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The Lagrange form of the nonlinear Schrödinger equation for low-vorticity waves in deep water: rogue wave aspect

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Abstract:

The nonlinear Schrödinger equation (NLS equation) describing weakly rotational wave packets in an infinity-depth fluid in the Lagrangian coordinates is derived. The vorticity is assumed to be an arbitrary function of the Lagrangian coordinates and quadratic in the small parameter proportional to the wave steepness. It is proved that the modulation instability criteria of the low-vorticity waves and deep water potential waves coincide. All the known solutions of the NLS equation for rogue waves are applicable to the low-vorticity waves. The effect of vorticity is manifested in a shift of the wave number in the carrier wave. In case of vorticity dependence on the vertical Lagrangian coordinate only (the Gouyon waves) this shift is constant. In a more general case, where the vorticity is dependent on both Lagrangian coordinates, the shift of the wave number is horizontally heterogeneous. There is a special case with the Gerstner waves where the vorticity is proportional to the square of the wave amplitude, and the resulting non-linearity disappears, thus making the equations of the dynamics of the Gerstner wave packet linear. It is shown that the NLS solution for weakly rotational waves in the Eulerian variables can be obtained from the Lagrangian solution by the ordinary change of the horizontal coordinates.

Keywords: nonlinear Schrödinger equation, vorticity, rogue waves

I. Introduction

The focusing nonlinear Schrödinger (NLS) equation is an effective model to study the emergence of rogue waves: single or sometimes a group of several waves, the amplitude of which exceeds the average surrounding background wave level more than twice. Considered initially for ocean waves [Osborne, 2010; Dysthe et al, 2008; Kharif and Pelinovsky, 2003; Kharif et al, 2009; Slunyaev et al, 2011], nowadays the rogue waves concept is extended to other fields of physics, such as nonlinear optics [Solli et al, 2007], plasma physics [Moslem et al, 2011],

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astrophysics [El-Labany et al, 2012; Sabry, 2015], superfluid helium [Efimov et al, 2010] and Bose-condensate systems [Zhao, 2013].

Numerous analytical and numerical solutions of the NLS equation demonstrate the formation of a few abnormally large crests and troughs with amplitudes corresponding to the rogue wave definition. The emergence of these extreme waves is associated with modulation instability of a wave train in relation to envelope long-wave modulation [Benjamin and Feir, 1967; Zakharov, 1968]. The role of the modulation instability in the rogue wave generation on the sea surface is generally accepted at present; see reviews: Dysthe et al., 2008; Kharif and Pelinovsky, 2009; Kharif et al., 2009; Slunyaev et al., 2011. A recent survey of rogue wave phenomena in various media supporting nonlinear wave self-modulation may be found in Akhmediev and Pelinovsky, 2010 and Onorato et al., 2013.

The NLS equation for deep water waves was first derived by Benny and Newell, 1967 and then Zakharov, 1968 who used the Hamiltonian formalism. Hashimoto and Ono, 1972 and Davey, 1972 independently obtained the same result. Like Benney and Newell, 1967 they use the method of multiple scale expansions in the Euler coordinates. In their turn, Yuen and Lake derived the NLS equation on the basis of the averaged Lagrangian method [Yuen and Lake, 1975]. Benney and Roskes extended these two-dimensional theories in the case of three-dimensional wave perturbations in finite depth fluid and obtained the equations which are now called the Davey-Stewartson equations [Benney and Roskes, 1969]. In this particular case the equation proves the existence of transverse instability of the flat wave which is much stronger than the longitudinal one. This circumstance diminishes the role and meaning of the NLS equation for sea applications. Meanwhile, the 1-D NLS equation has been successfully tested numerous times in laboratory wave tanks and under comparison of the natural observations with the numerical calculations. It is because of this fact the 1-D NLS equation is applied in many works devoted to rogue waves; see [Kharif et al, 2009; Slunyaev et al, 2011] and references herein.

In all of the above-mentioned works the wave motion was assumed as potential. However, the formation of rogue waves frequently occurs against the background of the shear flow possessing vorticity. Wave train modulations upon arbitrary vertically sheared currents were studied by Benney and his group. By using the method of multiple scales Johnson (1976) examined the slow modulation of the harmonic wave, moving over the surface of an arbitrary shear flow with the $U(y)$ velocity profile, where y is the vertical coordinate. He derived the NLS equation with the coefficients, which in a complicated way depend on the shear flow and gave the condition of linear stability of the nonlinear plane wave solution [Johnson, 1976]. Oikawa et al. considered the instability properties of weakly nonlinear wave packets to three-dimension perturbations in the presence of shear flow [Oikawa et al, 1985]. Their system of equations is reduced to the familiar NLS equation when confining the evolution to be purely two-dimensional. Li et al.

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(1987) and Baumstein (1998) studied the modulation instability of a Stokes wave train and derived the NLS equation for deep water in a uniform shear flow, when $U(y)=\Omega_0 y$, $\Omega_z=\Omega_0$ is constant vorticity (z is the horizontal coordinate, transversal to the flow plane X,Y; the wave propagates in the x direction). Thomas et al. (2012) generalized their results in the case of the finite-depth fluid and confirmed that linear shear flow may significantly modify the stability properties of weakly nonlinear Stokes waves. In particular, for the waves propagating in the direction of the flow the Benjamin-Feir (modulation) instability can vanish in the presence of positive vorticity ($\Omega_0<0$) for any depth.

