Lagrange form of the nonlinear Schrödinger equation for low-vorticity waves in deep water

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The nonlinear Schrödinger (NLS) equation describing propagation of weakly rotational wave packets in an infinitely deep fluid in the Lagrangian coordinates was derived. The vorticity is assumed to be an arbitrary function of the Lagrangian coordinates and quadratic in the small parameter proportional to the wave's steepness. The effects of vorticity are manifested in a shift of the wavenumber in the carrier wave as well as in variation of the coefficient multiplying the nonlinear term. In case of dependence of the vorticity on the vertical Lagrangian coordinate only (the Gouyon waves) the shift of the wavenumber and the respective coefficient are constant. When the vorticity is dependent on both Lagrangian coordinates the shift of the wavenumber is horizontally heterogeneous. There are special cases (the Gerstner wave is among them) when the vorticity is proportional to the square of the wave's amplitude and the resulting non-linearity disappears, thus making the equations of dynamics of the wave packet to be linear. It is shown that the NLS solution for weakly rotational waves in the Eulerian variables could be obtained from the Lagrangian solution by an ordinary change of the horizontal coordinates.

Key words: nonlinear Schrödinger equation, vorticity, water waves

1 Introduction

The nonlinear Schrödinger (NLS) equation was first derived by Benny and Newell (1967) and then Zakharov (1968), who used the Hamiltonian formalism for a description of waves propagation in deep water. Hashimoto and Ono (1972) and Davey (1972) independently obtained the same result. Like Benney and Newell (1967) they use the method of multiple scale expansions in the Euler coordinates. In their turn, Yuen and Lake (1975) derived the NLS equation on the basis of the averaged Lagrangian method. Benney and Roskes (1969) extended these two-dimensional theories in the case of three-dimensional wave perturbations in finite depth fluid and obtained the equations which are now called the Davey-Stewartson equations. In this particular case the equation proves the existence of transverse instability of the plane wave which is much stronger than longitudinal one. This circumstance diminishes the role and meaning of the NLS equation for sea applications. Meanwhile, the 1-D NLS equation has been successfully tested many times in laboratory wave tanks and in comparison of the natural observations with the numerical calculations.

In all of those works wave motion was considered to be potential. However, the formation and propagation of waves frequently occurs at the background of a shear flow possessing vorticity. Wave-train modulations at arbitrary vertically sheared currents were studied by Benney and his group. Using the method of multiple scales Johnson (1976) examined a slow modulation of the harmonic wave moving at the surface of an arbitrary shear flow with the velocity profile $U(y)$, where y is the vertical coordinate. He derived the NLS equation with the coefficients, which in a complicated way depend on the shear flow (Johnson, 1976). Oikawa et al. (1985) considered properties of instability of weakly nonlinear three-dimensional wave packets in the presence of a shear flow. Their simultaneous equations are reduced to the known NLS equation when requiring the wave's evolution to be purely two-dimensional. Li et al. (1987) and Baumstein (1998) studied the modulation instability of the Stokes wave-train and derived the NLS equation for uniform shear flow in deep water, when $U(y)=\Omega_0 y$ and $\Omega_z = \Omega_0$ is constant vorticity (z is the horizontal coordinate normal to the plane of the flow x, y; the wave propagates in x direction).

Thomas et al. (2012) generalized their results for the finite-depth fluid and confirmed that linear shear flow may significantly modify the stability properties of the weakly nonlinear Stokes waves. In particular, for the waves propagating in the direction of the flow the Benjamin-Feir (modulational) instability can vanish in the presence of positive vorticity ($\Omega_0 < 0$) for any depth.

In the traditional Eulerian approach to propagation of weakly nonlinear waves at the background current the shear flow determines the vorticity in a zero approximation. Depending on the flow profile $U(y)$ it may be sufficiently arbitrary and equals to $-U'(y)$. At the same time the vorticity of wave's perturbations $\Omega_n, n \geq 1$, i.e. the vorticity in a first and subsequent approximations by the parameter of the wave steepness $\varepsilon = kA_0$ (k is the wavenumber, A_0 is the wave amplitude) depends on its form. In the Eulerian coordinates the vorticity of wave perturbations are the functions not only of y, but depend on variables x and t as well. Plane waves on a shear flow with the linear vertical profile represent an exception of this statement (Li et al., 1987; Baumstein, 1998; Thomas et al., 2012). For such waves the vorticity of a zero approximation is constant, and all of the vorticities in wave perturbations equal to zero. For the arbitrary vertical profile of the shear flow (Johnson, 1976) expressions for the functions Ω_n could be hardly predicted even quantitatively.

The Lagrangian method allows one to apply a different approach. In the plane flow the vorticity of fluid particles is preserved and could be expressed via the Lagrangian coordinates only. Thus not only the vertical profile of the shear flow defining the vorticity of a zero approximation, but the expressions for the vorticity of the following orders of smallness could be given as known initial conditions as well. The expression for the vorticity is presented in the following form:

$$\Omega(a,b) = -U'(b) + \sum_{n\geq 1} \varepsilon^n \Omega_n(a,b),$$

here a,b - the horizontal and the vertical Lagrangian coordinates respectively, $U(b)$ - the vertical profile of the shear flow, and the particular conditions for definition of the function Ω_n could be found while solving the problem. For the given shear flow this approach allows one to study wave perturbations with the most general law of distribution of the vorticities Ω_n. In the present paper the shear flow and the vorticity are absent in the linear approximation ($U = 0; \Omega_1 = 0$), but the vorticity in the quadratic approximation is an arbitrary function. That corresponds to the rotational flow proportional to ε^2. We can define both the shear flow and the localized vortex.

