varepsilon^2),$$

$$x_4(t) = \frac{-1+\sqrt{6}}{2} + \frac{\varepsilon}{\|v\|} \operatorname{Re} v_0 e^{i(\sqrt{3}t-2\pi/3)} + O(\varepsilon^2),$$

$$x_5(t) = \frac{-1-\sqrt{6}}{2} + \frac{\varepsilon}{\|v\|} \operatorname{Re} v_1 e^{i(\sqrt{3}t-2\pi/3)} + O(\varepsilon^2).$$

(11)

Note that if $\varepsilon = \sqrt{-\frac{7}{2} - F}$ is sufficiently small, then $x_j(t)$ is always positive (resp. negative) for $j = 0, 2, 4$ (resp. $j = 1, 3, 5$). This implies that the periodic orbit represents a ~~standing~~ stationary wave rather than a traveling wave. The period of the wave

10   is $T = 2\pi/\sqrt{3}$ and the spatial wave number is 3. These spatiotemporal properties are clearly visible in the left panel of Fig. 5.

The computations for the equilibrium $x_{P,2}$ are similar and show that *another* Hopf bifurcation takes place at $F = -\frac{7}{2}$. This means that for $F < -\frac{7}{2}$ there exists a *second* stable ~~period~~ periodic orbit which coexists with the stable periodic orbit born at the Hopf bifurcation of $x_{P,1}$. Its first-order approximation is almost identical to Eq. (11): only the numerators $1 - \sqrt{6}$ and
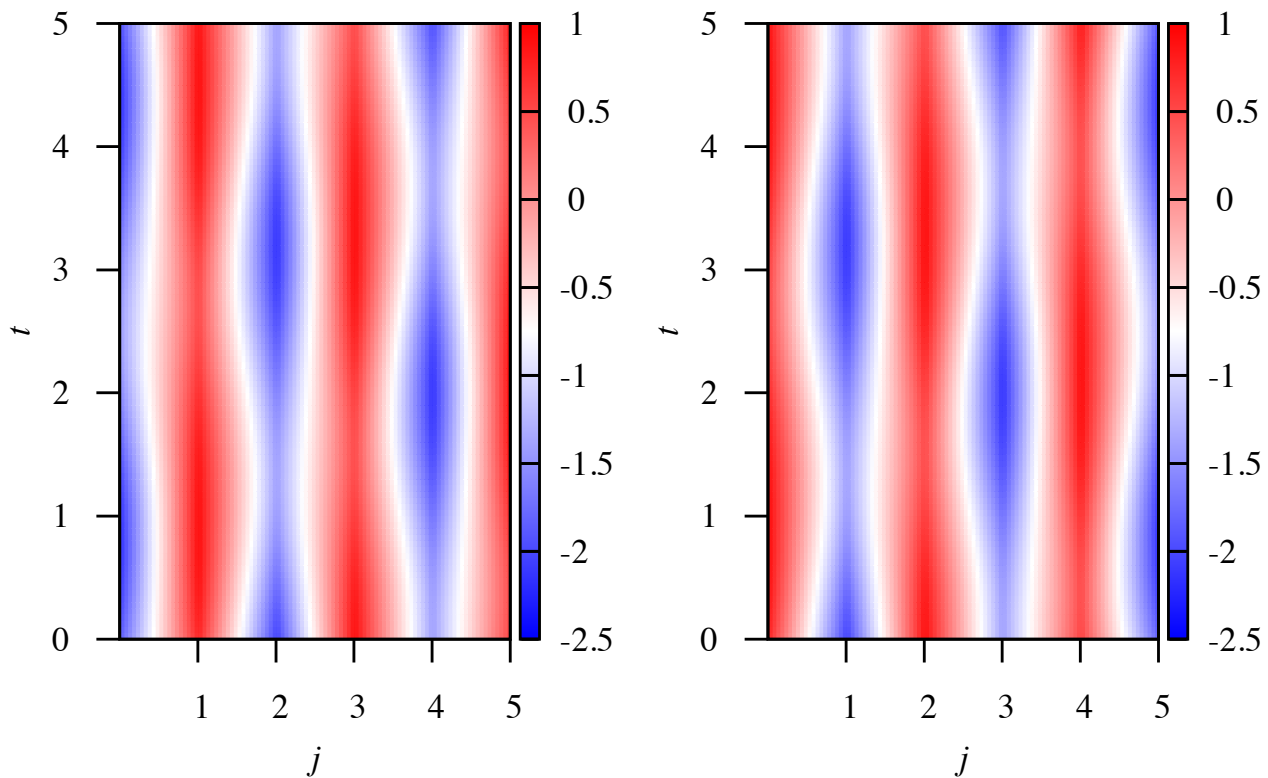
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Figure 5. As Fig. 1, but for two periodic attractors for $n = 6$ and $F = -3.6$. These attractors are born at Hopf bifurcations of the equilibria $x_{P,1}$ (left) and $x_{P,2}$ (right) at $F = -7/2$. Note that the waves do not travel "eastward" or "westward". The pitchfork bifurcation changed the mean flow which in turn changes the propagation of the wave.
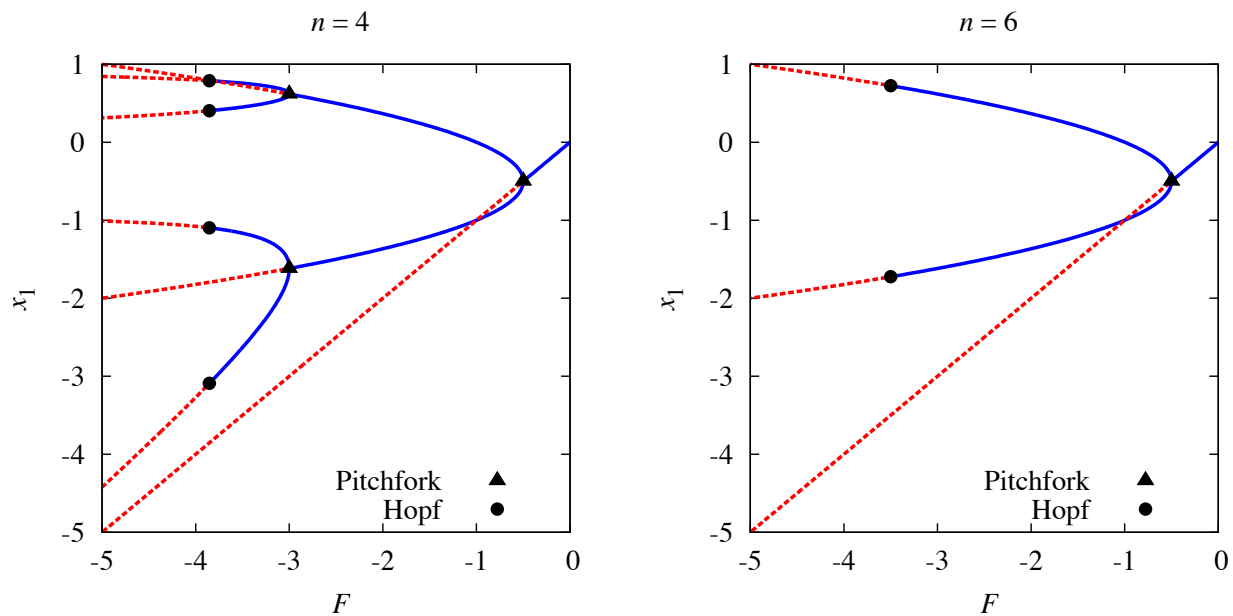
Figure 6. Bifurcation diagrams obtained by continuation of the equilibrium $\boldsymbol{x}_F = (F, \ldots, F)$ for $F < 0$ for the dimensions $n = 4$ and $n = 6$. Stable (unstable) branches are marked by solid (dashed) lines. For $n = 4$ two pitchforks in a row occur before the Hopf bifurcation, whereas for $n = 6$ only one pitchfork occurs before the Hopf bifurcation. The bifurcation diagram for $n = 4k$ with $k \in \mathbb{N}$ (resp. $n = 4k + 2$) is qualitatively similar to the bifurcation diagram for $n = 4$ (resp. $n = 6$), see the main text.

