Dear Professor Grimshaw!
We revised our manuscript according to the comments of the Reviewers which are definitely useful for the paper's content and are accepted with our gratitude.

Sincerely yours,
Anatoly Abrashkin and Efim Pelinovsky

Answers on review's comments on paper

The Lagrange form of the nonlinear Schrödinger equation for low vorticity waves in deep water: rogue wave aspect

by

Anatoly Abrashkin and Efim Pelinovsky

RESPONSE TO REVIEWER 1

THE MOTIVATION:

Review 1:

From the provided literature review it is not clear why this particular study is needed? ***What the specific questions that the authors want to clarify? Why these questions might be of interest and for what segments of the scholar community?***

Authors:

We found a new family of solutions for the wave train propagation in the deep water. Their novelty is **non-uniform distribution of the vorticity**.

Reviewer 1:

Wavetrain modulations upon arbitrary vertically sheared currents were thoroughly studied by Benny and his group. If the Benny asymptotic expansion becomes invalid for the range of small values of vorticity the present work is focusing upon, then it has to be shown and explained what is wrong with the Benny expansion. The same question applies to Jonhson (1976) results. The dependence of the cubic nonlinearity on vorticity in Jonhson (1976) and the works by Benny is not singular. Therefore similar expansion for the small vorticity can be carried out in the Eulerian framework as well using the known results, say, by Jonhson (1976) and/or the works by Benny group as the starting point. I think what the authors are doing is a re-derivation of the NLS for weak vorticity; the results were known, although implicitly, since nobody looked specifically into this case. Hence there is indeed a novelty here, but a comparison with the Eulerian results is necessary. In the Eulerian case vorticity can also be always presented as an explanation in *epsilon*, although in contrast with Lagrangian approach only the leading order vorticity will be constant.

Authors: We added (Sec. 5, lines 626-630):

Taking this into account one could conclude that the result will be the same in the Eulerian description if the vorticity Ω_2 will be set as a function of the coordinates x, y. Respectively when studying dynamics of wave packets in the vortical liquid in the Eulerian variables it is necessary to replace (ex. in Eq. (41) or (51)) the horizontal Lagrangian coordinate by the Eulerian one.

Reviewer 1:

In this context the most intriguing question is concerned with one of highlights of the work: the vanishing of the cubic nonlinearity in the NLS in the Lagrangian variables for the Gerstner wave. (This result is more significant than the authors it credit for: it shows that in principle an $O(\varepsilon^2)$ shear might kill the NLS nonlinearity. The question is: what is the manifold of Eulerian shear profiles (or vorticity distributions which would zero the NLS nonlinearity? I believe it could be answered by a straightforward analysis of the known expressions for the coefficient.

Authors: We added (Sec. 4.2, lines 477-489):

Let's consider some particular consequences of the obtained result. For the irrotaional ($\Omega_2 = 0$) stationary ($A = |A| = const$) wave Eq. (40) for the velocity of the drifting flow takes the form

$$\psi_{1t_1} = -\omega k A^2 e^{2kb}. \tag{45}$$

It coincides with the expression for the Stokes drift in the Lagrangian coordinates (in the Eulerian variables the profile of the Stokes current could be obtained by the substitution of b to y). Thus, our result may be interpreted as a compensation of the Stokes's drift by the shear flow induced by the Gerstner wave in a square approximation. This conclusion is also fair in the "differential" formulation for vorticities. From Eq. (22) it follows that the vorticity of the Stokes drift equals to the vorticity of the Gerstner wave with the inverse sign.

Reviewer 1: Also the similar question applies to the Lagrangian formulation: the vorticity distribution is arbitrary, what are other distributions for which the NLS nonlinearity vanishes? I doubt that the Gerstner is an isolated special case.

Authors: We added (Sec. 4.3, lines 513-556):

In our case $\Omega_1 = 0, \Omega_2 \neq 0$ and assuming the function Ω_2 to be independent of the coordinate a a description of the Gouyon waves could be obtained. The vorticity Ω_2 depends on the coordinate b only and has the following form

$$\Omega_{2Goyuon} = \omega k^2 |A|^2 H(kb), \tag{46}$$

here $H(kb)$ is an arbitrary function. In case of $H(kb) = -2\exp(2kb)$ the vorticity of the square-law Gerstner waves and the Gouyon waves coincide (compare Eqs. (44) and (46)). In the considered approximation the Gouyon wave generalize the Gerstner wave. From Eq. (22) it follows that the function ψ_{t_1} is equal to zero only when the vorticity of the Gouyon waves is equal to the vorticity of the Gerstner wave. Except of this case the average current ψ_{t_1} will be always present in the modulated Gouyon waves.