In the traditional Eulerian study of weakly nonlinear wave propagation on the current the shear flow determines the vorticity of the zero approximation. Depending on the flow profile $U(y)$, it may be sufficiently arbitrary and is equal to $-U'(y)$. At the same time the vorticity of wave perturbations $\Omega_n, n\geq 1$, i.e. the vorticity in the first and subsequent approximations by the parameter of wave steepness $\varepsilon=kA_0$ (k is the wave number, A_0 is the wave amplitude) depends on its type. In the Eulerian coordinates the vorticity of wave perturbations are the functions not only of *y*, but depend on variables *x* and *t* as well. Plane waves on a shear flow with the linear vertical profile represent an exception from this statement [Li et al, 1987; Baumstein, 1998; Thomas et al, 2012]. For such waves the vorticity of the zero approximation is constant, and all of the vorticities in wave perturbations are equal to zero. For the arbitrary vertical profile of the shear flow [Johnspn, 1976] expressions for the functions Ω_n could be hardly predicted even quantitatively.

The Lagrangian method allows applying a different approach. It is known that the fluid particle vorticity in the plane flow is preserved and can be expressed via the Lagrangian coordinates only. Thus, not only the vertical profile of the shear flow defining the zero approximation vorticity, but the expressions for the vorticity of the following orders of smallness can be given as the known initial conditions as well. The expression for the vorticity is presented in the following form:

$$\Omega(a,b)=-U'(b)+\sum_{n\geq 1}\varepsilon^n\Omega_n(a,b),$$

here a,b are the horizontal and the vertical Lagrangian coordinates respectively, $U(b)$ is the vertical profile of the shear flow, and the particular conditions for the function Ω_n definition can be found while solving of the problem. For the given shear flow this approach allows one to study wave perturbations with the most general law of the vorticities Ω_n distribution. In the present paper the shear flow and the vorticity in the linear approximation are absent ($U=0; \Omega_1=0$), but the vorticity in the quadratic approximation is an arbitrary function. This corresponds to the rotational flow proportional to ε^2. We can define both the shear flow and the

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localized vortex. The dynamics of plane wave trains on the background of flows with an arbitrary low vorticity has not been studied earlier.

An idea to study wave trains with the quadratic (with respect to the wave steepness parameter) vorticity has been realized earlier for the spatial problems in the Euler variables. Hjelmervik and Trulsen (2009) derived the NLS equation for the vorticity distribution [26]:

$$\Omega_y/\omega = O(\varepsilon^2); \quad (\Omega_x, \Omega_z)/\omega = O(\varepsilon^3),$$

here ω is the wave frequency. The vertical vorticity of wave perturbations by a factor of ten exceeds the other two vorticity components. This vorticity distribution corresponds to the low (order of ε) velocity of the horizontally inhomogeneous shear flow. The authors [Hjelmervik and Trulsen, 2009] used the NLS equation to study the rogue wave statistics on narrow current jets, and Onorato et al. (2011) used this equation to study the opposite flow rogue waves. The low vorticity effect (order of magnitude ε^2) in [Hjelmervik and Trulsen, 2009] is reflected in the NLS equation. This fact, in the same way as the NLS nonlinear term for plane potential waves, should be explained by the presence of an average current which is non-uniform in terms of depth.

Colin et al. (1995) considered the evolution of three-dimensional vortex disturbances in the finite depth fluid for a different type of vorticity distribution:

$$\Omega_y = 0; \quad (\Omega_x, \Omega_z)/\omega = O(\varepsilon^2)$$

and by means of the multiple scale expansion method in Eulerian variables reduced the problem to solving the Davey-Stewartson equations. In this case, after solving the problem, vorticity components are calculated by the found second approximation velocity components. As well as for the traditional Eulerian approach [Johnson, 1976], the form of quadratic vorticity distribution is very special and does not cover all of its numerous possible distributions.

In this paper we consider the plane problem of nonlinear wave packet propagation in an ideal incompressible fluid with the following form of vorticity distribution:

$$\Omega_z/\omega = O(\varepsilon^2).$$

In contrast to [Hjelmervik and Trulsen, 2009; Onorato et al., 2011; Colin et al, 1996], the flow is two-dimensional (respectively $\Omega_x = \Omega_y = 0$). The propagation of the potential wave packet causes the weak return flow underneath the free water surface which is proportional to the wave steepness square [McIntyre, 1982]. In the considered problem this potential flow is superimposed on the rotational one of the same order. It results in the appearance of the additional term in the NLS equation. The presence of the rotational flow changes the wave phase and its velocity but not

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the wave amplitude. So, we obtain a small correction to the NLS solutions derived for a strictly potential motion.

The examination is held in the Lagrangian variables. The Lagrangian variables are rarely used in fluid mechanics. This is due to a more complex type of nonlinear equations in the Lagrange form. However, when considering the vortex-induced oscillations of free fluid surface, the Lagrangian approach has two major advantages. Firstly, unlike the Euler description method, the shape of the free surface is known and determined by the condition of the vertical Lagrangian coordinate being equal to zero ($b=0$). Secondly, with a planar motion liquid particle vorticity is preserved and is a function of the Lagrangian variables $\Omega_z = \Omega_z(a,b)$, so the type of vorticity distribution in fluid can be set at the beginning. Euler's approach does not allow to do this. In this case the second-order vorticity is defined as a known function of the Lagrangian variables.

Here hydrodynamic equations in the Lagrange form are solved by the multiple scale expansion method. The variable-coefficient non-linear Schrödinger equation is derived. The ways to reduce it to the constant-coefficient of the NLS equation are studied. The vorticity role in the rogue wave formation is discussed.

