Lagrange form of the nonlinear Schrödinger equation for low-vorticity waves in deep water

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The nonlinear Schrödinger (NLS) equation describing the propagation of weakly rotational wave packets in an infinitely deep fluid in Lagrangian coordinates has been derived. The vorticity is assumed to be an arbitrary function of Lagrangian coordinates and quadratic in the small parameter proportional to the wave steepness. The vorticity effects manifest themselves in a shift of the wavenumber in the carrier wave as well as in variation of the coefficient multiplying the nonlinear term. In the case of the dependence of vorticity on the vertical Lagrangian coordinate only (the Gouyon waves), the shift of the wavenumber and the respective coefficient are constant. When the vorticity is dependent on both Lagrangian coordinates, the shift of the wavenumber is horizontally inhomogeneous. There are special cases (e.g., Gerstner waves) when the vorticity is proportional to squared wave amplitude and non-linearity disappears, thus making the equations for wave packet dynamics linear. It is shown that the NLS solution for weakly rotational waves in the Eulerian variables may be obtained from the Lagrangian solution by simply changing the horizontal coordinates.

Key words: nonlinear Schrödinger equation, vorticity, water waves

1 Introduction

The nonlinear Schrödinger (NLS) equation was first derived by Benny and Newell (1967) and then by Zakharov (1968), who used the Hamiltonian formalism for description of wave propagation in deep water. Hashimoto and Ono (1972) and Davey (1972) obtained the same result independently. Like Benney and Newell (1967) they used the method of multiple scale expansions in Euler coordinates. Yuen and Lake (1975), in turn, derived the NLS equation on the basis of the averaged Lagrangian method. Benney and Roskes (1969) extended those two-dimensional theories to the case of three-dimensional wave perturbations in a finite depth fluid and obtained equations that are now known as the Davey-Stewartson equations. In this particular case the equation proves the existence of transverse instability of a plane wave which is much stronger than a longitudinal one. This circumstance diminishes the role and meaning of the NLS equation for sea

45   applications. Meanwhile, the 1-D NLS equation has been successfully tested many
46   times in laboratory wave tanks and natural observations were compared with
47   numerical calculations in the framework of this equation.
48   In all the cited papers wave motion was considered to be potential. However,
49   wave formation and propagation frequently occur against the background of a
50   shear flow possessing vorticity. Using the method of multiple scales, Johnson
51   (1976) examined slow modulation of a harmonic wave moving at the surface of an
52   arbitrary shear flow with velocity profile $U(y)$, where y is vertical coordinate. He
53   derived the NLS equation with coefficients which depend in a complicated way on
54   a shear flow (Johnson, 1976). Oikawa et al. (1985) considered the properties of
55   instability of weakly nonlinear three-dimensional wave packets in the presence of a
56   shear flow. Their simultaneous equations reduce to the known NLS equation for
57   the case of purely two-dimensional wave evolution. Li et al. (1987) and Baumstein
58   (1998) studied the modulation instability of the Stokes wave-train and derived an
59   NLS equation for an uniform shear flow in deep water, when $U(y) = \Omega_0 y$ and
60    $\Omega_z = \Omega_0$ is constant vorticity (z is the horizontal coordinate normal to the x, y
61   plane of the flow; the wave propagates in the x direction).
62   Thomas et al. (2012) generalized their results for a finite-depth fluid and
63   confirmed that a linear shear flow may significantly modify the stability properties
64   of weakly nonlinear Stokes waves. In particular, for the waves propagating in the
65   direction of the flow, the Benjamin-Feir (modulational) instability can vanish in
66   the presence of positive vorticity ($\Omega_0 < 0$) for any depth.
67   In the traditional Eulerian approach to the propagation of weakly nonlinear
68   waves against the background current, a shear flow determines vorticity in a zero
69   approximation. Depending on the flow profile $U(y)$ it may be arbitrary and equal
70   to $-U'(y)$. At the same time, the vorticity of wave perturbations $\Omega_n, n \geq 1$, i.e. the
71   vorticity in the first and subsequent approximations in the wave steepness
72   parameter $\varepsilon = kA_0$ (k is wavenumber and A_0 is wave amplitude) depends on its
73   form. In Eulerian coordinates the vorticity of wave perturbations is a function not
74   only of y, but of x and t variables as well. Plane waves on a shear flow with a linear
75   vertical profile are regarded to be an exception (Li et al., 1987; Baumstein, 1998;
76   Thomas et al., 2012). For such waves the vorticity is constant in a zero
77   approximation, and all the vorticities in wave perturbations are equal to zero. For
78   an arbitrary vertical profile of the shear flow (Johnson, 1976), expressions for the
79   functions Ω_n can be hardly predicted even qualitatively.
80   The Lagrangian method allows applying a different approach. In the plane
81   flow the vorticity of fluid particles is preserved and can be expressed via
82   Lagrangian coordinates only. Thus, not only the vertical profile of the shear flow
83   defining the vorticity in a zero approximation, but the expressions for the vorticity
84   of the following orders of smallness can also be arbitrary. The expression for the
85   vorticity is written in the form
86

$$87 \qquad \Omega(a,b) = -U'(b) + \sum_{n \geq 1} \varepsilon^n \Omega_n(a,b),$$

88

89 where a, b are the horizontal and vertical Lagrangian coordinates, respectively,
90 $U(b)$ is the vertical profile of the shear flow, and particular conditions for defining
91 the Ω_n functions can be found. For the given shear flow this approach allows
92 studying wave perturbations under the most general law of distribution of
93 vorticities Ω_n. In the present paper we do not consider shear flow and vorticity in
94 the linear approximation ($U = 0; \Omega_1 = 0$), whereas vorticity in the quadratic
95 approximation is an arbitrary function. This corresponds to the rotational flow
96 proportional to ε^2. We can define both the shear flow and the localized vortex.
97 The dynamics of plane wave trains on the background flows with arbitrary
98 low vorticity has not been studied before. The idea to study wave trains with
99 quadratic (with respect to the wave steepness parameter) vorticity was realized
100 earlier for the spatial problems in the Euler variables. Hjelmervik and Trulsen
101 (2009) derived the NLS equation for vorticity distribution