The dynamics of plane wave-trains on the background flows with the arbitrary low vorticity was not studied earlier. An idea to study wave-trains with the quadratic (with respect to the parameter of the wave's steepness) vorticity was realized earlier for the spatial problems in the Euler variables. Hjelmervik and Trulsen (2009) derived the NLS equation for the vorticity distribution:

$$\Omega_y / \omega = O(\varepsilon^2); \quad (\Omega_x, \Omega_z)/\omega = O(\varepsilon^3),$$

here ω is the wave frequency. The vertical vorticity of wave perturbations by a factor of ten exceeds the other two components of the vorticity. This vorticity distribution corresponds to the low (order of ε) velocity of the horizontally inhomogeneous sheared flow. Hjelmervik and Trulsen (2009) used the NLS equation to study the statistics of rogue waves on narrow current jets, and Onorato et al. (2011) used this equation to study the opposite flow rogue waves. The effect of low vorticity (order of magnitude ε^2) in the paper by Hjelmervik and Trulsen (2009) is reflected in the NLS equation. This fact in the same way as the NLS nonlinear term for plane potential waves should be explained by the presence of an average current non-uniformed over the fluid depth.

Colin et al. (1995) have considered the evolution of three-dimensional vortex disturbances in the finite-depth fluid for a different type of vorticity distribution:

$$\Omega_y = 0; \quad (\Omega_x, \Omega_z)/\omega = O(\varepsilon^2)$$

and by means of the multiple scale expansion method in the Eulerian variables reduced the problem to a solution of the Davey-Stewartson equations. In this case vorticity components are calculated after the solution of the problem. As well as for the traditional Eulerian approach (Johnson, 1976) the form of distribution of the quadratic vorticity is very special and does not cover all of its numerous possible distributions.

In this paper we consider the plane problem of propagation of the nonlinear wave packet in an ideal incompressible fluid with the following form of vorticity distribution:

$$\Omega_z/\omega = O(\varepsilon^2).$$

In contrast to Hjelmervik and Trulsen (2009), Onorato et al. (2011) and Colin et al. (1996) the flow is two-dimensional (respectively $\Omega_x = \Omega_y = 0$). Propagation of the packet of potential waves causes the weak counter flow underneath the free water surface with its velocity proportional to the square of the wave's steepness (McIntyre, 1982). In the considered problem this potential flow is superimposed with the rotational one of the same order. It results in appearance of an additional term in the NLS equation and in changing of the coefficient in the nonlinear term. So a difference from the NLS solutions derived for strictly potential fluid motion was revealed.

The examination is held in the Lagrangian variables. The Lagrangian variables are rarely used in fluid mechanics. This is due to a more complex type of nonlinear equations in the Lagrange form. However, when considering the vortex-induced oscillations of the free fluid surface the Lagrangian approach has two major advantages. First, unlike the Euler description method the shape of the free surface is known and is determined by the condition of the vertical Lagrangian coordinate's being equal to zero ($b = 0$). Second, the vortical motion of liquid particles is confined within the plane and represents the function of the Lagrangian variables $\Omega_z = \Omega_z(a,b)$, so the type of the vorticity distribution in the fluid can be set initially. The Eulerian approach does not allow one to do this. In this case the second-order vorticity is defined as a known function of the Lagrangian variables.

Here hydrodynamic equations are solved in the Lagrange form by multiple scale expansion method. The nonlinear Schrödinger equation with the variable coefficients is derived. The ways to reduce it to the NLS equation with the constant coefficients are studied.

The paper is organized as follows. Section 2 describes the Lagrangian approach to the study of wave oscillations at the free surface of the fluid. Zero of the Lagrangian vertical coordinate is placed at the free surface, thus facilitating formulation of the pressure boundary conditions. The peculiarity of the suggested approach is the introduction of a complex coordinate of a fluid particle's trajectory. In Section 3 the nonlinear evolution equation on the basis of the method of multiple scale expansion is derived. In Section 4 different solutions of the NLS equation adequately describing various examples of vortex waves are considered.

In Section 5 the transform from the Lagrangian coordinates to the Euler description of the solutions of the NLS equation is shown. Section 6 summarizes the obtained results.

2 Basic equations in the Lagrangian coordinates

Consider the propagation of a packet of gravity surface wave in rotational infinitely deep fluid. The 2D hydrodynamic equations of an incompressible inviscid fluid in the Lagrangian coordinates have the following form (Lamb, 1932; Abrashkin and Yakubovich, 2006; Bennett, 2006):

$$\frac{D(X,Y)}{D(a,b)} = [X,Y] = 1, \tag{1}$$

$$X_{tt}X_a + (Y_{tt} + g)Y_a = -\frac{1}{\rho}p_a, \tag{2}$$

$$X_{tt}X_b + (Y_{tt} + g)Y_b = -\frac{1}{\rho}p_b, \tag{3}$$

where X, Y are the horizontal and vertical Cartesian coordinates and a, b are the horizontal and vertical Lagrangian coordinates of fluid particles, t is time, ρ is fluid density, p is pressure, g is acceleration due to gravity, the subscripts mean differentiation with respect to the corresponding variable. The square brackets denote the Jacobian. The axis b is directed upwards, and $b = 0$ corresponds to the free surface. Eq. (1) is a volume conservation equation. Eq. (2) and (3) are momentum equations. The problem geometry is presented in Fig. 1.