$1 + \sqrt{6}$ need to be interchanged and therefore the complete expression will be omitted. Hence, the two coexisting stable waves that arise from the two Hopf bifurcations of the equilibria $x_{P,1}$ and $x_{P,2}$ have the same spatiotemporal properties, but they differ in spatial phase which is indeed visible in the Hovmöller diagrams in Fig. 5. These results show how the pitchfork bifurcation ~~changed~~ changes the mean flow and hence also the propagation characteristics of the wave. In the next section we will explore spatiotemporal properties of waves for $F < 0$ and other even values of n.

3.4 ~~Standing~~ Stationary waves for even $n \geq 4$ and $F < 0$

The case $n = 4$ turns out to be more complicated than the case $n = 6$. If $n = 4$, then the equilibrium $\boldsymbol{x}_F = (F, F, F, F)$ undergoes a pitchfork bifurcation at $F = -\frac{1}{2}$ since $\lambda_2 = 0$. Just as in the case $n = 6$ two new branches of equilibria appear which are given by

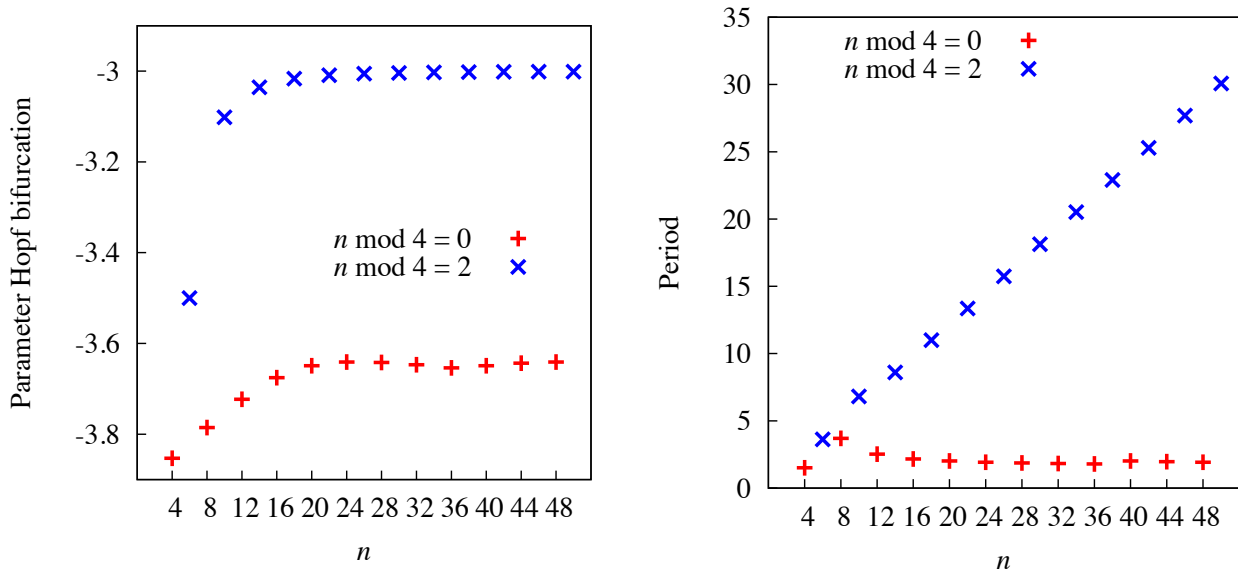$$x_{P,1} = (a, b, a, b), \quad x_{P,2} = (b, a, b, a),$$