102

$$103 \qquad \Omega_y/\omega = O(\varepsilon^2); \quad (\Omega_x, \Omega_z)/\omega = O(\varepsilon^3),$$

104

105 where ω is wave frequency. The vertical vorticity of wave perturbations exceeds
106 the other two vorticity components by a factor of ten. This vorticity distribution
107 corresponds to the low (of order ε) velocity of the horizontally inhomogeneous
108 shear flow. Hjelmervik and Trulsen (2009) used the NLS equation to study the
109 statistics of rogue waves on narrow current jets, and Onorato et al. (2011) used that
110 equation to study the opposite flow rogue waves. The effect of low vorticity (ε^2
111 order of magnitude) in the paper by Hjelmervik and Trulsen (2009) is reflected in
112 the NLS equation. This fact, like the NLS nonlinear term for plane potential waves,
113 may be attributed to the presence of an average current non-uniform over the fluid
114 depth.
115 Colin et al. (1995) considered the evolution of three-dimensional vortex
116 disturbances in a finite-depth fluid for a different type of vorticity distribution:
117

$$118 \qquad \Omega_y = 0; \quad (\Omega_x, \Omega_z)/\omega = O(\varepsilon^2)$$

119

120 and reduced the problem to a solution of the Davey-Stewartson equations by
121 means of the multiple scale expansion method in Eulerian variables. In this case,
122 vorticity components are calculated after the solution of the problem. Similarly to
123 the traditional Eulerian approach (Johnson, 1976) the form of the quadratic
124 vorticity distribution is very special and does not cover all of its numerous possible
125 distributions.
126 In this paper we consider the plane problem of nonlinear wave packet
127 propagating in an ideal incompressible fluid with the following form of vorticity
128 distribution

129

$$\Omega_z / \omega = O\!\left(\varepsilon^2\right).$$

131

In contrast to Hjelmervik and Trulsen (2009), Onorato et al. (2011) and Colin et al. (1996), the flow is two-dimensional ($\Omega_x = \Omega_y = 0$). The propagation of a packet of potential waves gives rise to a weak counterflow underneath the free water surface with velocity proportional to the square of the wave steepness (McIntyre, 1982). In the considered problem this potential flow is superimposed with the rotational one of the same order of magnitude. This results in appearance of an additional term in the NLS equation and in change of the coefficient in the nonlinear term. So, the difference from the NLS solutions derived for a strictly potential fluid motion was revealed.

The examination is made in the Lagrangian variables. The Lagrangian variables are rarely used in fluid mechanics because of a more complex type of nonlinear equations in Lagrangian form. However, when considering the vortex-induced oscillations of a free fluid surface, the Lagrangian approach has two major advantages. First, unlike the Euler description method the shape of the free surface is known and is determined by the condition of the equality to zero ($b = 0$) of the vertical Lagrangian coordinate. Second, the vortical motion of liquid particles is confined within the plane and is a function of Lagrangian variables $\Omega_z = \Omega_z(a,b)$, so the type of vorticity distribution in the fluid can be preset. The Eulerian approach does not allow this. In this case, the second-order vorticity is defined as a known function of Lagrangian variables.

Here, hydrodynamic equations are solved in Lagrangian form by the multiple scale expansion method. A nonlinear Schrödinger equation with variable coefficients is derived. Possible ways of reducing it to the NLS equation with constant coefficients are studied.

The paper is organized as follows. Section 2 describes the Lagrangian approach to studying wave oscillations at the free surface of a fluid. The zero of the Lagrangian vertical coordinate corresponds to the free surface, thus simplifying formulation of the pressure boundary conditions. The specific feature of the proposed approach is introduction of a complex coordinate of a fluid particle trajectory. In Section 3 a nonlinear evolution equation is derived on the basis of the method of multiple scale expansion. Different solutions of the NLS equation adequately describing various examples of vortex waves are considered in Section 4. The transform from the Lagrangian coordinates to the Euler description of solutions of the NLS equation is shown in Section 5. Section 6 summarizes the obtained results.

167

2 Basic equations in Lagrangian coordinates

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Consider the propagation of a packet of gravity surface waves in a rotational infinitely deep fluid. 2D hydrodynamic equations of an incompressible inviscid

fluid in Lagrangian coordinates have the following form (Lamb, 1932; Abrashkin and Yakubovich, 2006; Bennett, 2006):