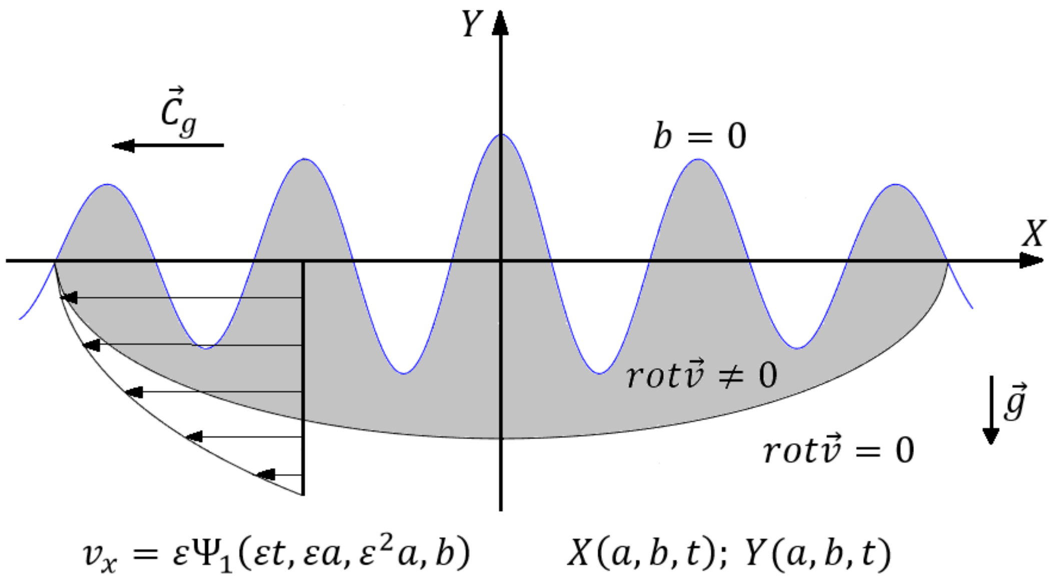

Fig. 1. Problem geometry: v_x is the average current.

By means of the cross differentiation it is possible to exclude the pressure and to obtain the condition of conservation of vorticity along the trajectory (Lamb, 1932; Abrashkin and Yakubovich, 2006; Bennett, 2006):

$$X_{ta}X_b + Y_{ta}Y_b - X_{tb}X_a - Y_{tb}Y_a = \Omega(a,b). \tag{4}$$

This equation is equivalent to the momentum Eq. (2) and (3), but it involves an explicit vorticity of liquid particles Ω, which in case of two-dimensional flows is the function of the Lagrangian coordinates only.

We introduce the complex coordinate of the trajectory of a fluid particle $W = X + iY \quad (\overline{W} = X - iY)$, the overline means complex conjugation. In the new variables the Eq. (1) and (4) take the following form:

$$\left[W,\overline{W}\right] = -2i, \tag{5}$$

$$\mathrm{Re}\left[W_t,\overline{W}\right] = \Omega(a,b), \tag{6}$$

Eqs. (2) and (3) after simple algebraic manipulations could be reduced to the following single equation:

$$W_{tt} = -ig + i\rho^{-1}[p,W]. \tag{7}$$

Eqs. (5) and (6) will be used further to find the coordinates of complex trajectories of fluid particles, and Eq. (7) determines the fluid pressure. The boundary conditions are the non-flowing condition at the bottom ($Y_t \to 0$ at $b \to -\infty$) and the constant pressure at the free surface (at $b = 0$).

The Lagrangian coordinates mark the position of fluid particles. In the Eulerian description the displacement of the free surface $Y_s(X,t)$ is calculated in an explicit form, but in the Lagrangian description it is defined parametrically with the following equalities: $Y_s(a,t) = Y(a,b=0,t)$; $X_s(a,t) = X(a,b=0,t)$, where the role of a parameter plays the Lagrangian horizontal coordinate a. Its value along the free surface $b = 0$ varies in the range $(-\infty;\infty)$. In the Lagrangian coordinates the function $Y_s(a,t)$ defines the displacement of the free surface.

3 Derivation of evolution equation

Let us present the function W using the multiple scales method in the following form:

$$W = a_0 + ib + w(a_l,b,t_l), \quad a_l = \varepsilon^l a, \quad t_l = \varepsilon^l t; \quad l = 0,1,2, \tag{8}$$

where ε - the a small parameter of the wave's steepness. All of unknown functions and the given vorticity can be represented as a series in this parameter:

$$w = \sum_{n=1} \varepsilon^n w_n; \quad p = p_0 - \rho g b + \sum_{n=1} \varepsilon^n p_n; \quad \Omega = \sum_{n=1} \varepsilon^n \Omega_n(a,b). \tag{9}$$

In the formula for the pressure a term with hydrostatic pressure is selected, p_0 - constant atmospheric pressure at the fluid surface. Let us substitute the representations (8) and (9) in Eqs. (5)-(7).

3.1 Linear approximation

In a first approximation in the small parameter we have the following simultaneous equations:

$$\operatorname{Im}\left(i w_{1a_0} + w_{1b}\right) = 0, \tag{10}$$

$$\operatorname{Re}\left(i w_{1a_0} + w_{1b}\right)_{t_0} = -\Omega_1, \tag{11}$$

$$w_{1t_0t_0} + \rho^{-1}\left(p_{1a_0} + i p_{1b}\right) = i g w_{1a_0}. \tag{12}$$

The solution satisfying the continuity Eq. (10) and the equation of conservation of vorticity (11) describes a monochromatic wave (for definiteness, we consider the wave propagating to the left) and the average horizontal current

$$w_1 = A(a_1, a_2, t_1, t_2)\exp\left[i(ka_0 + \omega t_0) + kb\right] + \psi_1(a_1, a_2, b, t_1, t_2); \quad \Omega_1 = 0, \tag{13}$$

here A is the complex amplitude of the wave, ω is its frequency, and k is the wave number. The function ψ_1 is real and it will be determined under consideration of the following approximation.