Figure 7. Parameter values of the first Hopf bifurcation (left) and the periods of the periodic attractor (right) that appears after the Hopf bifurcation for $F < 0$ and even values of the dimension n. For clarity the cases $n = 4k$ and $n = 4k + 2$ have been marked with different symbols in order to emphasize the differences between the two cases.

where a, b are again given by Eq. (10). The Jacobian matrix at the equilibrium $x_{P,1}$ is given by

$$5 \quad J = \begin{pmatrix} -1 & b & -b & b-a \\ a-b & -1 & a & -a \\ -b & b-a & -1 & b \\ a & -a & a-b & -1 \end{pmatrix}$$

For $F = -3$, in which case $a = (-1 + \sqrt{5})/2$ and $b = (-1 - \sqrt{5})/2$, the characteristic polynomial of the matrix J is given by

$$\det(J - \lambda I) = \lambda(30 + 13\lambda + 4\lambda^2 + \lambda^3),$$

which implies that a real eigenvalue of J becomes zero at $F = -3$. For the equilibrium $x_{P,2}$ we obtain the same result. Since the equilibria $x_{P,1}$ and $x_{P,2}$ continue to exist for $F < -3$ a saddle-node bifurcation is ruled out. Numerical continuation

10 using the software package AUTO-07p (Doedel and Oldeman, 2007) shows that again a pitchfork bifurcation takes place at $F = -3$. It is not feasible to derive analytic expressions for the new branches of equilibria as in Eq. (10). Continuation of the four branches while monitoring their stability indicates that at $F \approx -3.853$ in total four Hopf bifurcations occur (one at each branch). Figure 6 shows the bifurcation diagrams for the cases $n = 4$ and $n = 6$.

The question is whether the results described above persist for even dimensions $n > 6$. To that end we conducted the fol-

5 lowing numerical experiment. For all even dimensions $4 \leq n \leq 50$ we used the software package AUTO-07p to numerically

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continue the equilibrium $\boldsymbol{x}_F = (F, \ldots, F)$ for $F < 0$ while monitoring the eigenvalues to detect bifurcations. At each pitchfork bifurcation we performed a branch switch in order to follow the new branches of equilibria and detect their bifurcations. Once a Hopf bifurcation is detected we can compute the period of the wave as $T = 2\pi/\omega$ from the eigenvalue pair $\pm\omega i$. The results of this experiment reveal that the cases $n = 4k$ and $n = 4k + 2$ are different both qualitatively and quantitatively.

10 If $n = 4k + 2$ for some $k \in \mathbb{N}$, then one pitchfork bifurcation occurs at $F = -0.5$. This follows directly from Eq. (4) for the eigenvalues of the equilibrium $\boldsymbol{x}_F$: for even n we have $\lambda_{n/2} = -1 - 2F$ which changes sign at $F = -\frac{1}{2}$. From the pitchfork bifurcation two new branches of stable equilibria emanate. Each of these equilibria is of the form

$$(a, b, a, b, a, b, \ldots) \tag{12}$$

with $a > 0$ and $b < 0$; the other equilibrium just follows by interchanging a and b. Each of the two equilibria undergoes a Hopf

15 bifurcation, which leads to the coexistence of two stable waves. Figure 7 (left panel) suggests that the value of F at which this bifurcation occurs is not constant, but tends to -3 as $n \to \infty$. The period of the periodic attractor that is born at the Hopf bifurcation increases almost linearly with n: fitting the function $T(n) = \alpha + \beta n$ to the numerically computed periods gives $\alpha = 0.36$ and $\beta = 0.59$, see Fig. 7 (right panel).

If $n = 4k$ for some $k \in \mathbb{N}$, then *two* Pitchfork bifurcations in a row occur at $F = -0.5$ and $F = -3$. After the second

5 pitchfork bifurcation there are four branches of equilibria. Each of these equilibria is of the form

$$(a, b, c, d, a, b, c, d, \ldots) \tag{13}$$

where ~~a, b, c, d~~ a, b, c, and d alternate in sign; the other equilibria are obtained by applying a circulant shift. Each of the four stable equilibria undergoes a Hopf bifurcation at the same value of the parameter F, which leads to the coexistence of four stable waves. Figure 7 (left panel) suggests that the value of F at which this bifurcation occurs is not constant, but tends to

10 -3.64 as $n \to \infty$. Contrary to the case $n = 4k + 2$, the period of the periodic attractor that appears after the Hopf bifurcation settles down and tends to 1.92 as $n \to \infty$.