$$\frac{D(X,Y)}{D(a,b)} = [X,Y] = 1,\tag{1}$$

$$X_{tt}X_a + \left(Y_{tt} + g\right)Y_a = -\frac{1}{\rho}p_a,\tag{2}$$

$$X_{tt}X_b + \left(Y_{tt} + g\right)Y_b = -\frac{1}{\rho}p_b,\tag{3}$$

where X,Y are the horizontal and vertical Cartesian coordinates and a,b are the horizontal and vertical Lagrangian coordinates of fluid particles, t is time, ρ is fluid density, p is pressure, g is acceleration due to gravity, the subscripts mean differentiation with respect to the corresponding variable. The square brackets denote the Jacobian. The b axis is directed upwards, and $b = 0$ corresponds to the free surface. Equation (1) is a volume conservation equation. Equations (2) and (3) are momentum equations. The geometry of the problem is presented in Fig. 1.
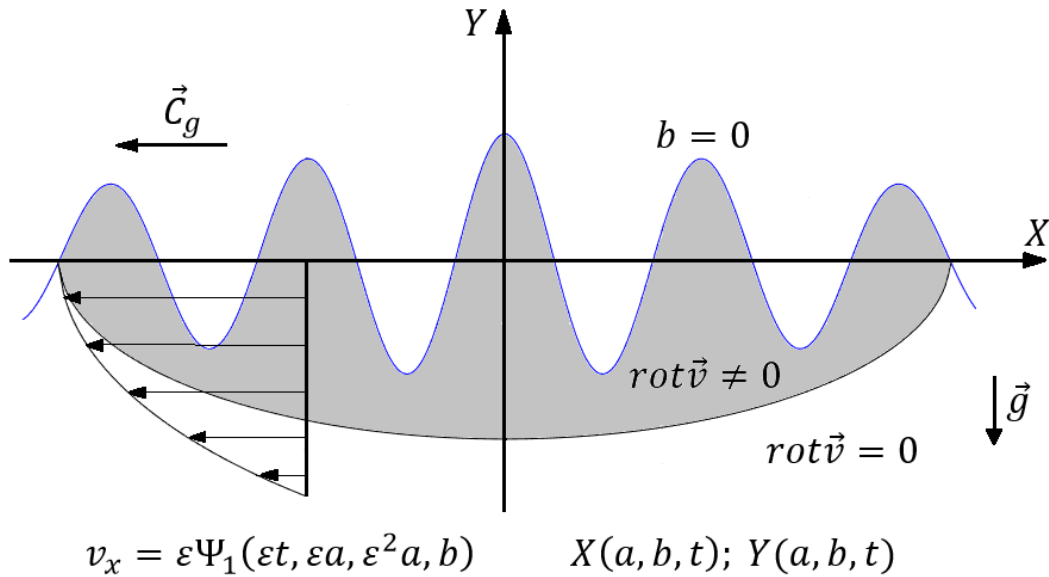
Fig. 1. Problem geometry: v_x **is average current.**

Making use of cross differentiation it is possible to exclude pressure and to obtain the condition of conservation of vorticity along the trajectory (Lamb, 1932; Abrashkin and Yakubovich, 2006; Bennett, 2006):

$$X_{ta}X_b + Y_{ta}Y_b - X_{tb}X_a - Y_{tb}Y_a = \Omega(a,b).\tag{4}$$

197 This equation is equivalent to the momentum equations (2) and (3) but involves
198 explicit vorticity of liquid particles, Ω, which in case of two-dimensional flows is
199 the function of Lagrangian coordinates only.
200   We introduce a complex coordinate of a fluid particle trajectory
201 $W = X + iY$ $\left(\overline{W} = X - iY \right)$, the overline means complex conjugation. In the new
202 variables Eqs. (1) and (4) take on the form

$$\left[W, \overline{W} \right] = -2i \, , \tag{5}$$

$$\mathrm{Re} \left[W_t, \overline{W} \right] = \Omega(a, b), \tag{6}$$

208 After simple algebraic manipulations Eqs. (2) and (3) reduce to the following
209 single equation

$$W_{tt} = -ig + i\rho^{-1} \left[p, W \right]. \tag{7}$$

212 Equations (5) and (6) will be further used to find the coordinates of complex
213 trajectories of fluid particles, and Eq. (7) determines the pressure of the fluid. The
214 boundary conditions are the non-flowing condition at the bottom ($Y_t \to 0$ at
215 $b \to -\infty$) and constant pressure at the free surface (at $b = 0$).
216   The Lagrangian coordinates mark the position of fluid particles. In the
217 Eulerian description the displacement of the free surface $Y_s(X, t)$ is calculated in
218 an explicit form, but in the Lagrangian description it is defined parametrically by
219 the following equalities: $Y_s(a, t) = Y(a, b = 0, t)$; $X_s(a, t) = X(a, b = 0, t)$, where the
220 Lagrangian horizontal coordinate a plays the role of parameter. Its value along the
221 free surface $b = 0$ varies in the $(-\infty; \infty)$ range . In Lagrangian coordinates the
222 function $Y_s(a, t)$ defines the displacement of the free surface.

3 Derivation of evolution equation

226 Let us represent the function W using the multiple scales method in the following
227 form

$$W = a_0 + ib + w(a_l, b, t_l), \quad a_l = \varepsilon^l a, \quad t_l = \varepsilon^l t; \quad l = 0, 1, 2, \tag{8}$$

231 where ε is the small parameter of wave steepness. All unknown functions and the
232 given vorticity can be represented as a series in this parameter:

$$w = \sum_{n=1} \varepsilon^n w_n; \quad p = p_0 - \rho g b + \sum_{n=1} \varepsilon^n p_n; \quad \Omega = \sum_{n=1} \varepsilon^n \Omega_n(a, b). \tag{9}$$

236 In the formula for the pressure, the term with hydrostatic pressure is selected, p_0
237 is the constant atmospheric pressure at the fluid surface. The representations (8)
238 and (9) are substituted into Eqs. (5)-(7).
239
240 **3.1 Linear approximation**
241
242 In a first approximation in the small parameter we have the following system of
243 equations
244

245 $$\mathrm{Im}\left(iw_{1a_0} + w_{1b} \right) = 0, \tag{10}$$

246 $$\mathrm{Re}\left(iw_{1a_0} + w_{1b} \right)_{t_0} = -\Omega_1, \tag{11}$$

247 $$w_{1t_0t_0} + \rho^{-1}\left(p_{1a_0} + ip_{1b} \right) = igw_{1a_0}. \tag{12}$$

248
249 The solution satisfying the continuity equation (10) and the equation of
250 conservation of vorticity (11) describes a monochromatic wave (for definiteness,
251 we consider the wave propagating to the left) and the average horizontal current
252

253 $$w_1 = A(a_1,a_2,t_1,t_2)\exp\left[i(ka_0 + \omega t_0) + kb\right] + \psi_1(a_1,a_2,b,t_1,t_2); \quad \Omega_1 = 0. \tag{13}$$

254
255 Here A is the complex amplitude of the wave, ω is its frequency, and k is the
256 wave number. The function ψ_1 is real and will be found in the next
257 approximation.
258   The substitution of solution (13) into Eq. (12) yields the equation for the
259 pressure
260

261 $$\rho^{-1}\left(p_{1a_0} + ip_{1b} \right) = \left(\omega^2 - gk \right)A\exp\left[i(ka_0 + \omega t_0) + kb\right], \tag{14}$$