Substitution of solution (13) in Eq. (12) yields the equation for the pressure

$$\rho^{-1}\left(p_{1a_0} + i p_{1b}\right) = \left(\omega^2 - gk\right) A \exp\left[i(ka_0 + \omega t_0) + kb\right], \tag{14}$$

which is solved analytically

$$p_1 = -\operatorname{Re}\frac{i\left(\omega^2 - gk\right)}{k} \rho A \exp\left[i(ka_0 + \omega t_0) + kb\right] + C_1(a_1, a_2, t_1, t_2), \tag{15}$$

where C_1 is an arbitrary function. The boundary condition at the free surface is $p_1\big|_{b=0} = 0$, which leads to $\omega^2 = gk$ as well as $C_1 = 0$. Thus, in the first approximation the pressure correction p_1 is equal to zero.

3.2 Quadratic approximation

The equations of the second order of the perturbation theory can be written as follows:

$$\mathrm{Im}\left(iw_{2a_0} + w_{2b} + iw_{1a_1} - w_{1a1}\overline{w_{1b}} \right) = 0\,, \qquad (16)$$

$$\mathrm{Re}\left[iw_{2t_0a_0} + w_{2t_0b} + i\left(w_{1t_0a_1} + w_{1t_1a_0}\right) - w_{1t_0a_0}\overline{w_{1b}} + w_{1t_1b} + w_{1t_0b}\overline{w_{1a_0}} \right] = -\Omega_2\,, \quad (17)$$

$$w_{2t_0t_0} + \rho^{-1}\left(p_{2a_0} + ip_{2b} \right) = ig\left(w_{2a_0} + w_{a_1} \right) - 2w_{1t_1t_0}\,. \qquad (18)$$

Substituting expression (13) for w_1 to Eq. (16) we have:

$$\mathrm{Im}\left[iw_{2a_0} + w_{2b} - i\left(k\psi_{1b}A - A_{a_1}\right)\exp\left[i\left(ka_0 + \omega\, t_0\right) + kb\right] - ik^2|A|^2 e^{2kb} + i\psi_{1a_1} \right] = 0\,, \quad (19)$$

which is integrated as follows:

$$w_2 = i\left(kA\psi_1 - bA_{a_1}\right)\exp\left[i\left(ka_0 + \omega\, t_0\right) + kb\right] + \psi_2 + if_2\,, \qquad (20)$$

here ψ_2, f_2 are functions of slow coordinates and the Lagrange vertical coordinate b and:

$$f_{2b} = k^2|A|^2\exp 2kb - \psi_{1a_1}\,, \qquad (21)$$

the function ψ_2 is an arbitrary real function. It will be determined by solving the following cubic approximation.

When substituting (13), (20) in (17) all of the terms containing the exponential factor neglect each other, and the remaining terms satisfy the equation:

$$\psi_{1t_1b} = -2k^2\omega|A|^2\exp(2kb) - \Omega_2\,. \qquad (22)$$

The expression for the function ψ_1 can be determined by a simple integration. It should be emphasized that the vorticity of the second approximation, being a part of Eq. (22), is an arbitrary function of the slow horizontal and vertical Lagrange coordinates, so that $\Omega_2 = \Omega_2\left(a_1, a_2, b\right)$.

Taking into account the solutions of the first two approximations we can write Eq. (18) as:

$$\rho^{-1}\left(p_{2a_0} + ip_{2b} \right) = i\left(gA_{a_1} - 2\omega A_{t_1}\right)\exp\left[i\left(ka_0 + \omega\, t_0\right) + kb\right] + ig\psi_{1a_1}\,. \qquad (23)$$

Its solution determines the pressure correction:

$$p_2 = \mathrm{Re}\left[\frac{1}{k}\left(gA_{a_1} - 2\omega A_{t_1}\right)\exp\left[i\left(ka_0 + \omega t_0\right) + kb\right]\right] + \rho g\int_0^b \psi_{1a_1} db + C_2\left(a_1, a_2, t_1, t_2\right) \quad (24)$$

The limits of integration in the penultimate term are chosen so that this integral term equals to zero at the free surface. Due to the boundary condition for pressure ($p_2(b=0)=0$), $C_2 = 0$, and

$$A_{t_1} - c_g A_{a_1} = 0; \quad c_g = \frac{g}{2\omega} = \frac{1}{2}\sqrt{\frac{g}{k}}, \quad (25)$$

here c_g is the group velocity of wave propagation in deep water, which in this approximation is independent of the fluid vorticity. As expected, the wave of this approximation moves with the group velocity c_g to the left (the “minus” sign in the Eq. (25)).

3.3 Cubic approximation

The equation of continuity and the condition of conservation of vorticity in the third approximation have the form

$$\mathrm{Im}\left[iw_{2a_0} + w_{3b} + i\left(w_{1a_2} + w_{2a_1} + w_{2a_0}\right) - \left(w_{1a_1} + w_{2a_2}\right)\overline{w_{1b}} - w_{1a_0}\overline{w_{2b}}\right] = 0, \quad (26)$$

$$\mathrm{Re}\left[iw_{3t_0a_0} + w_{3t_0b} + i\left(w_{1t_2a_0} + w_{1t_1a_1} + w_{1t_0a_2} + w_{2t_1a_0} + w_{2t_0a_1}\right) + w_{1t_2b} - \overline{w_{2b}}w_{1t_0a_0} - \right.$$
$$\left. + w_{2t_1b} - w_{1b}\left(w_{1t_0a_1} + w_{1t_1a_0} + w_{2t_0a_0}\right) + + \overline{w_{1a_0}}\left(w_{1t_1b} + w_{2t_0b}\right) + w_{1t_0b}\left(\overline{w_{1a_1}} + \overline{w_{2a_0}}\right)\right] = -\Omega_3. \quad (27)$$