In spite of the aforementioned quantitative differences between the cases $n = 4k + 2$ and $n = 4k$, the wave numbers depend in the same way on n in both cases. Equations (12) and (13) show that the n components of the equilibrium that undergoes the Hopf bifurcation alternate in sign. Therefore, sufficiently close to the Hopf bifurcation the components $x_0(t), \ldots, x_{n-1}(t)$

15 of the periodic orbit will also alternate in sign. Hence, the resulting ~~standing~~ stationary waves consists of $n/2$ "troughs" and "ridges" which means that their wave number equals $n/2$.

4 Multi-stability: coexistence of waves

The results of Sect. 3.4 show that for even n and $F < 0$ either two or four stable periodic orbits coexist for the same parameter values. This phenomenon is referred to as *multi-stability* in the dynamical systems literature. An overview of the wide range of

20 applications of multi-stability in different disciplines of science is given by Feudel (2008).

Multi-stability also occurs ~~for~~ when $F > 0$ ~~but due to~~, but for a very different reason. For $n = 12$, Theorem 1 implies that the first bifurcation of the equilibrium $\boldsymbol{x}_F = (F, \ldots, F)$ for $F > 0$ is *not* a Hopf bifurcation, but a double-Hopf bifurcation.

Indeed, at $F = 1$ we have two pairs of purely imaginary eigenvalues, namely $(\lambda_2, \lambda_{10}) = (-i\sqrt{3}, i\sqrt{3})$ and $(\lambda_3, \lambda_9) = (-i, i)$. Note that the double-Hopf bifurcation is a codimension-2 bifurcation which means that generically two parameters must be varied in order for the bifurcation to occur (Kuznetsov, 2004). However, symmetries such as those in the Lorenz-96 model can reduce the codimension of a bifurcation.

In previous work (Van Kekem and Sterk, 2018) we have introduced an embedding of the Lorenz-96 model in a 2-parameter family by adding a diffusion-like term multiplied by an additional parameter G:

$$\frac{dx_j}{dt} = x_{j-1}(x_{j+1} - x_{j-2}) - x_j + G(x_{j-1} - 2x_j + x_{j+1}) + F, \quad j = 0, \dots, n-1. \tag{14}$$

Note that by setting $G = 0$ we retrieve the original Lorenz-96 model in Eq. (1). Since the Jacobian matrix of Eq. (14) is again a circulant matrix we can use Eq. (4) to determine its eigenvalues:

$$\lambda_j = -1 - 2G(1 - \cos(2\pi j/n)) + F f(2\pi j/n) + F g(2\pi j/n) i \tag{15}$$

Also note that $\boldsymbol{x}_F = (F, \dots, F)$ remains an equilibrium solution of Eq. (14) for all (F, G). The Hopf bifurcations of $\boldsymbol{x}_F$ described in Theorem 1 now occur along the lines

$$G = \frac{F f(2\pi j/n) - 1}{2(1 - \cos(2\pi j/n))}, \tag{16}$$

and the intersection of two such lines leads to a double-Hopf bifurcation.

Figure 8 shows a local bifurcation diagram of the 2-parameter Lorenz-96 model in the (F, G)-plane for $n = 12$ which was numerically computed using MATCONT (Dhooge et al., 2011). A double-Hopf point is located at $(F, G) = (1, 0)$ which is indeed implied by Theorem 1. ~~Computing the normal form of this bifurcation~~ The normal form of a double-Hopf bifurcation depends on the values of two coefficients which determine the unfolding of the bifurcation. In total, there are eleven different bifurcation scenarios to consider. The normal form computation of the double-Hopf bifurcation for $n = 12$ and $(F, G) = (1, 0)$ in Van Kekem and Sterk (2018) shows that the unfolding of ~~the bifurcation~~ this particular case is of "type I in the simple case" as described by Kuznetsov (2004). This means that from the double-Hopf point only two curves of Neĭmark-Sacker bifurcations emanate. ~~These curves bound a~~ In discrete-time dynamical systems a Neĭmark-Sacker bifurcation is the birth of a closed invariant curve when a fixed point changes stability through a pair of complex eigenvalues crossing the unit circle in the complex plane. From a continuous-time system, such as eq. (14), we can construct a discrete-time dynamical system by defining a Poincaré return map of a periodic orbit, in which case a Neĭmark-Sacker bifurcation refers to the birth of an invariant 2-dimensional torus when the periodic orbit changes stability by a pair of Floquet multipliers crossing the unit circle in the complex plane.