262
263 which is solved analytically
264

265 $$p_1 = -\mathrm{Re}\,\frac{i\left(\omega^2 - gk\right)}{k}\rho A\exp\left[i(ka_0 + \omega t_0) + kb\right] + C_1\left(a_1,a_2,t_1,t_2\right), \tag{15}$$

266
267 where C_1 is an arbitrary function. The boundary condition at the free surface is
268 $p_1|_{b=0} = 0$, which leads to $\omega^2 = gk$ and to $C_1 = 0$. Thus, in the first approximation
269 the pressure correction p_1 is equal to zero.
270
271 **3.2 Quadratic approximation**
272

The equations of the second order of the perturbation theory can be written as follows

$$\mathrm{Im}\left(iw_{2a_0} + w_{2b} + iw_{1a_1} - w_{1a1}\overline{w_{1b}} \right) = 0, \qquad (16)$$

$$\mathrm{Re}\left[iw_{2t_0a_0} + w_{2t_0b} + i\left(w_{1t_0a_1} + w_{1t_1a_0} \right) - w_{1t_0a_0}\overline{w_{1b}} + w_{1t_1b} + w_{1t_0b}\overline{w_{1a_0}} \right] = -\Omega_2, \quad (17)$$

$$w_{2t_0t_0} + \rho^{-1}\left(p_{2a_0} + ip_{2b} \right) = ig\left(w_{2a_0} + w_{a_1} \right) - 2w_{1t_1t_0}. \qquad (18)$$

By substituting expression (13) for w_1 into Eq. (16) we obtain

$$\mathrm{Im}\left[iw_{2a_0} + w_{2b} - i\left(k\psi_{1b}A - A_{a_1} \right)\exp\left[i\left(ka_0 + \omega t_0 \right) + kb \right] - ik^2|A|^2 e^{2kb} + i\psi_{1a_1} \right] = 0, \quad (19)$$

which is integrated as follows

$$w_2 = i\left(kA\psi_1 - bA_{a_1} \right)\exp\left[i\left(ka_0 + \omega t_0 \right) + kb \right] + \psi_2 + if_2, \qquad (20)$$

where ψ_2, f_2 are the functions of slow coordinates and Lagrangian vertical coordinate b and

$$f_{2b} = k^2|A|^2 \exp 2kb - \psi_{1a_1}, \qquad (21)$$

ψ_2 is an arbitrary real function. It will be determined in a solution in the cubic approximation.

When (13) and (20) are substituted into (17), the sum of the terms containing the exponential factor becomes equal to zero, and the remaining terms satisfy the equation

$$\psi_{1t_1b} = -2k^2\omega|A|^2 \exp(2kb) - \Omega_2. \qquad (22)$$

The expression for the function ψ_1 can be found by a simple integration. It should be emphasized that the vorticity in the second approximation, that is part of Eq. (22), is an arbitrary function of slow horizontal and vertical Lagrange coordinates, so that $\Omega_2 = \Omega_2\left(a_1, a_2, b \right)$.

Taking into account the solutions in the first two approximations we can write Eq. (18) as

$$\rho^{-1}\left(p_{2a_0} + ip_{2b} \right) = i\left(gA_{a_1} - 2\omega A_{t_1} \right)\exp\left[i\left(ka_0 + \omega t_0 \right) + kb \right] + ig\psi_{1a_1}. \qquad (23)$$

Its solution determines the pressure correction

310

$$p_2 = \mathrm{Re}\left[\frac{1}{k}\left(gA_{a_1} - 2\omega A_{t_1}\right)\exp\left[i\left(ka_0 + \omega t_0\right) + kb\right]\right] + \rho g\int_0^b \psi_{1a_1}\,db + C_2\left(a_1, a_2, t_1, t_2\right) \quad (24)$$

312

313 The integration limits in the penultimate term are chosen so that this integral term
314 equals zero at the free surface. Due to the boundary condition for pressure
315 ($p_2(b=0)=0$), $C_2 = 0$, and

316

$$A_{t_1} - c_g A_{a_1} = 0; \quad c_g = \frac{g}{2\omega} = \frac{1}{2}\sqrt{\frac{g}{k}}, \quad (25)$$

318

319 where c_g is the group velocity of wave propagation in deep water, which in this

320 approximation is independent of fluid vorticity. As was expected, in this

321 approximation the wave moves with group velocity c_g to the left (the "minus"

322 sign in Eq. (25)).

323

324 **3.3 Cubic approximation**

325

326 The equation of continuity and the condition of conservation of vorticity in the
327 third approximation are written in the form

328

329

330
$$\mathrm{Im}\left[iw_{2a_0} + w_{3b} + i\left(w_{1a_2} + w_{2a_1} + w_{2a_0}\right) - \left(w_{1a_1} + w_{2a_2}\right)\overline{w_{1b}} - w_{1a_0}\overline{w_{2b}}\right] = 0, \quad (26)$$

$$\mathrm{Re}\left[iw_{3t_0a_0} + w_{3t_0b} + i\left(w_{1t_2a_0} + w_{1t_1a_1} + w_{1t_0a_2} + w_{2t_1a_0} + w_{2t_0a_1}\right) + w_{1t_2b} - \overline{w_{2b}}w_{1t_0a_0} - \right.$$

331
$$\quad (27)$$

$$\left. + w_{2t_1b} - w_{1b}\left(w_{1t_0a_1} + w_{1t_1a_0} + w_{2t_0a_0}\right) + +\overline{w_{1a_0}}\left(w_{1t_1b} + w_{2t_0b}\right) + w_{1t_0b}\left(\overline{w_{1a_1}} + \overline{w_{2a_0}}\right)\right] = -\Omega_3.$$