We substitute the solutions of the first and second approximations in the simultaneous equations:

$$\mathrm{Im}\left[iw_{3a_0} + w_{3b} + i\left(\psi_{1a_2} + \psi_{2a_1}\right) + 2k(kb+1)A\overline{A_{a_1}}e^{2b} + G_b e^{i\left(ka_0 + \omega t_0\right) + kb}\right] = 0, \quad (28)$$

$$\mathrm{Re}\left\{\left[iw_{3a_0}+w_{3b}+\left(G_b+2k\psi_{1t_1b}\omega^{-1}A\right)e^{i\left(ka_0+\omega t_0\right)+kb}\right]_{t_0}+\psi_{2t_1b}+\psi_{1t_2b}+\right. \tag{29}$$
$$\left.+i\omega k(4kb+5)A\overline{A_{a_1}}e^{2kb}\right\}=-\Omega_3,$$

$$G=ibA_{a_2}+\frac{b^2}{2}A_{a_1a_1}-(kb+1)\psi_1A_{a_1}-\left(ik\psi_2+kf_2-\frac{k^2}{2}\psi_1^2\right)A. \tag{30}$$

We sought the solution for the third approximation in the following form:

$$w_3=(G_1-G)e^{i\left(ka_0+\omega t_0\right)+kb}+G_2e^{-i\left(ka_0+\omega t_0\right)+kb}+\psi_3+if_3, \tag{31}$$

here G_1, G_2, ψ_3, f_3 are functions of slow coordinates and b. Substituting this expression in (28) and (29) we immediately find that:

$$f_{3b}+\psi_{2a_1}+\psi_{1a_2}+k(kb+1)\left(A\overline{A_{a_1}}-\overline{A}A_{a_1}\right)e^{2kb}=0, \tag{32}$$

$$\psi_{2t_1b}+\psi_{1t_2b}+\frac{1}{2}(4kb+5)\omega k\left(A\overline{A_{a_1}}-\overline{A}A_{a_1}\right)e^{2kb}=-\Omega_3. \tag{33}$$

The function ψ_2 according to Eq. (33) is determined by a known solution for A and ψ_1, and by the given distribution Ω_3. The expression for the function f_3 is derived then from Eq. (32). These functions determine the horizontal and vertical average movements respectively. But in this approximation they are not included in the evolution equation for the wave envelope. The function ψ_3 should be determined in the next approximation.

When solving (28) and (29) we found:

$$G_1=-k\omega^{-1}\psi_{1t_1}A;\quad G_2=k\omega^{-1}\left(2ke^{-2kb}\int\limits_{-\infty}^{b}\psi_{1t_1}e^{2kb'}db'-\psi_{1t_1}\right)\overline{A}. \tag{34}$$

These relationships should be substituted in the Eq. (7), which in this approximation has the following form:

$$w_{3t_0t_0}-igw_{3a_0}=i\rho^{-1}\left[i\left(p_{2a_1}+p_{3a_0}\right)-p_{3b}-p_{2b}w_{1a_0}+\rho g\left(w_{1a_2}+w_{2a_1}\right)\right]- \tag{35}$$
$$-2w_{1t_2t_0}-w_{1t_1t_1}-2w_{2t_0t_1}.$$

Taking into account (13), (20), (24), (31) and (34) we rewrite it as follows:

$$\rho^{-1}\left(p_{3a_0}+ip_{3b}\right)=\left(-2i\omega\frac{\partial A}{\partial t_2}+ig\frac{\partial A}{\partial a_2}-\frac{\partial^2 A}{\partial t_1^2}+2\omega k\psi_{1t_1}A\right)e^{i\left(ka_0+\omega t_0\right)+kb}+ \quad (36)$$

$$+2\omega^2 G_2\bar{A}e^{-i\left(ka_0+\omega t_0\right)+kb}+ig\left(\psi_{2a_1}+\psi_{1a_2}\right)+I;\quad I=-g\left(f_{2a_1}-\int_b^0\psi_{1a_1a_1}db\right)-\psi_{t_1t_1}.$$

Due to relationships (21), (22) and (25) the derivative of I by the vertical Lagrangian coordinate is zero ($I_b=0$), so I is the only function of the slow coordinates and time - $a_l,t_l,l\geq 1$. The contribution to the pressure of that term $I\left(a_l,t_l\right)\neq 0$ will be complex, so it requires $I=0$.

The solution of Eq. (36) yields the expression for the pressure perturbation in the third approximation:

$$\frac{p_3}{\rho}=\operatorname{Re} ik^{-1}\left(2i\omega\frac{\partial A}{\partial t_2}-ig\frac{\partial A}{\partial a_2}+\frac{\partial^2 A}{\partial t_1^2}-4\omega k^2 Ae^{-2kb}\int_{-\infty}^{b}\psi_{1t_1}e^{2kb'}db'\right)e^{i\left(ka_0+\omega t_0\right)+kb}+$$

$$+\rho g\int_0^b\left(\psi_{2a_1}+\psi_{1a_2}\right)db'. \quad (37)$$

In Eq. (37) the limits of integration for the second integral term have been pre-selected to satisfy the boundary condition at the free surface (the pressure p_3 should turn to zero). Then the factor before the exponent should be equal to zero:

$$2i\omega\frac{\partial A}{\partial t_2}-ig\frac{\partial A}{\partial a_2}+\frac{\partial^2 A}{\partial t_1^2}-4\omega k^2 A\int_{-\infty}^{0}\psi_{1t_1}e^{2kb}db=0. \quad (38)$$

Introducing the “running” coordinate $\zeta_2=a_2+c_g t_2$ we may reduce Eq. (38) in a compact form:

$$i\frac{\partial A}{\partial a_2}-\frac{k}{\omega^2}\frac{\partial^2 A}{\partial t_1^2}+\frac{4k^3 A}{\omega}\int_{-\infty}^{0}\psi_{1t_1}e^{2kb}db=0. \quad (39)$$

Further it will be shown that variables in Eqs. (38), (39) were chosen in the easiest form for their reduction (under the particular assumptions) to the classical NLS equation.