The paper is organized as follows. Section 2 describes the Lagrangian approach to the study of fluid free surface wave oscillations. The free surface corresponds to the value of the Lagrangian vertical zero coordinate, which facilitates the formulation of the pressure boundary conditions. The peculiarity of the suggested approach is the introduction of the fluid particle trajectory complex coordinate. In Section 3 the nonlinear evolution equation on the basis of the method of the multiple scale expansion is derived. It is shown that for Gerstner waves it becomes linear. In Section 4 it is shown that for the low-vorticity waves under consideration, the modulation instability criterion is the same as for potential waves in deep water. Vorticity changes the phase of the wave packet and, consequently, the number of individual waves in breather which is the analytical representation of rogue waves. Section 5 summarizes the obtained results.

2. Basic equations in the Lagrangian coordinates

We are interested in the propagation of gravity surface wave packet in rotational infinity-depth fluid. The 2D hydrodynamic equations of an incompressible inviscid fluid in the Lagrangian coordinates have the following form [Lamb, 1934; Bennet, 2006; Abrashkin and Yakubovich, 2006; Abrashkin and Soloviev, 2013; Abrashkin and Oshmarina, 2016]:

$$\frac{D(X,Y)}{D(a,b)} = [X,Y] = 1, \tag{1}$$

$$X_{tt} X_a + (Y_{tt} + g) Y_a = -\frac{1}{\rho} p_a, \tag{2}$$

$$X_{tt} X_b + (Y_{tt} + g) Y_b = -\frac{1}{\rho} p_b, \tag{3}$$

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where X,Y are horizontal and vertical Cartesian coordinates and a,b are the horizontal and vertical Lagrangian coordinates of fluid particles, t is time, ρ is fluid density, p is pressure, g is acceleration of gravity, the subscripts mean differentiation by the corresponding variable. The square brackets denote the Jacobian. The axis b is directed upwards, and $b=0$ corresponds to the free surface. The equation (1) is a volume conservation equation. Equations (2) and (3) are momentum equations. The problem geometry is presented in Fig. 1.

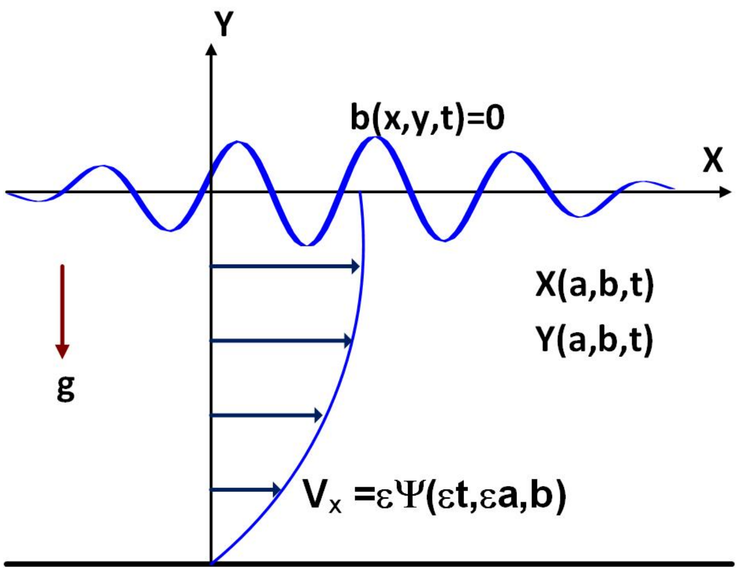

Fig. 1. Problem geometry: V_X **is the average current.**

By using the cross differentiation it is possible to exclude the pressure and to obtain the vorticity conservation condition along the trajectory [Lamb, 1934; Bennet, 2006; Abrashkin and Yakubovich, 2006; Abrashkin and Soloviev, 2013; Abrashkin and Oshmarina, 2016]:

$$X_{ta}X_b + Y_{ta}Y_b - X_{tb}X_a - Y_{tb}Y_a = \Omega(a,b). \tag{4}$$

This equation is equivalent to the momentum equations (2) and (3), but it involves explicit vorticity of liquid particles Ω, which in the case of two-dimensional flows is the function of the Lagrangian coordinates alone.

We introduce the complex coordinate of the fluid particle trajectory $W = X + iY$ $(\overline{W} = X - iY)$, the line is a sign of complex conjugation. In the new variables the equations (1) and (4) take the following form:

$$\left[W, \overline{W}\right] = -2i, \tag{5}$$

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$$\mathrm{Re}\left[W_t, \overline{W}\right] = \Omega(a,b), \tag{6}$$

The system of equations (2) and (3) after simple algebraic manipulations is reduced to the following single equation:

$$W_{tt} = -ig + i\rho^{-1}[p, W]. \tag{7}$$

Further on, equations (5) and (6) will be used to find the coordinates of the complex trajectories of fluid particles, and the equation (7) will be used to determine the fluid pressure. The boundary conditions are the conditions of impermeability at the bottom ($Y_t \to 0$ at $b \to -\infty$) and constant pressure on the free surface (at $b = 0$) .

The Lagrangian coordinates represent the labels of the fluid particles and are the functions of the variables X, Y, t (that is shown in Fig. 1 for the vertical coordinate). In the Eulerian description the displacement of the free surface $Y_s(X,t)$ is calculated in an explicit form, but in the Lagrangian description it is defined parametrically with the following equalities: $Y_s(a,t) = Y(a, b=0, t)$; $X_s(a,t) = X(a, b=0, t)$, where the role of a parameter is played by the Lagrangian horizontal coordinate a. Its value along the free surface $b = 0$ varies in the range $(-\infty;\infty)$. In the Lagrangian coordinates the function $Y_s(a,t)$ defines the free surface displacement.