332

333 We substitute the solutions in the first and second approximations into the
334 simultaneous equations

335

336

337
$$\mathrm{Im}\left[iw_{3a_0} + w_{3b} + i\left(\psi_{1a_2} + \psi_{2a_1}\right) + 2k(kb+1)A\overline{A_{a_1}}e^{2b} + G_b e^{i\left(ka_0 + \omega t_0\right) + kb}\right] = 0, \quad (28)$$

338

$$\text{Re}\left\{\left[iw_{3a_0}+w_{3b}+\left(G_b+2k\psi_{1t_1b}\omega^{-1}A\right)e^{i\left(ka_0+\omega t_0\right)+kb}\right]_{t_0}+\psi_{2t_1b}+\psi_{1t_2b}+\right.$$

$$\left.+i\omega k(4kb+5)A\overline{A_{a_1}}e^{2kb}\right\}=-\Omega_3, \tag{29}$$

$$G=ibA_{a_2}+\frac{b^2}{2}A_{a_1a_1}-(kb+1)\psi_1A_{a_1}-\left(ik\psi_2+kf_2-\frac{k^2}{2}\psi_1^2\right)A. \tag{30}$$

We seek a solution for the third approximation in the following form

$$w_3=\left(G_1-G\right)e^{i\left(ka_0+\omega t_0\right)+kb}+G_2e^{-i\left(ka_0+\omega t_0\right)+kb}+\psi_3+if_3, \tag{31}$$

where G_1,G_2,ψ_3,f_3 are functions of slow coordinates and b. By substituting this expression into (28) and (29) we immediately find

$$f_{3b}+\psi_{2a_1}+\psi_{1a_2}+k(kb+1)\left(A\overline{A_{a_1}}-\overline{A}A_{a_1}\right)e^{2kb}=0, \tag{32}$$

$$\psi_{2t_1b}+\psi_{1t_2b}+\frac{1}{2}(4kb+5)\omega k\left(A\overline{A_{a_1}}-\overline{A}A_{a_1}\right)e^{2kb}=-\Omega_3. \tag{33}$$

The function ψ_2 according to Eq. (33) is determined by known solutions for A and ψ_1, and by the given distribution of Ω_3. The expression for the function f_3 is then derived from Eq. (32). These functions determine the horizontal and vertical average motion, respectively. But in this approximation they are not included in the evolution equation for the wave envelope. The function ψ_3 will be found in the next approximation.

When solving (28) and (29) we have found

$$G_1=-k\omega^{-1}\psi_{1t_1}A; \quad G_2=k\omega^{-1}\left(2ke^{-2kb}\int_{-\infty}^{b}\psi_{1t_1}e^{2kb'}db'-\psi_{1t_1}\right)\overline{A}. \tag{34}$$

These relationships should be substituted into Eq. (7), which in this approximation has the form

$$w_{3t_0t_0}-igw_{3a_0}=i\rho^{-1}\left[i\left(p_{2a_1}+p_{3a_0}\right)-p_{3b}-p_{2b}w_{1a_0}+\rho g\left(w_{1a_2}+w_{2a_1}\right)\right]-$$

$$-2w_{1t_2t_0}-w_{1t_1t_1}-2w_{2t_0t_1}. \tag{35}$$

Taking into account (13), (20), (24), (31) and (34) we rewrite it as follows

370

$$\rho^{-1}\left(p_{3a_0}+ip_{3b}\right)=\left(-2i\omega\frac{\partial A}{\partial t_2}+ig\frac{\partial A}{\partial a_2}-\frac{\partial^2 A}{\partial t_1^2}+2\omega k\psi_{1t_1}A\right)e^{i\left(ka_0+\omega t_0\right)+kb}+ \tag{36}$$

371 $$+2\omega^2 G_2\overline{A}e^{-i\left(ka_0+\omega t_0\right)+kb}+ig\left(\psi_{2a_1}+\psi_{1a_2}\right)+I;\quad I=-g\left(f_{2a_1}-\int_b^0\psi_{1a_1a_1}db\right)-\psi_{t_1t_1}.$$

372 By virtue of the relationships (21), (22) and (25) the derivative of I along the
373 vertical Lagrangian coordinate is zero ($I_b=0$), so I is the only function of the
374 slow coordinates and time - $a_l,t_l,l\geq1$. The contribution of the term $I\left(a_l,t_l\right)\neq0$ to
375 the pressure is complex, so it demands $I=0$.
376   The solution of Eq. (36) yields the expression for the pressure perturbation
377 in the third approximation:
378

$$\frac{p_3}{\rho}=\mathrm{Re}\,ik^{-1}\left(2i\omega\frac{\partial A}{\partial t_2}-ig\frac{\partial A}{\partial a_2}+\frac{\partial^2 A}{\partial t_1^2}-4\omega k^2 Ae^{-2kb}\int_{-\infty}^b\psi_{1t_1}e^{2kb'}\,db'\right)e^{i\left(ka_0+\omega t_0\right)+kb}+$$

379

$$+\rho g\int_0^b\left(\psi_{2a_1}+\psi_{1a_2}\right)db'. \tag{37}$$

380
381 In Eq. (37) the integration limits for the second integral term have been preset to
382 satisfy the boundary condition at the free surface (the pressure p_3 should turn to
383 zero). Then the factor before the exponent should be equal to zero:
384

385 $$2i\omega\frac{\partial A}{\partial t_2}-ig\frac{\partial A}{\partial a_2}+\frac{\partial^2 A}{\partial t_1^2}-4\omega k^2 A\int_{-\infty}^0\psi_{1t_1}e^{2kb}\,db=0. \tag{38}$$

386
387 By introducing the "running" coordinate $\zeta_2=a_2+c_g t_2$ we can reduce Eq.
388 (38) to a compact form
389

390 $$i\frac{\partial A}{\partial a_2}-\frac{k}{\omega^2}\frac{\partial^2 A}{\partial t_1^2}+\frac{4k^3 A}{\omega}\int_{-\infty}^0\psi_{1t_1}e^{2kb}\,db=0. \tag{39}$$

