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Nonlinear Processes in Geophysics
Discussions
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Interactive comment on "Fractional Brownian motion, the Matérn process, and stochastic modeling of turbulent dispersion" *by* Jonathan M. Lilly et al.

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The paper contains a collection of facts about the fractional Brownian motion and the Matérn process. These processes are used to model the velocity of particles moving in the complex plane. The paper explores the properties of the spectral density and their relation to the process diffusivity and memory. In Section 5 the authors propose a new approximate simulation algorithm for the Matérn process.

The main virtue of this paper is that it links statistical concepts and objects with physical ones, providing the intuition behind mathematical formulae and explaining their physical meaning. Mots of explanations are intuitively clear, calculations are detailed. This results in the size of the paper; the paper is very long and I recommend to split it into two papers. Sometimes the authors use objects before introducing them (for example,

the Matérn process or autocovariance function; please, see also the comments below).

General remarks and questions

- Statistically fBM and the Matérn process are very different in nature: fBM is not stationary and starts from 0; on the contrary, the Matérn process is stationary and its value at 0 value is random. Please clarify if this difference important for the physical model used in the paper.
- Page 3, line 1: please explain what 'damped version' of a process mean. This term is used many times throughout the paper, but it is never defined.
- Page 3, line 14: what parameters are meant here? Hurst index for fbm and a scale parameter for fbm? What is the third parameter of the Matérn process?
- Page 5, lines 15-16: in what sense the derivative (stochastically, pathwise, in L^p sense) is taken?
- Page 5, line 23: you use here the terms 'autocovariance function' and 'spectral density', but you define them only in the next subsection. The rearrangement of this subsection might be helpful for the reader.
- Page 7, line 5: please define the isotropy of a process to avoid the confusion with isotropy in spatial statistics, where it means the invariance of the process under rotation of coordinates.
- Page 9, line 20: please give the definition of the Matérn process and formula for its spectrum here. Is it a Gaussian process?
- Page 9, line 21: in what sense the derivative is taken?
- Page 10, line 17: what is periodic domain?

- Page 13, line 11: please write the range of the parameters.
- Page 15, line 23: the equality sign here without the reference to Section 3.3 is misleading. In Section 2.3 you define the spectrum for stationary process. Fractional Brownian motion is not stationary, so at that stage it is not clear what this equality means.
- Pages 18-19, beginning of Section 3.3. it is worth mentioning that the spectrum of stationary process and its time-independent spectrum coincide. Thus, time-independent spectrum is a more general notion than spectrum. Otherwise it is not clear, why one can compare the spectrum of stationary process and time-independent spectrum of non-stationary process.
- Page 24, lines 13-14: the sentence is not clear. There are two separate integrals in equation (44). Why the integral in (47) cannot be split into two integrals?
- Page 25, lines 26-27: the following statement is not clear. 'The first is that there is no upper bound on α, so processes can become still smoother than the α = 3/2 case that defines the upper limit of the slope parameter for fBm'. If $\alpha_M > \alpha_{fBM}$, where $alpha_M$ is the smoothness parameter of the Matérn process and α_{fBM} is the smoothness parameter of the fBM, it does not mean that the corresponding Matérn process is smoother than the corresponding fBM.
- It would be interesting to have the comparison of your simulation algorithm with known approximate algorithms (for example, spectral method). Note, that exact realizations for some values of α can be generated fast using a modification of circulant embedding called cut-off circulant embedding, see ().
- Please explain how to choose parameters k and $\hat{N}$ optimally. How can one estimate the error of the simulation for given k and $\hat{N}$?

- Please explain why you do not use classical maximum likelihood estimation in the time domain. What is the benefit of using frequency-domain maximum likelihood? It would be interesting to see the box-plots and bootstrap confidence intervals for the estimated parameters of the times series generated by the proposed algorithm.
- For the reader's convenience, when citing books, please include the number of the relevant chapter or pages.

Minor issues and typos

- Page 3, line 2: 'a uniform rotation rate to is shown'
- Page 3, line 30: 'This paper was inspired by the need to a develop'
- Page 9, Table 1: "The term in the box is the Matérn process", you probably mean that the term in the box is the spectrum of the Matérn process. Please write the ranges for λ, Ω, α.
- Page 9, line 17: if a function is absolutely integrable, then it is integrable; not vice versa.
- Page 17, lines 5-6: the sentence "The exponent 2α-1 varies from ..." is very difficult to understand, however it has a very simple meaning, namely $0 < 2\alpha$-$1 < 2$.
- Page 19, line 3: 'faveraging'
- Page 27, lines 1-3: the sentence is unclear. What vanishes at $|w| = \lambda$?
- Page 41, line 2: we proceed as values
- Page 41, line 16: turbulence turbulence.

References

Gneiting, T. and Ševčíková, H. and Percival, D. B. and Schlather, M. and Jiang, Y. *Fast and exact simulation of large Gaussian lattice systems in $\mathbb{R}^2$: exploring the limits*. Journal of Computational and Graphical Statistics, 15, 483–501, 2006.