3. Evolution equation derivation

We use the multiple scales method. Let us present the function W as follows:

$$W = a_0 + ib + w(a_l, b, t_l), \quad a_l = \varepsilon^l a, \quad t_l = \varepsilon^l t; \quad l = 0,1,2, \tag{8}$$

where ε is the small parameter of the wave steepness, all the unknown functions and the given vorticity can be represented as a series in this parameter:

$$w = \sum_{n=1} \varepsilon^n w_n; \quad p = p_0 - \rho g b + \sum_{n=1} \varepsilon^n p_n; \quad \Omega = \sum_{n=1} \varepsilon^n \Omega_n(a,b). \tag{9}$$

In the formula for the pressure the term with hydrostatic pressure is selected, p_0 is constant atmospheric pressure on the fluid surface. Let us substitute the representations (8) and (9) in the equations (5)-(7).

Linear approximation. In the first approximation in the small parameter we have the following system of equations:

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$$\mathrm{Im}\left(iw_{1a_0} + w_{1b} \right) = 0, \tag{10}$$

$$\mathrm{Re}\left(iw_{1a_0} + w_{1b} \right)_{t_0} = -\Omega_1, \tag{11}$$

$$w_{1t_0t_0} + \rho^{-1}\left(p_{1a_0} + ip_{1b} \right) = igw_{1a_0}. \tag{12}$$

The solution satisfying the continuity equation (10) and the vorticity conservation equation (11) describes a monochromatic wave (for definiteness, we consider the wave propagating to the left) and the average horizontal current

$$w_1 = A(a_1, a_2, t_1, t_2)\exp[i(ka_0 + \omega t_0) + kb] + \psi_1(a_1, a_2, b, t_1, t_2); \quad \Omega_1 = 0, \tag{13}$$

here A is the complex amplitude of the wave, ω is its frequency, and k is the wave number. The function ψ_1 is real, and it will be determined upon consideration the following approximation.

Substituting the solution (13) in the equation of motion (12), we obtain the equation for the pressure

$$\rho^{-1}\left(p_{1a_0} + ip_{1b} \right) = \left(\omega^2 - gk \right) A\exp[i(ka_0 + \omega t_0) + kb], \tag{14}$$

which is solved analytically

$$p_1 = -\mathrm{Re}\,\frac{i(\omega^2 - gk)}{k}\rho A\exp[i(ka_0 + \omega t_0) + kb] + C_1(a_1, a_2, t_1, t_2), \tag{15}$$

where C_1 is an arbitrary function. On the free surface the boundary condition is $p_1|_{b=0} = 0$, which leads to $\omega^2 = gk$, as well as $C_1 = 0$. Thus, in the first approximation the pressure correction p_1 is identically equal to zero.

Quadratic approximation. The equations of the second order of the perturbation theory can be written as follows:

$$\mathrm{Im}\left(iw_{2a_0} + w_{2b} + iw_{1a_1} - w_{1a1}\overline{w_{1b}} \right) = 0, \tag{16}$$

$$\mathrm{Re}\left[iw_{2t_0a_0} + w_{2t_0b} + i\left(w_{1t_0a_1} + w_{1t_1a_0} \right) - w_{1t_0a_0}\overline{w_{1b}} + w_{1t_1b} + w_{1t_0b}\overline{w_{1a_0}} \right] = -\Omega_2, \tag{17}$$

$$w_{2t_0t_0} + \rho^{-1}\left(p_{2a_0} + ip_{2b} \right) = ig\left(w_{2a_0} + w_{a_1} \right) - 2w_{1t_1t_0}. \tag{18}$$

Substituting the expression (13) for w_1 in the equation of continuity (16), we get:

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$$\mathrm{Im}\left[iw_{2a_0}+w_{2b}-i\left(k\psi_{1b}A-A_{a_1}\right)\exp\left[i\left(ka_0+\omega t_0\right)+kb\right]-ik^2|A|^2e^{2kb}+i\psi_{1a_1}\right]=0, \quad (19)$$

which is integrated as follows:

$$w_2=i\left(kA\psi_1-bA_{a_1}\right)\exp\left[i\left(ka_0+\omega t_0\right)+kb\right]+\psi_2+if_2, \quad (20)$$

here ψ_2, f_2 are the slow coordinates functions and the Lagrange vertical coordinate b and:

$$f_{2b}=k^2|A|^2\exp 2kb-\psi_{1a_1}, \quad (21)$$

the function ψ_2 is an arbitrary real function. It will be determined by solving the following cubic approximation.

When substituting (13), (20) in (17), all terms containing the exponential factor cancel each other, and the remaining terms satisfy the equation:

$$\psi_{1t_1b}=-2k^2\omega|A|^2\exp(2kb)-\Omega_2. \quad (22)$$

The expression for the function ψ_1 can be determined by simple integration. It should be emphasized that the vorticity of the second approximation, being a part of the equation (22), is an arbitrary function of the slow horizontal and vertical Lagrange coordinates, that is $\Omega_2=\Omega_2(a_1,a_2,b)$.

Taking into account the solutions of the first two approximations, we can write the equation (18) as:

$$\rho^{-1}\left(p_{2a_0}+ip_{2b}\right)=i\left(gA_{a_1}-2\omega A_{t_1}\right)\exp\left[i\left(ka_0+\omega t_0\right)+kb\right]+ig\psi_{1a_1}. \quad (23)$$

Its solution determines the pressure correction:

$$p_2=\mathrm{Re}\left[\frac{1}{k}\left(gA_{a_1}-2\omega A_{t_1}\right)\exp\left[i\left(ka_0+\omega t_0\right)+kb\right]\right]+\rho g\int_0^b\psi_{1a_1}db+C_2\left(a_1,a_2,t_1,t_2\right) \quad (24)$$

The integration limits in the penultimate term are chosen so that this integral term is zero on the free surface. Due to the boundary condition for pressure ($p_2(b=0)=0$), $C_2=0$, and

$$A_{t_1}-c_gA_{a_1}=0;\quad c_g=\frac{g}{2\omega}=\frac{1}{2}\sqrt{\frac{g}{k}}, \quad (25)$$

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Here c_g is the group velocity of wave propagation in deep water, which in this approximation is independent on the fluid vorticity. As expected, the wave of this approximation moves with the group velocity c_g to the left (the "minus" sign in the equation (25)).