In order to explain the dynamics in a neighbourhood around the double-Hopf point, we now use Fig. 8 to describe the successive bifurcations that occur for $G = 0.1$ fixed and increasing F. At $F = 1.10$ the equilibrium x_F becomes unstable through a supercritical Hopf bifurcation (the blue line given by $G = F - 1$) and a stable periodic orbit with wave number 2 is born. At $F = 1.20$ the now unstable equilibrium undergoes second Hopf bifurcation (the red line given by $G = \frac{1}{2}(F - 1)$) and an unstable periodic orbit with wave number 3 is born. The latter periodic orbit becomes *stable* at $F \approx 1.58$ through
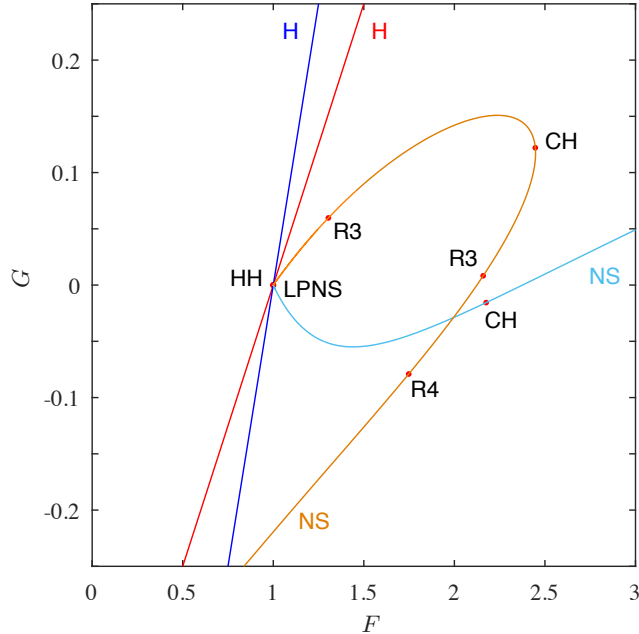
Figure 8. Bifurcation diagram of the 2-parameter system (14) in the (F, G)-plane for $n = 12$. A double-Hopf bifurcation point is located at the point $(F, G) = (1, 0)$ due to the intersection of two Hopf bifurcation lines. From this codimension-2 point two Neĭmark-Sacker bifurcation curves emanate which bound a "lobe-shaped" region in which two periodic attractors coexist.

a subcritical Neĭmark-Sacker bifurcation (orange curve) and an unstable 2-dimensional invariant torus is born. Hence, for parameter values $F > 1.58$ two stable waves with wave numbers 2 and 3 coexist until one of these waves becomes unstable in a bifurcation. For all fixed values of $0 < G < 0.15$ the same bifurcation scenario occurs, but the values of F are different. For $-0.06 < G < 0$ the roles of the two Hopf bifurcations and periodic orbits have to be interchanged.

30   The scenario described above shows how the presence of two subcritical Neĭmark-Sacker bifurcation emanating from a double-Hopf bifurcation determines a region of the (F, G)-plane in which two stable periodic ~~attractors~~ orbits coexist with an unstable ~~2-torus~~2-dimensional invariant torus. We will refer to this region as the "multi-stability lobe". ~~Figure ?? shows two periodic attractors with wave numbers 2 and 3 in the multi-stability lobe for n = 12 and (F,G) = (1.5,0)~~The scenario described above is not limited to the special case of the Lorenz-96 model, but occurs near a double-Hopf bifurcation of type I in any dynamical system (Kuznetsov, 2004).

~~Projections onto the (x₁,x₂)-plane of coexisting periodic attractors for dimensions n = 12 and (F,G) = (1.5,0), which is inside the multi-stability lobe of Fig. 8.~~

5   Double-Hopf bifurcations are abundant in the 2-parameter Lorenz-96 model of Eq. (14). The lines described in Eq. (16) have a different slope for all $0 < j < n/2$ and $j \neq n/3$, and hence they mutually intersect each other. This implies that the number of double-Hopf points in the (F, G)-plane grows quadratically with n, see Appendix A3. However, not all these points will have an influence on the dynamics: if $\boldsymbol{x}_F$ is already unstable, then any dynamical object born through the double-Hopf bifurcation
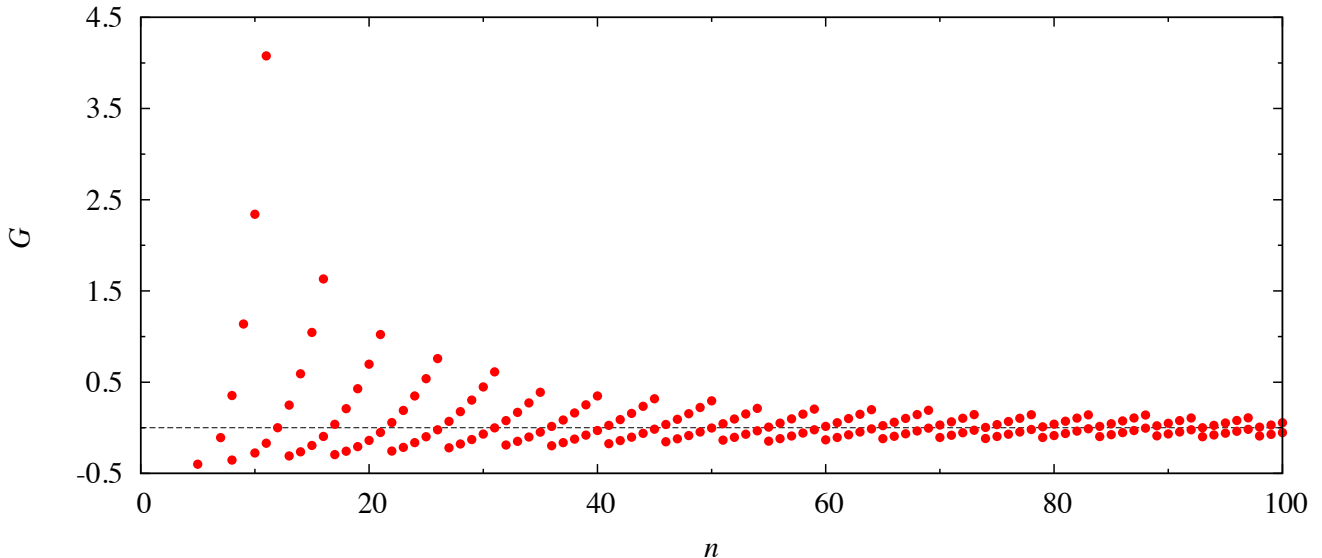
Figure 9. G-coordinates of double-Hopf points as a function of n. Only those double-Hopf points are shown which destabilize the equilibrium $\boldsymbol{x}_F$. For large values of n the double-Hopf bifurcations are close to the F-axis in the (F,G)-plane, which means that these points are likely to affect the dynamics of the Lorenz-96 model for $G = 0$.

