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MULTISCALED SOLITARY WAVES

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Abstract. It is analytically shown how competing nonlinearities yield new multiscaled (multi humped) structures for internal solitary waves in shallow fluids. These solitary waves only exist for large amplitudes beyond the limit of applicability of the KdV equation or its usual extensions. Multiscaling phenomenon exists or do not exist for almost identical density profiles. Trapped core inside the wave prevents appearance of such multiple scales within the core area. It is anticipated that multiscaling phenomena exist for solitary waves in various physical origins.

1 *Introduction*

The typical horizontal scale (or scales) is a major characteristic of a plane disturbance propagating in a nonuniform medium. Usually, in an ideal, density stratified shallow fluid, a wave of small, albeit finite amplitude has one typical scale resulting from the (local) balance between nonlinearity and dispersion like in the realm of Korteveg-de Vries (KdV) equation (Helfrich and Melville , 2006). If the wavelength of the disturbance is too small for displaying, for instance, capillary dispersion, multiscaled solitary waves are possible as shown by Benjamin (1992). They exist due to the competition of gravitational and capillary dispersion. Similar effect is observed if viscosity is taken into account. The KdV type equation with viscosity term governing the behaviour of a disturbance in that system allows solution in the form of oscillatory bore as mentioned, for example, by Grimshaw et al. (2010) . Oscillations of smaller scale superimposed on the smooth front clearly display a two-scale structure of the bore. This structure is the result of the combined manifestation of gravitational and dissipative effects in the dispersion relation. In the present note it is shown that for gravitational dispersion, ignoring all the other earlier mentioned effects, multiscaled solitary waves are possible. These solutions exist only for disturbances of finite amplitude exceeding the range of applicability of eKdV model, which incorporates both quadratic and cubic nonlinearities. Existing small amplitude models can not predict such waves, because they account for higher nonlinearities generating longer scales only as a correction to the lowest one Grimshaw et al. (2010). For multiscales to occur the various competitive nonlinearities should be of the same order and that order needs to be higher then the cubic one as analytically discussed below. This effect was initially noticed by Derzho and Borisov (1990) in Russian journal. Recently Dunphy et al. (2011) presented numerical results on two humped and usual one humped solitary internal waves for nearly identical density profiles. However, neither specific nonlinearity in terms of power series in wave amplitude necessary to reveal a two humped structure or regions of density profiles at which such structures exist were not presented. It is worth noting that family of solutions is richer then two-humped structures. It is

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expectable that such multiscaling solitary waves exist in other physical systems where complicated competitive nonlinearities are balanced by dispersion.

2 *Model for internal waves*

Let us consider the two-dimensional steady motion of an ideal stratified fluid in a frame of reference moving with phase speed of wave c . Assuming stratification to be in the form

$$\rho_0\left(z\right) = \rho_{00}(1-\sigma(z+\delta f(z))), \delta << 1, \sigma << 1, f \sim 1, \tag{1}$$

where σ denotes Boussinesq parameter. In (Derzho and Velarde, 1995) it was shown that for this case the dimensionless (primed) streamfunction $\psi' = -\psi/cH$ of a solitary disturbance obeys the equation

$$\psi_{zz} + \mu^2\psi_{xx} + \lambda(\psi - z) - \frac{\sigma}{2}({\psi_z}^2 - 1 - 2\psi\lambda(\psi - z)) + \delta\lambda(\psi - z)f_\psi(\psi) = o(\sigma, \delta, \mu^2), \tag{2}$$

where μ is the aspect ratio H/L and $\lambda = \frac{\sigma g H}{c^2}$.

In Eq.(2) z denotes the vertical axis, taken positive upwards and x corresponds to the horizontal axis; z and x are scaled with H and L, the given vertical and horizontal scales respectively. Expecting no confusion we have, for simplicity, dropped the primes in Eq.(2). Let us locate the bottom and the surface at the dimensionless heights $z = -0.5$ and $z = 0.5 + \eta(x)$, respectively, where $\eta(x)$ denotes surface displacement. The boundary conditions at the bottom and surface are

$$\psi_x = 0 \text{ at } z = -0.5 \tag{3}$$

$$\sigma(\psi_x\psi_z\psi_{zz} - \psi_z^2\psi_{zx}) + \lambda\psi_x = o(\sigma) \text{ at } z = 0.5 + \eta(x) \tag{4}$$