The explicit form of the function ψ_{1t_1} is found by integration of Eq. (22):

$$\psi_{1t_1}=-k\omega|A|^2 e^{2kb}-\int_{-\infty}^{b}\Omega_2\left(a_2,b'\right)db'-U\left(a_2,t_1\right), \quad (40)$$

This expression includes three terms. All of them describe a certain component of the average current. The first one is proportional to square of the amplitude modulus and describes the classical potential drift of fluid particles (see (Henderson et al. (1999) for example). The second one is caused by the presence of low vorticity in the fluid. And, finally, the third item, including $U(a_2,t_1)$ term, describes an additional potential flow. It appears while integrating Eq. (22) over the vertical coordinate b and will evidently not disappear in case of $A=0$ as well. This is a certain external flow which must be attributed with the definite physical sense in each specific problem. Note that a term of that kind arises in the Eulerian description of potential wave oscillations of the free surface as well. In the paper by Stocker and Peregrine (1999) it was chosen $U = U_* \sin(kx - \omega t)$ and was interpreted as a harmonically changing surface current induced by the internal wave. We shall consider further $U = 0$.

Eq. (39) may be written in the final form after substitution of Eq. (40):

$$i\frac{\partial A}{\partial a_2} - \frac{k}{\omega^2}\frac{\partial^2 A}{\partial t_1^2} - k\left(k^2|A|^2 + \beta(a_2)\right)A = 0,$$
$$\beta(a_2) = \frac{4k^2}{\omega}\int_{-\infty}^{0} e^{2kb}\left(\int_{-\infty}^{b}\Omega_2(a_2,b')db'\right)db. \qquad (41)$$

This is the nonlinear Schrӧdinger equation for the packet of surface gravity waves propagating in the fluid with vorticity distribution $\Omega = \varepsilon^2\Omega_2(a_2,b)$. The function $\Omega_2(a_2,b)$ determining flow vorticity may be an arbitrary function setting the initial distribution of vorticity. When integrating it twice we find the vortex component of the average current which is in no way related to the average current induced by the potential wave.

4 Examples of the waves

Let us consider some special cases arising from Eq. (41).

4.1 Potential waves

In this case $\Omega_2 = 0$ and Eq. (41) becomes the classical nonlinear Schrӧdinger equation for waves in deep water. Three kinds of analytical solutions of the NLS equation are usually discussed regarding to water waves. The first is the Peregrine breather propagated in space and time (Peregrine, 1983). This wave may be considered as a long wave limit of a breather - a pulsating mode of an infinite wavelength (Grimshaw et al., 2010). Two another ones are the Akhmediev breather - the solution periodic in space and localized in time (Akhmediev et al., 1985) and the Kuznetsov-Ma breather - the solution periodic in time and localized in space (Kuznetsov, 1977; Ma, 1979). Both latter solutions evolve at the background of the unperturbed sine wave.

4.2 Gerstner wave

The exact Gerstner solution in the complex form is written as (Lamb, 1932; Abrashkin and Yakubovich, 2006; Bennett, 2006):

$$W = a + ib + iA\exp\left[i\left(ka + \omega t\right) + kb\right]. \tag{42}$$

It describes a stationary traveling rotational wave with a trochoidal profile. Their dispersion characteristic coincides with the dispersion of linear waves in the deep water $\omega^2 = gk$. Fluid particles are moving in circles and the drift current is absent.

Eq. (42) represents the exact solution of the problem. Following Eqs. (8), (9) the Gerstner wave should be written as follows

$$W = a_0 + ib + \sum_{n \geq 1} \varepsilon^n \cdot iA\exp\left[i\left(ka_0 + \omega t_0\right) + kb\right]. \tag{43}$$

All of the functions w_n in Eqs. (8), (9) have the same form. To derive the vorticity of the Gerstner wave Eq. (43) should be substituted in Eq. (6). Then in could be found that in the linear approximation the Gerstner wave is potential ($\Omega_1 = 0$), but in the quadratic approximation it possesses vorticity

$$\Omega_{2Gerstner} = -2\omega k^2 |A|^2 e^{2kb}. \tag{44}$$

For this type of the vorticity distribution the first two terms in the parentheses in Eq. (41) neglect each other. From the physical point of view this is due to the fact that the average current induced by the vorticity compensates the potential drift exactly. The packet of weakly nonlinear Gerstner waves in this approximation is not affected by their non-linearity, and the effect of the modulation instability for the Gerstner wave is absent.

Generally speaking this result is quite obvious. As there are no particle's drift in the Gerstner wave the function ψ_1 equals to zero. So the multiplier of the wave's amplitude in Eqs. (38), (39) may be neglected initially without derivation of the vorticity of the Gerster wave.

Let's consider some particular consequences of the obtained result. For the irrotaional ($\Omega_2 = 0$) stationary ($A = |A| = const$) wave Eq. (40) for the velocity of the drifting flow takes the form

$$\psi_{1t_1} = -\omega k A^2 e^{2kb}. \tag{45}$$

It coincides with the expression for the Stokes drift in the Lagrangian coordinates (in the Eulerian variables the profile of the Stokes current could be obtained by the substitution of b to y). Thus, our result may be interpreted as a compensation of the Stokes's drift by the shear flow induced by the Gerstner wave in a square approximation. This conclusion is also fair in the "differential" formulation for vorticities. From Eq. (22) it follows that the vorticity of the Stokes drift equals to the vorticity of the Gerstner wave with the inverse sign.