Substitution of ratio (46) in Eq. (41) yields the NLS equation for the modulated Gouyon wave possessing the square-law in amplitude vorticity:

$$\begin{gathered} i\frac{\partial A}{\partial a_2} - \frac{k}{\omega^2}\frac{\partial^2 A}{\partial t_1^2} - \beta_G k^3 |A|^2 A = 0; \\ \beta_G = 1 + 4\int_{-\infty}^{0} e^{2\tilde{b}} \left(\int_{-\infty}^{\tilde{b}} H(\tilde{b}') d\tilde{b}' \right) d\tilde{b}; \quad \tilde{b} = kb, \end{gathered} \tag{47}$$

here $\tilde{b}$ is a dimensionless vertical coordinate. The coefficient at the nonlinear term in the NLS equation varies when taking into account the wave's vorticity. For the Gerstner wave it could be equal to zero as well as for the Gouyon waves when satisfying the condition

$$\int_{-\infty}^{0} e^{2\tilde{b}} \left(\int_{-\infty}^{\tilde{b}} H(\tilde{b}') d\tilde{b}' \right) d\tilde{b} = -\frac{1}{4}. \tag{48}$$

Obviously an infinite number of distributions of the vorticity $H(\tilde{b})$ meeting this condition are possible. And such distributions represent just a small part of all possible ones. Therefore a realization of one of them seems to be improbable. Most likely that in the natural conditions distributions of the vorticity with a certain sign of β_G are implemented. Its negative values correspond to the defocusing NLS equation and positive ones relate to the focusing NLS equation. In the latter case the maximal value of the increment as well as the width of the modulation instability zone of a uniform train of vortex waves vary depending on the value of β_G.

Eqs. (39) and (47) will be focusing for $\psi_{1t_1} < 0,\ b \le 0$ and defocusing if $\psi_{1t_1} > 0,\ b \le 0$. The case of the sign-variable function ψ_{1t_1} requires an additional research. From the physical viewpoint the sign of this function is defined by a ratio of the velocity of the Stokes drift (45) to the velocity of the current induced by the vorticity (the integral term in Eq. (40)). For $\psi_{1t_1} < 0$ the Stokes drift either dominates over a vortex current or both of them have the same direction. When $\psi_{1t_1} > 0$ the vortex current dominates over the counter Stokes drift. In case of the sign-variable ψ_{1t_1} a ratio between these currents varies at different vertical levels, so requiring a direct calculation of β_G.

Reviewer 1: It follows from the works by Benny and his group that transverse instability is much stronger than the longitudinal one, therefore, the studies of strictly longitudinal instabilities are limited interest from the viewpoint of sea applications and could be applied only to narrow wave tanks. I'd like this point to be mentioned more explicitly in the introduction. This is important since it squarely places the derived NLS into the realm of toy models. This does not mean that the results cannot be of interest or should not be published, it just means that the results might interest a different community.

Authors: We mentioned Benny's result in the introduction (lines 41-49). And suppose that it is quite enough.

Reviewer 1: The original element of the work is the asymptotic derivation of the NLS in Lagrangian variables. In my view this is complementary to the existing Eulerian works and it remains unclear what new features/aspects this might reveal.

Authors: The original aspect of our study is **horizontal non-uniformity of vorticity's distribution**. As a consequence, in contrast to Benny and his group and Johnson we derived the evolutional equation with variable coefficients.

THE NLS:

<u>Reviewer 1</u>: In contrast to the NLS in Euler variables where we know that the equation describes evolution of the envelope amplitude in the (x,t) space and how the actual elevation can be expressed as a Stokes-like series in wave amplitude up to cubic order, here the NLS in Lagrangian variables is an object which is much less straightforward to interpret. Obviously, A is the envelope amplitude, but what are the independent variables (a,b)?

<u>Authors</u>: Lagrangian variables are the labels of the fluid particles, nothing more over.