will also be unstable. In what follows, we only consider the double-Hopf bifurcations through which $\boldsymbol{x}_F$ can change from

10   stable to unstable. We can find such points as follows. Starting from the line in Eq. (16) with $j = l_1^+(n)$ as defined by Eq. (8), we first compute double-Hopf points by computing the intersections with all other lines. From these intersections we select those that satisfy the condition $\max\{\mathrm{Re}\,\lambda_j : j = 0,\ldots,n-1\} = 0$.

Figure 9 shows the G-coordinates of these double-Hopf points as a function of n. Clearly, for large n there exist double-Hopf points which are very close to the F-axis which suggests that the multi-stability lobe that emanates from such points can

15   intersect the F-axis and hence influence the dynamics of the original Lorenz-96 model for $G = 0$. Moreover, Fig. 9 shows that for $n > 12$ there are always *two* double-Hopf points by which $\boldsymbol{x}_F$ can change from stable to unstable. It is then possible that two multi-stability lobes intersect each other which leads to a region in the (F,G)-plane in which at least three stable waves coexist.

Figures 10–12 show bifurcation diagrams of three periodic orbits as a function of F for $G = 0$ for $n = 40, 60, 80$. For each periodic orbit the continuation is started from a Hopf bifurcation of the equilibrium $\boldsymbol{x}_F$. If $\boldsymbol{x}_F$ is unstable, then so will be the

5   periodic orbit. However, when the boundary of a multi-stability lobe is crossed a Neĭmark-Sacker bifurcation occurs by which a periodic orbit can gain stability. For specific intervals of the parameter F three stable periodic orbits coexist. Since Fig. 9 shows that for large values of n the double-Hopf bifurcations are close to the F-axis, we expect that the coexistence of three or more stable waves is typical for the Lorenz-96 model.
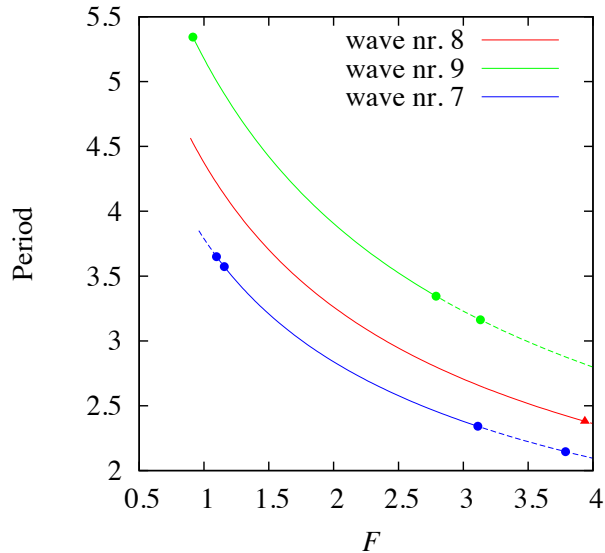
Figure 10. Continuation of periodic orbits for $n = 40$ and $G = 0$. The period of the orbit is plotted as a function of F. Stable (resp. unstable) orbits are indicated by solid (resp. dashed) lines. Circles denote Neĭmark-Sacker bifurcations and triangles denote period doubling bifurcations. The Hopf bifurcations generating the waves with wave numbers 8, 9, 7 occur at respectively $F = 0.894$, $F = 0.902$, and $F = 0.959$. Clearly, for $1.15 < F < 2.79$ three stable periodic orbits coexist.

The double-Hopf bifurcation has been reported in many works on fluid dynamical models. A few examples are baroclinic flows (Moroz and Holmes, 1984), rotating cylinder flows (Marqués et al., 2002, 2003), Poiseuille flows (Avila et al., 2006), rotating annulus flows (Lewis and Nagata, 2003; Lewis, 2010), and quasi-geostrophic flows (Lewis and Nagata, 2005). In all of these examples, the coexistence of multiple waves is reported where the nature of these waves depends on the specific model.