The absence of a nonlinear term in the NLS equation for the Gerstner waves obtained here in the Lagrangian formulation is a robust result and should appear in the Euler description as well. This follows from the famous Lighthill criterion for the modulation instability because the dispersion relation for the Gerstner wave is linear and do not include terms proportional to the wave's amplitude.

4.3 Gouyon waves

As it has been shown by Dubreil - Jacotin (1934) the Gerstner wave is a special case of a wide class of stationary waves with the vorticity $\Omega = \varepsilon \Omega_*(\psi)$, where Ω_* is an arbitrary function, and ψ is the stream function. These results have been obtained and then developed by Gouyon (1958) who explicitly represented the vorticity in the form of a power series $\Omega = \sum_{n=1}^{\infty} \varepsilon^n \Omega_n(\psi)$ (see also the monograph by Sretensky (1977)).

When considering the plane steady flow in the Lagrange variables the stream lines ψ coincide with the isolines of the Lagrangian vertical coordinate b (Abashkin and Yakubovich, 2006; Bennett, 2006). We are going to consider a steady-state wave at the surface of an indefinitely deep water. Let us assume that there is no undisturbed shear current, but the wave's disturbances have the vorticity. Then, the formula for the vorticity has the form $\Omega = \sum_{n=1}^{\infty} \varepsilon^n \Omega_n(b)$. Now we name the steady-state waves propagating in such low-vorticity fluid the Gouyon waves. In the Lagrangian description properties of the Gouyon wave for the first two approximations were studied by Abrashkin and Zen'kovich (1990).

In our case $\Omega_1 = 0, \Omega_2 \neq 0$ and assuming the function Ω_2 to be independent of the coordinate a a description of the Gouyon waves could be obtained. The vorticity Ω_2 depends on the coordinate b only and has the following form

$$\Omega_{2Goyuon} = \omega k^2 |A|^2 H(kb), \qquad (46)$$

here $H(kb)$ is an arbitrary function. In case of $H(kb) = -2\exp(2kb)$ the vorticity of the square-law Gerstner waves and the Gouyon waves coincide (compare Eqs. (44) and (46)). In the considered approximation the Gouyon wave generalize the Gerstner wave. From Eq. (22) it follows that the function ψ_{t_1} is equal to zero only when the vorticity of the Gouyon waves is equal to the vorticity of the Gerstner wave. Except of this case the average current ψ_{t_1} will be always present in the modulated Gouyon waves.

Substitution of ratio (46) in Eq. (41) yields the NLS equation for the modulated Gouyon wave possessing the square-law in amplitude vorticity:

$$i\frac{\partial A}{\partial a_2} - \frac{k}{\omega^2}\frac{\partial^2 A}{\partial t_1^2} - \beta_G k^3 |A|^2 A = 0;$$
$$\beta_G = 1 + 4\int_{-\infty}^{0} e^{2\tilde{b}} \left(\int_{-\infty}^{\tilde{b}} H(\tilde{b}')\,d\tilde{b}' \right) d\tilde{b}; \quad \tilde{b} = kb, \tag{47}$$

here $\tilde{b}$ is a dimensionless vertical coordinate. The coefficient at the nonlinear term in the NLS equation varies when taking into account the wave's vorticity. For the Gerstner wave it could be equal to zero as well as for the Gouyon waves when satisfying the condition

$$\int_{-\infty}^{0} e^{2\tilde{b}} \left(\int_{-\infty}^{\tilde{b}} H(\tilde{b}')\,d\tilde{b}' \right) d\tilde{b} = -\frac{1}{4}. \tag{48}$$

Obviously an infinite number of distributions of the vorticity $H(\tilde{b})$ meeting this condition are possible. And such distributions represent just a small part of all possible ones. Therefore a realization of one of them seems to be improbable. Most likely that in the natural conditions distributions of the vorticity with a certain sign of β_G are implemented. Its negative values correspond to the defocusing NLS equation and positive ones relate to the focusing NLS equation. In the latter case the maximal value of the increment as well as the width of the modulation instability zone of a uniform train of vortex waves vary depending on the value of β_G.

Eqs. (39) and (47) will be focusing for $\psi_{1t_1} < 0,\ b \le 0$ and defocusing if $\psi_{1t_1} > 0,\ b \le 0$. The case of the sign-variable function ψ_{1t_1} requires an additional research. From the physical viewpoint the sign of this function is defined by a ratio of the velocity of the Stokes drift (45) to the velocity of the current induced by the vorticity (the integral term in Eq. (40)). For $\psi_{1t_1} < 0$ the Stokes drift either dominates over a vortex current or both of them have the same direction. When $\psi_{1t_1} > 0$ the vortex current dominates over the counter Stokes drift. In case of the sign-variable ψ_{1t_1} a ratio between these currents varies at different vertical levels, so requiring a direct calculation of β_G.