Cubic approximation. The continuity equation and the vorticity conservation condition in the third approximation have the form

$$\mathrm{Im}\left[iw_{2a_0} + w_{3b} + i\left(w_{1a_2} + w_{2a_1} + w_{2a_0} \right) - \left(w_{1a_1} + w_{2a_2} \right)\overline{w_{1b}} - w_{1a_0}\overline{w_{2b}} \right] = 0, \quad (26)$$

$$\begin{aligned}\mathrm{Re}\Big[& iw_{3t_0a_0} + w_{3t_0b} + i\left(w_{1t_2a_0} + w_{1t_1a_1} + w_{1t_0a_2} + w_{2t_1a_0} + w_{2t_0a_1} \right) + w_{1t_2b} - \overline{w_{2b}}w_{1t_0a_0} - \\ & + w_{2t_1b} - w_{1b}\left(w_{1t_0a_1} + w_{1t_1a_0} + w_{2t_0a_0} \right) + +\overline{w_{1a_0}}\left(w_{1t_1b} + w_{2t_0b} \right) + w_{1t_0b}\left(\overline{w_{1a_1}} + \overline{w_{2a_0}} \right)\Big] = -\Omega_3.\end{aligned} \quad (27)$$

We substitute the solutions of the first and second approximations in the system of equations:

$$\mathrm{Im}\left[iw_{3a_0} + w_{3b} + i\left(\psi_{1a_2} + \psi_{2a_1} \right) + 2k(kb+1)A\overline{A_{a_1}}e^{2b} + G_b e^{i\left(ka_0 + \omega t_0\right) + kb} \right] = 0, \quad (28)$$

$$\begin{aligned}\mathrm{Re}\Big\{ & \left[iw_{3a_0} + w_{3b} + \left(G_b + 2k\psi_{1t_1b}\omega^{-1}A \right)e^{i\left(ka_0 + \omega t_0\right) + kb} \right]_{t_0} + \psi_{2t_1b} + \psi_{1t_2b} + \\ & + i\omega k\left(4kb+5\right)A\overline{A_{a_1}}e^{2kb} \Big\} = -\Omega_3,\end{aligned} \quad (29)$$

$$G = ibA_{a_2} + \frac{b^2}{2}A_{a_1a_1} - (kb+1)\psi_1 A_{a_1} - \left(ik\psi_2 + kf_2 - \frac{k^2}{2}\psi_1^2 \right)A. \quad (30)$$

We get the solution to the third approximation in the following form:

$$w_3 = \left(G_1 - G\right)e^{i\left(ka_0 + \omega t_0\right) + kb} + G_2 e^{-i\left(ka_0 + \omega t_0\right) + kb} + \psi_3 + if_3, \quad (31)$$

here G_1, G_2, ψ_3, f_3 are the slow coordinates functions and b. Substituting this expression in (28) and (29), we immediately find that:

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$$f_{3b} + \psi_{2a_1} + \psi_{1a_2} + k(kb+1)\left(A\overline{A_{a_1}} - \overline{A}A_{a_1}\right)e^{2kb} = 0, \tag{32}$$

$$\psi_{2t_1 b} + \psi_{1t_2 b} + \frac{1}{2}(4kb+5)\omega k\left(A\overline{A_{a_1}} - \overline{A}A_{a_1}\right)e^{2kb} = -\Omega_3. \tag{33}$$

The function ψ_2, according to the relation (33) is determined by a known solution for A and ψ_1, and the given distribution Ω_3. The expression for the function f_3 is derived then from the equation (32). These functions determine respectively the horizontal and vertical average movements, but in this approximation they are not included in the evolution equation for the envelope. The function ψ_3 should be determined in the next approximation.

Solving (28) and (29), we find:

$$G_1 = -k\omega^{-1}\psi_{1t_1}A; \quad G_2 = k\omega^{-1}\left(2ke^{-2kb}\int_{-\infty}^{b}\psi_{1t_1}e^{2kb'}db' - \psi_{1t_1}\right)\overline{A}. \tag{34}$$

These relationships should be substituted in the motion equation (7), which in this approximation has the following form:

$$w_{3t_0t_0} - igw_{3a_0} = i\rho^{-1}\left[i\left(p_{2a_1} + p_{3a_0}\right) - p_{3b} - p_{2b}w_{1a_0} + \rho g\left(w_{1a_2} + w_{2a_1}\right)\right] - \\ -2w_{1t_2t_0} - w_{1t_1t_1} - 2w_{2t_0t_1}. \tag{35}$$

Taking into account (13), (20), (24), (31) and (34) we rewrite it as follows:

$$\rho^{-1}\left(p_{3a_0} + ip_{3b}\right) = \left(-2i\omega\frac{\partial A}{\partial t_2} + ig\frac{\partial A}{\partial a_2} - \frac{\partial^2 A}{\partial t_1^2} + 2\omega k\psi_{1t_1}A\right)e^{i(ka_0+\omega t_0)+kb} + \\ + 2\omega^2 G_2\overline{A}e^{-i(ka_0+\omega t_0)+kb} + ig\left(\psi_{2a_1} + \psi_{1a_2}\right) + I; \quad I = -g\left(f_{2a_1} - \int_b^0 \psi_{1a_1a_1}db\right) - \psi_{t_1t_1}. \tag{36}$$

Due to the relationships (21), (22) and (25) the derivative of I by the vertical Lagrangian coordinate is zero ($I_b = 0$), so I is the only function of the slow coordinates and time - $a_l, t_l, l \geq 1$. The contribution to the pressure of that member $I(a_l, t_l) \neq 0$ will be complex, so it should be $I = 0$.