5 Conclusions

In this paper we have studied spatiotemporal properties of waves in the Lorenz-96 model and their dependence on the dimension n. For $F > 0$ the first bifurcation of the equilibrium $\boldsymbol{x}_F = (F, \ldots, F)$ is either a supercritical Hopf or a double-Hopf bifurcation and the periodic attractor born at the Hopf bifurcation represents a traveling wave. The spatial wave number is determined by the index of the eigenpair that crosses the imaginary axis and increases linearly with n, but the period tends to a finite limit as $n \to \infty$. For $F < 0$ and n *odd* the first bifurcation of $\boldsymbol{x}_F$ is always a supercritical Hopf bifurcation and the periodic attractor that appears after the bifurcation is again a traveling wave. In this case the wave number equals $(n-1)/2$ and the period is $O(4n)$.
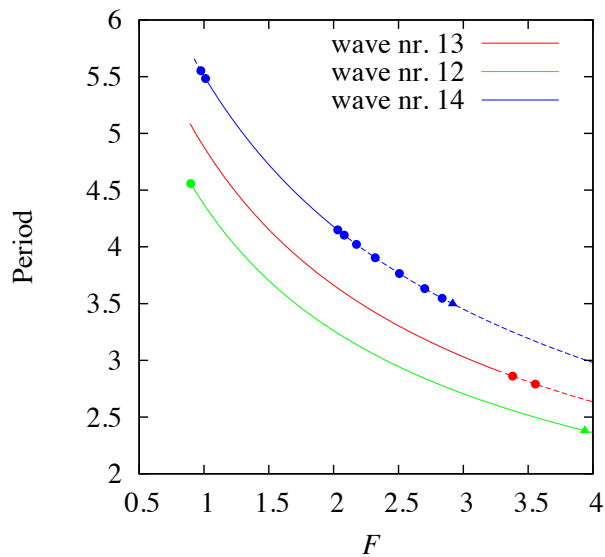
Figure 11. As ~~Figure~~ Fig. 10, but for $n = 60$. For $1.01 < F < 2.03$ three stable periodic orbits coexist. The Hopf bifurcations generating the waves with wave numbers 13, 12, 14 occur at respectively $F = 0.891$, $F = 0.894$, and $F = 0.923$.
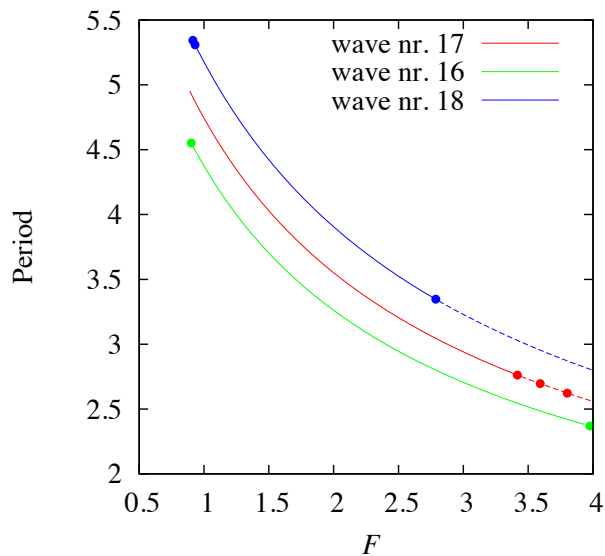

Figure 12. As ~~Figure~~ Fig. 10, but for $n = 80$. For $0.93 < F < 2.78$ three stable periodic orbits coexist. The Hopf bifurcations generating the waves with wave numbers 17, 16, 18 occur at respectively $F = 0.889$, $F = 0.894$, and $F = 0.902$.

For n *even* and $F < 0$ the first bifurcation of $\boldsymbol{x}_F$ is a pitchfork bifurcation which occurs at $F = -\frac{1}{2}$ and leads to two stable equilibria. If $n = 4k + 2$ for some $k \in \mathbb{N}$, then each of these equilibria undergoes a Hopf bifurcation which leads to the coexistence of two ~~standing~~ stationary waves. The role of the pitchfork bifurcation is to change the mean flow which in turn changes the propagation of the wave. If $n = 4k$ for some $k \in \mathbb{N}$, then *two* pitchfork bifurcations take place at $F = -\frac{1}{2}$ and $F = -3$ before a Hopf bifurcation occurs which leads to the coexistence of four ~~standing~~ stationary waves.

The occurrence of pitchfork bifurcations before the Hopf bifurcation leads to multi-stability, i.e. the coexistence of different waves for the same parameter settings. A second scenario that leads to multi-stability is via the double-Hopf bifurcation. For $n = 12$ the equilibrium $\boldsymbol{x}_F$ loses stability through a double-Hopf bifurcation. By adding a second parameter G to the Lorenz-96 model we have studied the unfolding of this codimension-2 bifurcation. Two Neĭmark-Sacker bifurcation curves emanating from the double-Hopf point bound a lobe-shaped region in the (F, G)-plane in which two stable traveling waves with different wave numbers coexist. For dimensions $n > 12$ we find double-Hopf bifurcations *near* the F-axis, which can create two multi-stability lobes intersecting each other, and in turn this can lead to the coexistence of *three* stable waves coexisting for $G = 0$ and a range of F-values. Hence, adding a parameter G to the Lorenz-96 model helps to explain the dynamics which is observed in the original model for $G = 0$.