<u>Reviewer 1</u>: Their link to the standard Eulerian variables (x,y) is not known. Although, it is straightforward, at least in principle, to provide this link in terms of series in ε, the authors choose not to do this. The effectively use the zero order approximation where the difference between the Eulerian and Lagrangian description vanishes. Then the rationale for using the Lagrangian approach apparently disappears.

<u>Authors</u>: We derived a new family of solutions due to Lagrange approach. A problem of their Eulerian description has not been solved yet.

<u>Reviewer 1</u>: I suspect (this is the most interesting point), that if the authors make transformation to return to the Euler variables, they will get a higher order NLS type equation since the transformation itself is nonlinear (see e.g. F. Nouguier, B. Chapron, C-A, Guérin Second-order Lagrangian description of tri-dimensional gravity wave interactions, JFM 772, 165-196 (2015) and references therein).

<u>Authors</u>: That is a special problem. We are ready to discuss it further.

<u>Reviewer 1</u>: If the authors do not want to go through this straightforward but quite time consuming pass I suggested above, then they can handle the comparison numerically. The Lagrangian solution yields X,Y in terms of a,b,t. Hence the surface elevation $Y(a,0,t)$ and position of a parcel on the free surface, $X(a,0,t)$, which are found in terms of series, provide implicit function $Y(a,0,t)$ which can be easily plotted for a typical $Y(X,t)$, say, a breather. This plot has to compared to the Eulerian solution with the cubic terms retained.

<u>Authors</u>: That is a good programme, but nobody has calculated the Eulerian solution with the cubic terms. All authors are restricted to the derivation of the NLS equation. With what solution do we have to compare our results? Or we must study our problem in Euler variables too? Besides, we are interested in rogue waves in this paper and study the leading order of the solution only. The terms of the second and cubic orders are out of our attention.

Reviewer 1: The obtained NLS is presented in an "optical" form (with space rather than time chosen as the propagation variable), which is somewhat strange choice for a hydrodynamic work. Dependence on t in this context means dependence on running variable. I do not understand why the authors choose this form and stick to it, they give no clue. They have either argue for their preference or switch to the conventional form.

Authors: We added (Sec. 3.3, after Eq. (39), lines 398-400):

Further it will be shown that variables in Eqs. (38), (39) were chosen in the easiest form for their reduction (under the particular assumptions) to the classical NLS equation.

Reviewer 1: The authors consider the NLS derivation allowing for horizontal non-uniformity, which raises a host of questions. How arbitrary the dependence on a_2 is? What does it mean? Are the a_2 dependencies of these vertical and potential parts of the Doppler correction linked to satisfy the Lagrange equations? How these dependencies can be specified?

Authors: The vorticity $\Omega_2(a_2, b)$ is an arbitrary bounded function. The vertical and horizontal parts of the Doppler correction don't link. It is obvious from the comparison of the equations (41) and (44).

ROGUE WAVES:

Reviewer 1: As I've already mentioned, the strong transverse instability of the wavetrains does not allow one to speak seriously about ocean applications. I found nothing new and specific adding to our understanding of rogue waves. The fact that the NLS is formulated in the Lagrangian variables and only the leading order term is used makes this equation equivalent (to this order) to the Eulerian NLS. The fact that in the focusing NLS there is modulational instability and that such NLS admits breather solutions is known for about thirty years. The term "rogue wave" is used in the manuscript as synonymous with the term breather, just because the latter satisfy the rogue wave criterion. Although the NLS breather solutions are indeed often used as prototypes of rogue waves, this could be done only with appropriate explicitly spelled out caveats. The weakest point in the rogue wave aspect of the paper is that I don't see any new insight into the nature of rogue waves even in the framework of the chosen toy model. In my view the following question might be of interest in the context of rogue waves and would have an element of novelty: what is the profile and maximal height of the found Akhmediev Lagrangian breather in the Eulerian variables. To answer this question the authors have to sum up all orders of their expansion and then perform the transformation to the Eulerian variables. The results will differ from the corresponding expansion in the Eulerian variables. I re-iterate that it would be of interest to discuss this difference. I've mentioned already the simplest way to get it.

Authors: We excluded the discussion of the problem of rogue waves from our paper.