391
392 Further it will be shown that the variables in Eqs. (38), (39) have been chosen so
393 that they should be easily reduced (under particular assumptions) to the classical
394 NLS equation.
395 The explicit form of the function ψ_{1t_1} is found by integrating Eq. (22):
396

$$\psi_{1t_1}=-k\omega|A|^2 e^{2kb}-\int_{-\infty}^b\Omega_2(a_2,b')db'-U(a_2,t_1), \tag{40}$$

397

398   This expression includes three terms. All of them describe a certain component of
399 the average current. The first one is proportional to the square of the amplitude
400 modulus and describes the classical potential drift of fluid particles (see
401 (Henderson et al. (1999) for example). The second one is caused by the presence of
402 low vorticity in the fluid. Finally, the third item, including $U(a_2, t_1)$, describes an
403 additional potential flow. It appears in the integration of Eq. (22) over the vertical
404 coordinate b and will evidently not disappear in case of $A = 0$ either. This is a
405 certain external flow which is chosen depending on a specific problem. Note that a
406 term of that kind arises in the Eulerian description of potential wave oscillations of
407 the free surface as well. In the paper by Stocker and Peregrine (1999),
408 $U = U_* \sin(kx - \omega t)$ was chosen and interpreted as a harmonically changing
409 surface current induced by an internal wave. We shall further take $U = 0$.
410   After the substitution of Eq. (40), Eq. (39) may be written in the final form
411

$$i\frac{\partial A}{\partial a_2} - \frac{k}{\omega^2}\frac{\partial^2 A}{\partial t_1^2} - k\left(k^2|A|^2 + \beta(a_2)\right)A = 0,$$

412   (41)

$$\beta(a_2) = \frac{4k^2}{\omega}\int_{-\infty}^{0} e^{2kb}\left(\int_{-\infty}^{b}\Omega_2(a_2, b')db'\right)db.$$

413

414 It is the nonlinear Schrödinger equation for the packet of surface gravity waves
415 propagating in the fluid with vorticity distribution $\Omega = \varepsilon^2 \Omega_2(a_2, b)$. The function
416 $\Omega_2(a_2, b)$ determining flow vorticity may be an arbitrary function setting the initial
417 distribution of vorticity. On integrating it twice we find the vortex component of
418 the average current which is in no way related to the average current induced by
419 the potential wave.
420

421 **4 Examples of the waves**
422

423 Let us consider some special cases following from Eq. (41).
424

425 **4.1 Potential waves**
426

427 In this case $\Omega_2 = 0$ and Eq. (41) becomes the classical nonlinear Schrödinger
428 equation for waves in deep water. Three kinds of analytical solutions of the NLS
429 equation are usually discussed regarding water waves. The first one is the
430 Peregrine breather propagated in space and time (Peregrine, 1983). This wave may
431 be considered as a long wave limit of a breather – a pulsating mode of infinite
432 wavelength (Grimshaw et al., 2010). Two other ones are the Akhmediev breather –
433 the solution periodic in space and localized in time (Akhmediev et al., 1985) and
434 the Kuznetsov-Ma breather – the solution periodic in time and localized in space
435 (Kuznetsov, 1977; Ma, 1979). Both latter solutions evolve against the background
436 of an unperturbed sine wave.

4.2 Gerstner wave

The exact Gerstner solution in complex form is written as (Lamb, 1932; Abrashkin and Yakubovich, 2006; Bennett, 2006):

$$W = a + ib + iA\exp\left[i(ka + \omega t) + kb\right]. \tag{42}$$

It describes a stationary traveling rotational wave with a trochoidal profile. Its dispersion characteristic coincides with the dispersion of linear waves in deep water $\omega^2 = gk$. The fluid particles are moving in circles and there is no drift current.

Equation (42) is the exact solution of the problem. Following Eqs. (8) and (9) the Gerstner wave should be written as

$$W = a_0 + ib + \sum_{n \geq 1} \varepsilon^n \cdot iA\exp\left[i\left(ka_0 + \omega t_0\right) + kb\right]. \tag{43}$$

All the functions w_n in Eqs. (8), (9) have the same form. To derive the vorticity of the Gerstner wave, Eq. (43) should be substituted into Eq. (6). Then one can find that in the linear approximation the Gerstner wave is potential ($\Omega_1 = 0$), but in the quadratic approximation it possesses vorticity:

$$\Omega_{2Gerstner} = -2\omega k^2 |A|^2 e^{2kb}. \tag{44}$$

For this type of the vorticity distribution, the sum of the first two terms in the parentheses in Eq. (41) is equal to zero. From the physical point of view this is due to the fact that the average current induced by the vorticity compensates the potential drift exactly. The packet of weakly nonlinear Gerstner waves in this approximation is not affected by their non-linearity, and the effect of the modulation instability for the Gerstner wave does not occur.

Generally speaking this result is quite obvious. As there is no particle drift in the Gerstner wave, the function ψ_1 equals zero. So, the multiplier of the wave amplitude in Eqs. (38), (39) may be neglected without finding vorticity of the Gerster wave.

Let us consider some particular consequences of the obtained result. For the irrotational ($\Omega_2 = 0$) stationary ($A = |A| = const$) wave, Eq. (40) for the velocity of the drifting flow takes on the form

$$\psi_{1t_1} = -\omega kA^2 e^{2kb}. \tag{45}$$

477 It coincides with the expression for the Stokes drift in Lagrangian coordinates (in
478 the Eulerian variables the profile of the Stokes current may be obtained by the
479 substitution of b for y). Thus, our result may be interpreted as a compensation of
480 the Stokes drift by the shear flow induced by the Gerstner wave in a quadratic
481 approximation. This conclusion is also fair in the "differential" formulation for
482 vorticities. From Eq. (22) it follows that the vorticity of the Stokes drift equals the
483 vorticity of the Gerstner wave with the inverse sign.

484 The absence of a nonlinear term in the NLS equation for the Gerstner waves
485 obtained here in the Lagrangian formulation is a robust result and should appear in
486 the Euler description as well. This follows from the famous Lighthill criterion for
487 the modulation instability because the dispersion relation for the Gerstner wave is
488 linear and does not include terms proportional to the wave amplitude.