$$\psi_x = -\eta_x\psi_z \tag{5}$$

The solution of Eqs.(2-5) is searched in the form

$$\psi = \psi^{(0)} + \mu^2\psi^{(1)} + ..., \lambda = \lambda^{(0)} + \mu^2\lambda^{(1)} + ..., \eta = \eta^{(0)} + \mu^2\eta^{(1)} + ..., \tag{6}$$

where zeroth order variables are of order unity. Below we shall provide solution for the first mode, which is most frequently observed in nature. The analysis for the higher modes is similar. In the zeroth order

$$\psi^{(0)} = z + A(x)cos(\pi z), \lambda^{(0)} = \pi^2, \eta^{(0)} = 0 \tag{7}$$

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where the amplitude function $A(x)$ is to be determined at a higher order. For the solution to the first order equation to exist the solvability condition (Fredholm alternative) demands

$$A_{xxx}+\lambda^{(1)}A_x-\frac{\sigma}{\mu^2}(2A_x-8\pi AA_x+2\pi^2A^2A_x)+2\frac{\delta}{\mu^2}Q_x(A)=0 \tag{8}$$

$$Q(A)=A\int\limits_{-0.5}^{0.5}cos^2(\pi z)f_\psi(\psi=\psi^{(0)})dz \tag{9}$$

In order to (locally) balance nonlinearity and dispersion we have to require $max(\sigma/\mu^2,\delta/\mu^2)\sim 1$ thus determining L. Benney and Ko (1978) suggested to consider the nonlinear terms as power series in the Boussinesq parameter instead of small amplitude parameter. Derzho and Velarde (1995) somewhat generalized this idea. After straightforward integrations for remaining solitary wave solution, Eqs.(8-9) can be reduced to

$$A_x^2+\lambda^{(1)}A^2+2\frac{\delta}{\mu^2}\int\limits_0^A Q(A')dA'+A^2(\frac{8\pi A}{3}-2-\frac{\pi^2A^2}{3})=0. \tag{10}$$

Since an analytical function within a limited interval can be represented with prescribed accuracy in the form of some Nth-order polynomal (Korn and Korn , 1968) it follows that

$$\int\limits_0^A Q(A')dA'=A^2P_N(A), \tag{11}$$

For the wave of amplitude A_0 Eq.(10) yields

$$\frac{A_x^2}{A^2}=(A_0-A)\Phi(A), \tag{12}$$

$$\Phi(A)=2\frac{\delta}{\mu^2}\frac{P_N(A_0)-P_N(A)}{A_0-A}+\frac{\delta}{\mu^2}\frac{\pi^2}{3}(\frac{8}{\pi}-A-A_0) \tag{13}$$

$$\lambda^{(1)}=\frac{\sigma}{\mu^2}(-\frac{8\pi A}{3}+2+\frac{\pi^2A^2}{3})-2\frac{\delta}{\mu^2}P_N(A_0) \tag{14}$$

Equations (12-14) determine completely both profile and phase velocity of a solitary wave with amplitude A_0.

3 *Multiscaling*

The function f in the form of a Mth-order polynomial generates P_N with the index $N=M-1$. The power index of Φ is thus $max(1,M-2)$. The condition for Eq.(8) to possess a multiscaled solution reduces to the condition that $\Phi(A)$ must

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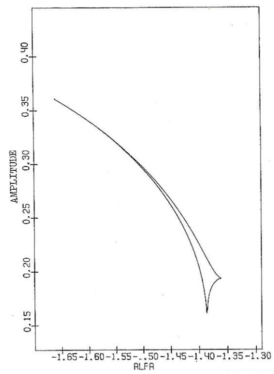

Figure 1. Existence domain for two humped solitary wave

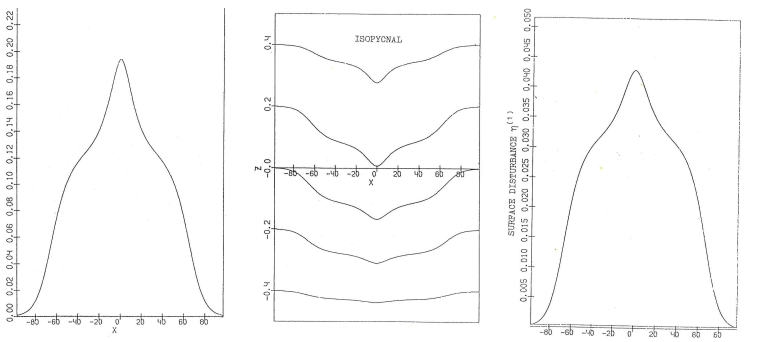

Figure 2. Amplitude function, streamlines and surface profile for typical two humped solitary wave