Solving the equation (36), we find pressure perturbation in the third approximation:

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$$\frac{p_3}{\rho} = \mathrm{Re}\, ik^{-1}\left(2i\omega\frac{\partial A}{\partial t_2} - ig\frac{\partial A}{\partial a_2} + \frac{\partial^2 A}{\partial t_1^2} - 4\omega k^2 A e^{-2kb}\int_{-\infty}^{b}\psi_{1t_1} e^{2kb'}\,db'\right)e^{i\left(ka_0+\omega t_0\right)+kb} + \\ + \rho g\int_0^b\left(\psi_{2a_1} + \psi_{1a_2}\right)db'. \quad (37)$$

In the expression (37) the limits of integration for the second integral term have been pre-selected to satisfy the boundary condition on the free surface (the value of pressure p_3 should turn zero). Then the factor before the exponent should be equal to zero:

$$2i\omega\frac{\partial A}{\partial t_2} - ig\frac{\partial A}{\partial a_2} + \frac{\partial^2 A}{\partial t_1^2} - 4\omega k^2 A\int_{-\infty}^{0}\psi_{1t_1} e^{2kb}\,db = 0. \quad (38)$$

Introducing the "running" coordinate $\zeta_2 = a_2 + c_g t_2$, we rewrite (38) in a compact form:

$$i\frac{\partial A}{\partial a_2} - \frac{k}{\omega^2}\frac{\partial^2 A}{\partial t_1^2} + \frac{4k^3 A}{\omega}\int_{-\infty}^{0}\psi_{1t_1} e^{2kb}\,db = 0. \quad (39)$$

The explicit form of the function ψ_{1t_1} is found by integrating the equation (22):

$$\psi_{1t_1} = -k\omega|A|^2 e^{2kb} - \int_{-\infty}^{b}\Omega_2(a_2,b')db' - U(a_2,t_1), \quad (40)$$

Here $U(a_2,t_1)$ is an arbitrary function describing the heterogeneous horizontally and homogeneous vertically (independent of the coordinate b) unsteady flow. Substituting the formula (40), we write the equation (39) in the final form:

$$i\frac{\partial A}{\partial a_2} - \frac{k}{\omega^2}\frac{\partial^2 A}{\partial t_1^2} - k\left(k^2|A|^2 + \beta(a_2) + \frac{2k}{\omega}U(a_2,t_1)\right)A = 0, \\ \beta(a_2) = \frac{4k^2}{\omega}\int_{-\infty}^{0} e^{2kb}\left(\int_{-\infty}^{b}\Omega_2(a_2,b')db'\right)db. \quad (41)$$

This is the nonlinear Schrödinger equation for the packet of surface gravity waves propagating in fluid with vorticity distribution $\Omega = \varepsilon^2\Omega_2(a_2,b)$ and additional heterogeneous, non-stationary potential flow $\varepsilon^2 U(a_2,t_1)$.

The factor before the complex amplitude A in the NLS equation (41) is obtained through equation integration (22). It includes three items. All of them

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describe a certain component of the average current. The first one of these, which is proportional to the amplitude modulus square, describes the classical potential drift of fluid particles (see [Henderson et al, 1999], for example). The second one is caused by the presence of low vorticity in the fluid. And, finally, the third item, including $U(a_2, t_1)$ term, describes an additional potential flow. It appears during equation integration (22) along the vertical coordinate b and will evidently not disappear in case of $A = 0$, as well. This is a certain external flow, which must be attributed definite physical sense in each specific problem. We should note that the term of this kind arises in the Eulerian description of potential wave oscillations of the free surface, as well. In the paper [Stocker and Peregrine, 1999] it was chosen as $U = U_* \sin(kx - \omega t)$and interpreted as a harmonically changing surface current, induced by the internal wave.

The function $\Omega_2(a_2, b)$, determining flow vorticity, can be an arbitrary continuous bounded function. This function sets the vorticity initial distribution. On integrating it twice, we find the vortex component of the average current, which is in no way related to the average current potential components.

Let us consider two limiting cases arising from (41).

a) Potential waves: $\Omega_2 = 0, U = 0$. In this case the equation (41) becomes the classical nonlinear Schrödinger equation for waves in deep water, which has been repeatedly derived for the potential waves.

With regard to rogue waves, three kinds of analytical solutions of the NLS equation are usually discussed. The first is the Peregrine breather, localized in space and time [Peregrine, 1983; Shrira and Geogjaev, 2010]. This rogue wave can be regarded as a long wave (infinite wavelength) limit of a breather (a pulsating mode) [Grimshaw et al, 2010]. Two others are the Akhmediev breather, the solution that is periodic in space and localized in time [Akhmediev et al, 1985], and the Kuznetsov-Ma breather, the solution that is periodic in time and localized in space [Kuznetsov, 1977; Ma, 1979]. Both evolve against a background of the unperturbed sine wave. The formation of the rogue waves was rather often carried out in the framework of the classical Schrödinger equation; see reviews: Akhmediev and Pelinovsky (2010), Slunyaev et al (2011), Onorato et al (2013); papers: Gelash and Zakharov (2014), Ruban (2012), Dubard and Matveev (2013), Slunyaev and Sergeeva (2011) and many others. We will not reproduce these results in the present paper.