489

490 **4.3 Gouyon waves**

491

492 As was shown by Dubreil - Jacotin (1934), the Gerstner wave is a special case of a
493 wide class of stationary waves having vorticity $\Omega = \varepsilon \Omega_*(\psi)$, where Ω_* is an
494 arbitrary function, and ψ is a stream function. Those results were later developed
495 by Gouyon (1958) who explicitly represented the vorticity in the form of a power
496 series $\Omega = \sum\limits_{n=1}^{\infty} \varepsilon^n \Omega_n(\psi)$ (see also the monograph by Sretensky (1977)).

497 When a plane steady flow is considered in the Lagrangian variables, the
498 stream lines ψ coincide with the isolines of the Lagrangian vertical coordinate b
499 (Abrashkin and Yakubovich, 2006; Bennett, 2006). We are going to consider a
500 steady-state wave at the surface of an indefinitely deep water. Assume that there is
501 no undisturbed shear current, but the wave disturbances have vorticity. Then, the
502 formula for the vorticity is written as $\Omega = \sum\limits_{n=1}^{\infty} \varepsilon^n \Omega_n(b)$. Here we will refer to the
503 steady-state waves propagating in such a low-vorticity fluid as to the Gouyon
504 waves. The properties of the Gouyon wave for the first two approximations were
505 studied by Abrashkin and Zen'kovich (1990) in the Lagrangian description.

506 In our case, $\Omega_1 = 0, \Omega_2 \neq 0$ and assuming the function Ω_2 to be independent
507 of the coordinate a we can describe the Gouyon waves. The vorticity Ω_2 depends
508 on the coordinate b only and has the following form

509

$$\Omega_{2Goyuon} = \omega k^2 |A|^2 H(kb),\qquad(46)$$

511

512 where $H(kb)$ is an arbitrary function. In case of $H(kb) = -2\exp(2kb)$, the vorticities
513 of the Gerstner and Gouyon waves in the quadratic approximation coincide
514 (compare Eqs. (44) and (46)). In the considered approximation the Gouyon wave
515 generalizes the Gerstner wave. From Eq. (22) it follows that the function ψ_{t_1} is
516 equal to zero only when the vorticity of the Gouyon wave is equal to the vorticity

517 of the Gerstner wave. Except for this case, the average current ψ_{t_1} will be always
518 present in the modulated Gouyon waves.
519   The substitution of the ratio (46) into Eq. (41) yields the NLS equation for
520 the modulated Gouyon wave:
521

$$i\frac{\partial A}{\partial a_2}-\frac{k}{\omega^2}\frac{\partial^2 A}{\partial t_1^2}-\beta_G k^3|A|^2 A=0;$$

522 (47)

$$\beta_G=1+4\int_{-\infty}^{0}e^{2\tilde{b}}\left(\int_{-\infty}^{\tilde{b}}H(\tilde{b}')d\tilde{b}'\right)d\tilde{b}; \quad \tilde{b}=kb,$$

523

524 where $\tilde{b}$ is a dimensionless vertical coordinate. The coefficient of the nonlinear
525 term in the NLS equation varies when the wave vorticity is taken into account. For
526 the Gerstner wave it may be equal to zero like for the Gouyon wave when the
527 condition
528
529

530

$$\int_{-\infty}^{0}e^{2\tilde{b}}\left(\int_{-\infty}^{\tilde{b}}H(\tilde{b}')d\tilde{b}'\right)d\tilde{b}=-\frac{1}{4}.$$

(48)

531
532

533 is satisfied. Obviously, an infinite number of distributions of the vorticity $H(\tilde{b})$
534 meeting this condition are possible. However, realization of one of them seems to
535 be hardly probable. In the real ocean, distributions of the vorticity with a certain
536 sign of β_G are more likely to be implemented. Its negative values correspond to
537 the defocusing NLS equation and the positive ones are related to the focusing NLS
538 equation. In the latter case, the maximum value of the increment as well as the
539 width of the modulation instability zone of a uniform train of vortex waves vary
540 depending on the value of β_G.

541   Equations (39) and (47) will be focusing for $\psi_{1t_1}<0,\ b\leq 0$ and defocusing if
542 $\psi_{1t_1}>0,\ b\leq 0$. The case of the sign-variable function ψ_{1t_1} requires an additional
543 research. From the physical viewpoint the sign of this function is defined by the
544 ratio of the velocity of the Stokes drift (45) to the velocity of the current induced
545 by the vorticity (the integral term in Eq. (40)). For $\psi_{1t_1}<0$, the Stokes drift either
546 dominates over a vortex current or both of them have the same direction. When
547 $\psi_{1t_1}>0$, the vortex current dominates over the counter Stokes drift. In case of the
548 sign-variable ψ_{1t_1}, the ratio of these currents varies at different vertical levels,
549 thereby requiring direct calculation of β_G.

550
551 **4.4 Waves with inhomogeneous vorticity distribution along both coordinates**

552

553 Neither a vorticity expression nor methods of its definition were discussed when
554 deriving the NLS equation. In Sections 4.2 and 4.3 devoted to the problems of the
555 Gerstner and Gouyon waves the vorticity was set to be proportional to a square
556 modulus of the wave amplitude. Note that waves can propagate against the
557 background of some vortex current, for example, the localized vortex. In this case
558 the vorticity may be presented in the form

559

560
$$\Omega_2(a_2,b) = \omega\left[\varphi_v(a_2,b) + k^2|A|^2 \varphi_w(a_2,b)\right],$$

561

562 where the function $\omega\varphi_v$ defines the vorticity of the background vortex current and

563 the function $\omega k^2|A|^2 \varphi_w$ defines the vorticity of waves. In the most general case
564 both functions depend on the horizontal Lagrangian coordinate as well. Then,
565 Eq.(41) takes a form
566

$$i\frac{\partial A}{\partial a_2} - \frac{k}{\omega^2}\frac{\partial^2 A}{\partial t_1^2} - k\beta_v(a_2)A - k^3\left(1 + \beta_w(a_2)\right)|A|^2 A = 0,$$