be sign-defined with several extrema within $[0, A_0\ ]$. Thus it must have more than two imaginary roots in that interval. It determines $M \geq 4$, i.e. for a stratification in the form of cubic polynomial or if wave amplitude is small enough to neglect A^4 and higher order nonlinearities, multiscaled solitary waves do not exist because f has no imaginary roots for this case. This is why classical KdV or mKdV can not provide multiscaled solitary waves over flat bottom. The region of existence of two humped solitary waves is shown in Fig.1.

The profile of stratification for this case is

$$\rho_0(z) = \rho(1 - \sigma z + 0.5\sigma^2 z^2 + \alpha\sigma^2 z^4), \tag{15}$$

which produces quadratic, cubic and quartic terms in Eq.(8). A two-humped solitary wave with amplitude A_0= 0.194 for the particular stratification profile Eq.(15) with $\alpha = -1.39$ and $\sigma = 0.01$ is shown in Fig. 2.

Indeed the maximum derivative on x in the dimensionless coordinates is of order unity. However, the wave has a pronounced two-scale structure with typical lengthscales which are much larger than the length L used to scale the derivative. A solitary wave with three typical lengthscales (tree-humped one) is shown in Fig. 3.

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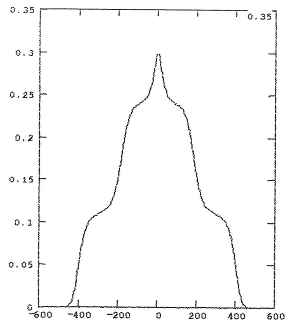

Figure 3. Amplitude function for three humed solitary wave

For this case the stratification profile is

$$\rho_0(z) = \rho(1 - \sigma z + \sigma^2(1.206z^2 - 4.37z^3 - 3.435z^4 - 33.407z^6)),$$

which produces in Eq. (8) nonlinear terms up to A^6. Generally, one can expect at most $M/2$ different scales for a stratification in the form of polynomial with even power index M, and $(M-1)/2$ otherwise. The theory described above is valid for wave amplitudes below A_*, a certain amplitude at which a vortex core started to appear inside the wave. For nearly linear density profile $A_* = 1/\pi$. Derzho and Grimshaw (1997) have shown that

$$B_x^2 \sim R(A_*)(1-B) - \frac{8\nu}{15}(1 - B^{5/2}), \tag{16}$$

where ν is the supercriticality parameter defined such that B varies from zero to one as wave amplitude does from A_* to the maximum value allowed predicted there. $R(A_*)$ depends on stratification profile and is fixed. It is straightforward to notice that $B(x)$ is monotonic and therefore multiscaling does not exist when $A > A_*$. Multiscaling effects similar to the discussed above, could be observed in various physical media. Derzho and Grimshaw (2005) reported that solitary Rossby waves in channels obey the same KdV type equation with complicated nonlinearity due to mean shear variations. Coriolis force for Rossby waves plays the same role as gravitational force for the internal gravity waves. The results on multiscalng for Rossby waves with and without trapped core will be reported elsewhere.

4 *Conclusion*

For the particular case of a nonlinear dispersive medium like a shallow stratified fluid embedded in the gravity field, we have obtained multiscaled solitary waves which are predicted when there exists competition of several different types of nonlinearity. The mechanism leading to these solutions differs from the mechanism of multiscaling due to the competition of different types of dispersion or effects due to the dissipation. We have shown that the length used to scale the x-derivative does not simply coincide with the typical length scale of the wave, as for KdV. Moreover, multiscaled (multi humped) disturbances exist for sufficiently large amplitudes, at least terms in forth order of waves amplitude should be accounted . Multiscaling (multi humped) phenomenon exists or do not exist for almost identical density profiles. Trapped core inside the wave prevents appearance of such multiple scales within the core area. It is noted that multiscaling phenomena exist for solitary waves in various physical origins, for example, for Rossby waves on a shear flow.

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