To obtain the value for the free surface elevation we substitute expressions (8), (9), (13) and $b = 0$ to the equation for $Y = \operatorname{Im} W$, which is written in the following form

$$Y_L = \varepsilon \operatorname{Im} A(a_2, t_1) \exp i(ka_0 + \omega t_0),$$

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here $A(a_2,t_1)$ is the solution of the equation (41). This expression defines the wave profile in the Lagrangian coordinates (refer to subscript "L" for Y). To rewrite this equation in the Eulerian variables it is necessary to define a via X. From the relation (8) it follows

$$X = a + \varepsilon \operatorname{Re}\left(w_1 + \sum_{n=2} \varepsilon^{n-1} w_n \right) = a + O(\varepsilon),$$

and the free surface elevation in the Eulerian variables Y_E is written as:

$$Y_E = \varepsilon \operatorname{Im} A(X_2,t_1) \exp i(kX_0 + \omega\, t_0) + O\left(\varepsilon^2\right); \quad X_l = \varepsilon^l X\,.$$

For example, the solution of the Akhmediev Lagrangian breather to the epsilon square order coincides with the classical Akhmediev solution in the Eulerian coordinates. The coordinate a plays the role of X, so the following substitutions are valid for the Lagrangian approach

$$a_0 \to X_0; \quad a_1 \to X_1; \quad a_2 \to X_2\,.$$

This result could be named an "accordance principle" between the Lagrange and the Euler descriptions for solutions in the linear approximation. This principle is valid for both the potential and rotational waves $(\beta \neq 0)$.

The solutions of the considered problem in the Lagrange and the Euler forms in the quadratic and cubic approximations differ from each other. To obtain the full solution in the Lagrange form we should obtain the functions $\psi_1, \psi_2, \psi_3, f_2, f_3$. This problem should be considered within a special study.

b) Gerstner waves: $\Omega_2 = -2k^2\omega|A|^2 e^{2kb}, U = 0$. The pressure at the surface is constant. The exact Gerstner solution in the complex form is written as [Lamb, 1932; Bennet, 2006; Abrashkin and Yakubovich, 2006; Nouguier et al, 2015]:

$$W = a_0 + ib + iA \exp\left[i(ka_0 + \omega t_0) + kb\right]. \tag{42}$$

It describes a stationary traveling rotational wave with a trochoidal profile. Their dispersion characteristic coincides with the dispersion of linear waves in deep water: $\omega^2 = gk$. Fluid particles are moving in circles, and the drift current is absent. In the linear approximation the Gerstner waves are potential ($\Omega_1 = 0$), but in the quadratic approximation they already possess vorticity.

For this type of vorticity distribution the first two terms in the parentheses of the equation (41) mutually cancel each other. From the physical point of view, this is due to the fact that the average current induced by vorticity exactly

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compensates the Stokes drift. The packet of weakly nonlinear Gerstner waves in this approximation is not affected by non-linearity, and the effect of the modulation instability for the Gerstner wave is absent. The packet of weakly nonlinear Gerstner waves in this approximation is not affected by non-linearity, and the effect of the modulation instability for the Gerstner wave is absent.

The absence of the nonlinear term in the NLS for the Gerstner waves obtained here in the Lagrangian formulation is a robust result and should also be included in the Euler description. This follows from the famous Lighthill criterion for the modulation instability because the dispersion relation for the Gerstner wave is linear and does not include proportional wave amplitude terms.

4. Rogue waves

The NLS equation admits multiple solutions in the form of non-steady-state breathers. We will be interested in the breathers which satisfy the rogue wave amplitude criterion. Let us consider a few examples.

a) The Gouyon modulated waves: $\Omega = \varepsilon^2 \Omega_2(b)$, $U = 0$. As shown by Dubreil – Jacotin (1934), the Gerstner wave is a special case of a wide class of stationary waves with vorticity $\Omega = \varepsilon \Omega_*(\psi)$, where Ω_* is arbitrary function and ψ is stream function. These results were rediscovered and then developed by Gouyon (1958) who explicitly presented the vorticity in the form of a power series $\Omega = \sum_{n=1}^{\infty} \varepsilon^n \Omega_n(\psi)$; see also the monograph [Sretensky, 1977].

When considering the plane steady flow in the Lagrange variables, the stream lines ψ coincide with the isolines of the Lagrangian vertical coordinate b [Bennet, 2006; Abrashkin and Yakubovich, 2006]. We are going to consider a steady-state wave on an indefinitely deep water surface. Let us assume that there is no undisturbed shear current, but wave disturbances have vorticity. Then, the vorticity formula is as follows: $\Omega = \sum_{n=1}^{\infty} \varepsilon^n \Omega_n(b)$. We are going to call the steady-state waves propagating in such low-vorticity fluid the Gouyon waves. In the Lagrangian description the Gouyon wave properties for the first two approximations were studied in [Abrashkin and Zenkovich, 1990].

Assuming for the value β in the formula (41) $\Omega_1 = 0, \Omega_2 = \Omega_2(b)$, we obtain:

$$\beta = \frac{4k^2}{\omega} \int_{-\infty}^{0} e^{2kb} \left(\int_{-\infty}^{b} \Omega_2(b')db' \right) db = \beta_* = \text{const} \tag{43}$$

And after replacing $A = A' \exp(-i\beta_* k a_2)$ we come to the NLS equation which describes the Gouyon modulated waves:

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$$i\frac{\partial A'}{\partial a_2}-\frac{k}{\omega^2}\frac{\partial^2 A'}{\partial t_1^2}-k^3|A'|^2A'=0. \tag{44}$$

It has the same coefficients as the NLS equation for potential waves in deep water. From it follows that the conditions of the modulation instability for the Gouyon waves will be exactly the same as for the potential waves.