567 (49)

$$\beta_{v,w}(a_2) = 4\int_{-\infty}^{0} e^{2\tilde{b}}\left(\int_{-\infty}^{\tilde{b}}\varphi_{v,w}(a_2,\tilde{b}')d\tilde{b}'\right)d\tilde{b}.$$

568

569 The substitution

570

571
$$A_* = A\exp\left(-ik\int_{-\infty}^{a_2}\beta_v(a_2)da_2\right)$$
(50)

572

573 reduces Eq. (49) to the NLS equation with a non-uniform multiplier for the
574 nonlinear term:

575

576
$$i\frac{\partial A^*}{\partial a_2} - \frac{k}{\omega^2}\frac{\partial^2 A^*}{\partial t_1^2} - k^3\left(1 + \beta_w(a_2)\right)\left|A^*\right|^2 A^* = 0.$$
(51)

577

578 Let us consider the propagation of the Gouyon wave, when $\beta_w = const = \beta_G - 1$
579 and Eq.(51) turns into the classical NLS equation (47). As shown in Sec. 4.3, it
580 describes the modulated Gouyon waves. Therefore, on the substitution of Eq. (50)
581 one can conclude that the propagation of the Gouyon waves against the
582 background of the non-uniform vortex current results in the variation of the wave
583 number of the carrier wave. For $\beta_w = 0$, Eq. (51) describes the propagation of a
584 packet of potential waves against the background of the non-uniform weakly
585 vortical current. The specific features of the wave propagation related to the
586 variable β_w require special investigation.

5 On equivalence of Lagrangian and Eulerian approaches

Consider the correlation between the Eulerian and the Lagrangian description of wave packets. To obtain the value for elevation of the free surface we substitute the expressions (8), (9), (13) and $b = 0$ into the equation for $Y = \text{Im}\, W$ written in the following form

$$Y_L = \varepsilon\, \text{Im}\, A(a_2, t_1) \exp i(ka_0 + \omega\, t_0),$$

where $A(a_2, t_1)$ is the solution of Eq. (41). This expression defines the wave profile in Lagrangian coordinates. To rewrite this equation in the Eulerian variables it is necessary to define a via X. From the relation (8) follows

$$X = a + \varepsilon\, \text{Re}\left(w_1 + \sum_{n=2} \varepsilon^{n-1} w_n \right) = a + O(\varepsilon),$$

and the elevation of the free surface in the Eulerian variables Y_E will be written as

$$Y_E = \varepsilon\, \text{Im}\, A(X_2, t_1) \exp i(kX_0 + \omega\, t_0) + O(\varepsilon^2); \quad X_l = \varepsilon^l X.$$

The coordinate a plays the role of X, so the following substitutions are valid for the Lagrangian approach:

$$a_0 \to X_0; \quad a_1 \to X_1; \quad a_2 \to X_2.$$

This result may be called an "equivalence principle" between the Lagrange and the Euler descriptions for solutions in the linear approximation. This principle is valid for both the potential and rotational waves.

To express the solution of Eq. (41) in the Eulerian variables it is necessary to use the equivalence principle and to replace the horizontal Lagrangian coordinate a_2 by the X_2 coordinate. So, there are no discrepancies between the Eulerian and the Lagrangian estimations of the NLS equation for the free surface elevation.

Taking this into account we can conclude that the result will be the same in the Eulerian description, if the vorticity Ω_2 is a function of the x, y coordinates. So, when studying the wave packets dynamics in the vortical liquid in the Eulerian variables it is necessary to replace (ex. in Eq. (41) or (51)) the horizontal Lagrangian coordinate by the Eulerian one.

Equation (47) can also be derived in Eulerian variables. The key idea is to take into consideration a weak shear flow. This approach is similar to the method used in the paper by Trulsden and Hejervick (2009), where the wave propagates along a weak horizontal shear current. Shrira and Slunyaev (2014) used this technique to study trapped waves in an uniform jet stream. They derived the NLS

equation for a single mode. Later, Slunyaev (2016) generalized the result to the case of a vortex jet flow. Our result was obtained with a weak vertical shear flow taking into account. In particular, to describe modulated Guyon waves, the Johnson approach (1976) should be modified, assuming a shear flow of the order of epsilon.

The solutions of the considered problem in the Lagrange and the Euler forms in the quadratic and cubic approximations differ from each other. To obtain a full solution in the Lagrange form one should find functions $\psi_1, \psi_2, \psi_3, f_2, f_3$. This problem should be considered within a special study.

6 Conclusion

We have derived the vortex-modified nonlinear Schrödinger equation using the method of multiple scale expansions in the Lagrange variables. The fluid vorticity Ω is specified as an arbitrary function of the Lagrangian coordinates, which is quadratic in the small parameter of the wave steepness. The calculations have been performed introducing a complex coordinate of the fluid particle trajectory.

The nonlinear evolution equation for the wave packet in the form of the nonlinear Schrödinger equation has been derived as well. From the mathematical viewpoint, the novelty of this equation is related to the emergence of a new term proportional to the envelope amplitude and the variance of the coefficient of the nonlinear term. If the vorticity depends on the vertical Lagrangian coordinate only (the Gouyon waves), this coefficient is constant. There are special cases, when the coefficient of the nonlinear term equals zero and the resulting non-linearity disappears. The Gerstner wave belongs to the latter case. Another effect revealed in the present study is the relation of the vorticity to the wave number shift in the carrier wave. This shift is constant for the modulated Gouyon wave. If the vorticity depends on both Lagrangian coordinates, the shift of the wave number is horizontally inhomogeneous. It is shown that the solution of the NLS equation for weakly rotational waves in the Eulerian variables may be obtained from the Lagrangian solution by an ordinary change of the horizontal coordinates.

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