Our results provide ~~an~~ a coherent overview of the spatiotemporal properties of the Lorenz-96 model for $n \geq 4$ and $F \in \mathbb{R}$. Since the Lorenz-96 model is often used as a model for testing purposes, our results can be used to select the most appropriate values of n and F for a particular application. The periodic attractors representing traveling or ~~standing~~ stationary waves can bifurcate into chaotic attractors representing irregular versions of these waves, and their spatiotemporal properties are inherited from the periodic attractor, see for example Fig. 1 and Fig. 2. This means that our results on the spatiotemporal properties of waves apply to broader parameter ranges of the parameter F than just in a small neighbourhood of the Hopf bifurcation.

The results presented in this paper also illustrate another important point: both qualitative and quantitative aspects of the dynamics of the Lorenz-96 model depend on the parity of n. This phenomenon also manifests itself in discretized partial differential equations. For example, for discretisations of Burgers' equation Basto et al. (2006) observed that for odd degrees of freedom the dynamics was confined to an invariant subspace, whereas for even degrees of freedom this was not the case. For the Lorenz-96 model the parity of n also determines the possible symmetries of the model. We will investigate these symmetries, and their consequences on bifurcation sequences using techniques from equivariant bifurcation theory in forthcoming work (Van Kekem and Sterk, 2017).

Appendix A

A1 Bounds on the wave number for $F > 0$

First note that for all $n \geq 4$, with the exception of $n = 7$, there exists at least one integer $j \in [\frac{n}{6}, \frac{n}{4}]$. Indeed, for $n = 4, 5, 6$ this follows by simply taking $j = 1$, and for $n = 8, 9, 10, 11$ this follows by taking $j = 2$. For $n \geq 12$ it follows from the fact that the interval $[\frac{n}{6}, \frac{n}{4}]$ has a width larger than 1 and hence must contain an integer. We now claim that these observations also imply that

$$l^+(n) = \underset{0 < j < n/3}{\arg\max} f(2\pi j/n) \in [\tfrac{n}{6}, \tfrac{n}{4}], \quad n \neq 7.$$

5   Note that $x \in [\frac{\pi}{3}, \frac{\pi}{2}]$ implies that $f(x) \geq 1$ and $x \in (0, \frac{\pi}{3}) \cup (\frac{\pi}{2}, \frac{2\pi}{3})$ implies $0 < f(x) < 1$. Moreover, $j \in [\frac{n}{6}, \frac{n}{4}]$ implies that $\frac{2\pi j}{n} \in [\frac{\pi}{3}, \frac{\pi}{2}]$. Therefore, $f(2\pi j/n)$ is maximized for some integer $j \in [\frac{n}{6}, \frac{n}{4}]$.

A2 Asymptotic period for even n and $F < 0$

Using l'Hopital's $0/0$ rule gives

$$
\begin{aligned}
\lim_{x \to \pi/2} (\tfrac{1}{2}\pi - x)\tan(x) &= \lim_{x \to \pi/2} \frac{(\tfrac{1}{2}\pi - x)\sin(x)}{\cos(x)} \\
&= \lim_{x \to \pi/2} \frac{-\sin(x) + (\tfrac{1}{2}\pi - x)\cos(x)}{-\sin(x)} \\
&= 1.
\end{aligned}
$$

10   Writing $x = \pi/2 - \pi/2n$ implies

$$\lim_{n \to \infty} \frac{2\pi \tan\left(\frac{\pi}{2} - \frac{\pi}{2n}\right)}{4n} = 1,$$

which in particular implies that

$$2\pi \tan\left(\frac{\pi}{2} - \frac{\pi}{2n}\right) = O(4n).$$

A3 The number of Hopf and double-Hopf bifurcations

15   The number of Hopf bifurcations of the equilibrium $x_F = (F, \ldots, F)$ for a given dimension n is exactly equal to the number of conjugate eigenvalue pairs which satisfy Theorem 1:

$$
N_H = \begin{cases} \lceil \frac{n}{2} \rceil - 1 & \text{if } n \neq 3m, \\ \lceil \frac{n}{2} \rceil - 2 & \text{if } n = 3m, \end{cases}
\tag{A1}
$$

where we need the ceiling-function if n is odd. Note that if n is a multiple of 3, then $f(\frac{2\pi n}{3}) = g(\frac{2\pi n}{3}) = 0$ which does give a proper complex conjugate pair crossing the imaginary axis and hence the number of Hopf bifurcations has to be decreased by 20   1.

For the 2-parameter system we can count the number of double-Hopf bifurcations by counting the intersections of the lines in Eq. (16). Since all the lines have a different slope, the number of such intersections is given by

$$N_{HH} = \frac{1}{2} N_H (N_H - 1) = \begin{cases} \frac{1}{2}(\lceil \frac{n}{2} \rceil - 1)(\lceil \frac{n}{2} \rceil - 2) & \text{if } n \neq 3m, \\ \frac{1}{2}(\lceil \frac{n}{2} \rceil - 2)(\lceil \frac{n}{2} \rceil - 3) & \text{if } n = 3m, \end{cases} \tag{A2}$$

which shows that the number of double-Hopf points grows quadratically with n.

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