All the known analytical and numerical calculations of rogue waves for potential waves can be transferred to the Gouyon modulated waves. We present here the modification of the solution in the form of the Peregrine breather which is the analytical model of the rogue waves [Akhmediev et al, 1985; Kharif et al, 2009]

$$A(t_1,a_2)=A_0\left[-1+4\frac{1+2ik^3A_0^2a_2}{1+2k^2\omega^2A_0^2t_1^2+4k^6A_0^4a_2^2}\right]\exp\left[ik\left(k^2A_0^2-\beta_*\right)a_2\right], \tag{45}$$

where A_0 is the amplitude of the unperturbed monochromatic wave. As it can be seen, the vorticity affects only the spatial wave number shift, reducing it in comparison with the Stokes wave. In comparison with the Peregrine breather in the inviscid fluid vorticity leads to a change in the wavelength of the carrier wave, which affects the number of individual waves in the rogue wave packet in space which will be reduced.

b) Waves with heterogeneous vorticity distribution in both coordinates: $\Omega=\varepsilon^2\Omega_2(a_2,b)$, $U=0$**.** In this case function β is already variable and it is the function of the Lagrange horizontal coordinate. But by replacing

$$A=A'\exp\left(-ik\int_{-\infty}^{a_2}\beta(a_2)da_2\right) \tag{46}$$

the equation (38) is again reduced to the equation (41) with constant coefficients. In fact, there are no fundamental differences from the Gouyon waves, and a new effect here is the heterogeneity of the spatial distribution of the individual wave lengths in the extreme packet.

To express the solution of the equation (41) in the Eulerian variables it is necessary to use the accordance principle and to change the horizontal Lagrangian coordinate a_2 to the coordinate X_2. So the discrepancies between the Eulerian and the Lagrangian NLS estimations for the free surface elevation are absent.

c) Waves in low-vorticity fluid in the presence of additional potential flow $\Omega=\varepsilon^2\Omega_2(a_2,b)$, $U\neq 0$. In this case, the equation (38) is one of the variants of the variable-coefficient nonlinear Shrödinger equation (VCNLSE) which is now being actively studied in optics and hydrodynamics. Under certain conditions, it

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has a solution in the form of breathers, showing the possibility of the rogue wave phenomenon. With regard to the optical problems, the review of the cases when VCNLSE can be reduced to the constant-coefficient NLS equation is given in [He and Li, 2011], however, it also includes the wave packet linear damping. It is clear that the large-amplitude wave generation is possible in more general cases when a breather solution cannot be received.

Note: for example, an important case when the function U is the linear function of time. Introducing the dimensionless variables

$$E=\frac{1}{\sqrt{2}}k\bar{A}\exp\left(-ik\int_{-\infty}^{a_2}\beta(a_2)da_2\right),\quad \tau=\omega t_1=\omega kA_0t,\quad q=ka_2=k^3A_0^2a,\quad U(t_1)=-\alpha\frac{\omega}{k}\tau, \tag{47}$$

where α is a constant, we reduce the equation (41) to the following equation

$$i\frac{\partial E}{\partial q}+\frac{\partial^2E}{\partial\tau^2}+\left(-2\alpha\tau+2|E|^2\right)E=0, \tag{48}$$

which has the exact one-soliton solution [Chen and Liu, 1976]:

$$E=E_0(q,\tau)\exp i\varphi(q,\tau);\quad E_0=2\eta\ \mathrm{sech}\ 2\eta\left(\tau+2\alpha q^2-4\xi q-\tau_0\right);\quad \varphi=2(\xi-\alpha q)\tau-4\left[\frac{1}{3}\alpha^2q^3-\alpha\xi q^2+\left(\xi^2-\eta^2\right)q\right], \tag{49}$$

where ξ,η are constants, τ_0 is initial time. It describes a non-uniformly moving soliton with the amplitude 2η. The parameter ξ sets the point where the soliton velocity $dq/d\tau$ changes sign (the caustic point). The existence of the soliton with the constant amplitude is determined by the competition of two effects: the dispersion of the wave pulse compression due to frequency modulation and spreading in a heterogeneous medium. The existence of a soliton envelope is characteristic of the focusing nonlinear Schrödinger equation which indicates the possibility of the appearance of modulation instability and rogue waves.

5. Conclusion

In the given paper the vortex-modified nonlinear Schrödinger equation is derived. To obtain it the method of multiple scale expansions in the Lagrange variables is used. The fluid vorticity Ω is set as an arbitrary function of the Lagrangian coordinates which is quadratic in the small wave steepness parameter ($\Omega=\varepsilon^2\Omega_2(a,b)$). The calculations are carried out by introducing the fluid particle trajectory complex coordinate.

The nonlinear evolution equation for the wave envelope in the form of the nonlinear Schrödinger equation is derived. From the mathematical point of view the novelty of the equation is related to the emergence of a new term that is proportional to the amplitude of the envelope, with the factor that depends on the spatial coordinate. It determines the average flow, connected with the vorticity presence in the fluid. By a simple replacement it is reduced to the NLS equation with the same coefficients as for the potential waves in deep water. The vorticity effect is associated with the wave number shift of the carrier wave. In the case of vorticity depending only on the vertical Lagrangian coordinate b (the modulated Gouyon wave) this shift is constant. In a more general case when the vorticity is dependent on both Lagrangian coordinates, the wave number shift is horizontally heterogeneous.

The criteria of the modulation instability of the considered low-vorticity waves and the potential waves in deep water are the same. The all-known analytical and numerical solutions of the NLS equation for rogue waves are also applicable to the given low-vorticity waves. It should be noted that for the Gerstner waves the vortical average current exactly compensates the Stokes drift; therefore, the modulation instability effect for them is absent.

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