4.4 Waves with heterogeneous vorticity distribution in both coordinates

An expression for the vorticity as well as any methods of its definition were not discussed while deriving the NLS equation. In Sections 4.2 and 4.3 for the problems on the Gerstner and the Gouyon waves the vorticity was set proportional to a square modulus of the wave's amplitude. Note that waves can propagate at the background of some vortex current, for example, at the localized vortex. In that case the vorticity could be presented in the form

$$\Omega_2(a_2,b)=\omega\left[\varphi_v(a_2,b)+k^2|A|^2\varphi_w(a_2,b)\right],$$

where the function $\omega\varphi_v$ defines the vorticity of the background vortex current and the function $\omega k^2|A|^2\varphi_w$ defines the vorticity of waves. In the most general case both functions depend on the horizontal Lagrangian coordinate as well. Then Eq.(41) takes a form

$$\begin{gathered} i\frac{\partial A}{\partial a_2}-\frac{k}{\omega^2}\frac{\partial^2 A}{\partial t_1^2}-k\beta_v(a_2)A-k^3\left(1+\beta_w(a_2)\right)|A|^2A=0, \\ \beta_{v,w}(a_2)=4\int_{-\infty}^{0}e^{2\tilde{b}}\left(\int_{-\infty}^{\tilde{b}}\varphi_{v,w}(a_2,\tilde{b}')d\tilde{b}'\right)d\tilde{b}. \end{gathered} \qquad (49)$$

By the following substitution

$$A* = A\exp\left(-ik\int_{-\infty}^{a_2}\beta_v(a_2)da_2\right) \qquad (50)$$

Eq. (49) is reduced to the NLS equation with the non-uniform multiplier for the nonlinear term:

$$i\frac{\partial A^*}{\partial a_2}-\frac{k}{\omega^2}\frac{\partial^2 A^*}{\partial t_1^2}-k^3\left(1+\beta_w(a_2)\right)\left|A^*\right|^2A^*=0. \qquad (51)$$

Let's consider propagation of the Gouyon wave when $\beta_w=const=\beta_G-1$ and Eq.(51) turns into the classical NLS equation Eq. (47). As it is shown in Sec. 4.3 it describes the modulated Gouyon waves. Therefore in view of substitution Eq. (50) one can conclude that the propagation of the Gouyon waves at the background of the non-uniform vortex current yields variation of the wave number of the carrier wave. For $\beta_w=0$ Eq. (51) describes propagation of a packet of potential waves at the background of the non-uniform weakly vortical current. Peculiarities of propagation of waves related to the variable β_w require a special investigation.

5 On correlation of Lagrangian and Eulerian approaches

Let us consider correlation between the Eulerian and the Lagrangian description of wave's packets. To obtain the value for elevation of the free surface we substitute expressions (8), (9), (13) and $b = 0$ to the equation for $Y = \operatorname{Im} W$ written in the following form

$$Y_L = \varepsilon \operatorname{Im} A(a_2, t_1) \exp i(ka_0 + \omega t_0),$$

here $A(a_2, t_1)$ is the solution of Eq. (41). This expression defines the wave's profile in the Lagrangian coordinates (refer to subscript "L" for Y). To rewrite this equation in the Eulerian variables it is necessary to define a via X. From relation (8) it follows

$$X = a + \varepsilon \operatorname{Re}\left(w_1 + \sum_{n=2} \varepsilon^{n-1} w_n \right) = a + O(\varepsilon),$$

and the elevation of the free surface in the Eulerian variables Y_E will be written as:

$$Y_E = \varepsilon \operatorname{Im} A(X_2, t_1) \exp i(kX_0 + \omega t_0) + O\left(\varepsilon^2\right); \quad X_l = \varepsilon^l X .$$

The coordinate a plays the role of X, so the following substitutions are valid for the Lagrangian approach

$$a_0 \to X_0; \quad a_1 \to X_1; \quad a_2 \to X_2 .$$

This result could be named an "accordance principle" between the Lagrange and the Euler descriptions for solutions in the linear approximation. This principle is valid both for the potential and rotational waves.

To express the solution of Eq. (41) in the Eulerian variables it is necessary to use the accordance principle and to replace the horizontal Lagrangian coordinate a_2 by the coordinate X_2. So the discrepancies between the Eulerian and the Lagrangian estimations of the NLS equation for elevation of the free surface are absent.

Taking this into account one could conclude that the result will be the same in the Eulerian description if the vorticity Ω_2 will be set as a function of the coordinates x, y. Respectively when studying dynamics of wave packets in the vortical liquid in the Eulerian variables it is necessary to replace (ex. in Eq. (41) or (51)) the horizontal Lagrangian coordinate by the Eulerian one.

The solutions of the considered problem in the Lagrange and the Euler forms in the quadratic and cubic approximations differ from each other. To obtain the full solution in the Lagrange form we should obtain the functions $\psi_1, \psi_2, \psi_3, f_2, f_3$. This problem should be considered within a special study.

6 Conclusion

In this paper we derived the vortex-modified nonlinear Schrödinger equation. To obtain it the method of multiple scale expansions in the Lagrange variables was applied. The fluid vorticity Ω was set as an arbitrary function of the Lagrangian coordinates, which is quadratic in the small parameter of the wave's steepness $\Omega = \varepsilon^2 \Omega_2(a,b)$. The calculations were carried out by introduction of the complex coordinate of trajectory of a fluid particle.

The nonlinear evolution equation for the wave packet in the form of the nonlinear Schrödinger equation was derived as well. From the mathematical viewpoint the novelty of this equation relates to the emergence of a new term proportional to the amplitude of the envelope and the variance of the coefficient of the nonlinear term. In case of the vorticity's dependence on the vertical Lagrangian coordinate only (the Gouyon waves) this coefficient will be constant. There are special cases when the coefficient of the nonlinear term equals to zero and the resulting non-linearity disappears. The Gerstner wave belongs to the latter case. Another effect revealed in the present study is the vorticity's relation to the shift of the wave number in the carrier wave. This shift is constant for the modulated Gouyon wave. In case of the vorticity's dependence on both Lagrangian coordinates the shift of the wave number is horizontally heterogeneous. It is shown that the solution of the NLS equation for weakly rotational waves in the Eulerian variables could be obtained from the Lagrangian solution by an ordinary change of the horizontal coordinates.

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