Answers on the comments of the Reviewer 2 on paper

The Lagrange form of the nonlinear Schrödinger equation for low vorticity waves in deep water: rogue wave aspect

by

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Reviewer 2:

First, it suffers a lack of illustration. Indeed, a single figure appears, and intends to show the full geometry of the problem. For instance, from the figure, I cannot understand what is this "average current" (average in time, in 'a' coordinate? In 'b' coordinate?) Neither can I see a weak vorticity. Thus, the definition of vorticity is confusing. Another way to say the same thing is that the Euler to Lagrange coordinate transform is not clear. Is a background vertical flow included? Or do we only consider the vertical flow induced by the waves?

Authors:

We are highly appreciated this comment. We shall add the information in the figure. The horizontal current ψ_{1t_1} depends on slow coordinates only. So we name it "average current" as the average in fast variables a_0, t_0 (see formula (13)).The weak vorticity is set as an arbitrary function of Lagrange coordinates in some region. For example, the vorticity can differ from zero inside the bounded region. This case corresponds to the interaction of the wave with the localized vortex. **We drew a new figure.** We concentrate on the studying of horizontal current because it is a term of the NLS equation. But in the quadratic approximation there exists the vertical flow too. This one is described by the function f_2 depending on the wave amplitude and the horizontal current (see the equation (21)). In the Eulerian coordinates the background flow has two components of the velocity depending on the variables x, y, t.

Reviewer 2:

Presentation of the results is a little bit confusing.

For instance, it is shown that the absence of vorticity and current, the Akhmediev soliton solution in Lagrange coordinates does correspond to the Akhmediev soliton in Euler coordinates, up to the second order in epsilon. But then, for quadratic and cubic terms, it is claimed the solutions differ. Here could be an interesting result. Could be the authors consider obtaining these solutions and present differencies?

Authors:

That is a good idea, but there are two serious problems. Firstly, it is necessary to get a solution up to the third order in Euler coordinates. As we know nobody did that. Secondly, it is necessary to transform our solution to the Eulerian form, i.e. to solve the equations for the functions $\psi_1, \psi_2, \psi_3, f_2, f_3$ and then to express the obtained solutions in Euler variables. This program requires the very unwieldy calculations. So we establish that the realization of the reviewer's idea has to be a subject of a new paper.

Reviewer 2:

When considering the Gerstner wave, where does the vorticity profile comes from? Thus, the following sentence is disturbing: "From the physical point of view, this is due to the fact that the average current induced by the vorticity compensates the stokes drift". Is it only true when integrated? The result associated is very interesting (finding Gerstner waves not affected by modulational instability), but its explanation is not straightforward and should be developed. Still, these waves are a very specific case, and this is not clear from the text.

Authors: We added in the next (Sec. 4.2, lines 460-489):
To derive the vorticity of the Gerstner wave Eq. (43) should be substituted in Eq. (6). Then in could be found that in the linear approximation the Gerstner wave is potential ($\Omega_1 = 0$), but in the quadratic approximation it possesses vorticity

$$\Omega_{2Gerstner} = -2\omega k^2 |A|^2 e^{2kb}. \qquad (44)$$

For this type of the vorticity distribution the first two terms in the parentheses in Eq. (41) neglect each other. From the physical point of view this is due to the fact that the average current induced by the vorticity compensates the potential drift exactly. The packet of weakly nonlinear Gerstner waves in this approximation is not affected by their non-linearity, and the effect of the modulation instability for the Gerstner wave is absent.

Generally speaking this result is quite obvious. As there are no particle's drift in the Gerstner wave the function ψ_1 equals to zero. So the multiplier of the wave's amplitude in Eqs. (38), (39) may be neglected initially without derivation of the vorticity of the Gerster wave.

Let's consider some particular consequences of the obtained result. For the irrotaional ($\Omega_2 = 0$) stationary ($A = |A| = const$) wave Eq. (40) for the velocity of the drifting flow takes the form

$$\psi_{1t_1} = -\omega k A^2 e^{2kb}. \qquad (45)$$

It coincides with the expression for the Stokes drift in the Lagrangian coordinates (in the Eulerian variables the profile of the Stokes current could be obtained by the substitution of b to y). Thus, our result may be interpreted as a compensation of the Stokes's drift by the shear flow induced by the Gerstner wave in a square approximation. This conclusion is also fair in the "differential" formulation for vorticities. From Eq. (22) it follows that the vorticity of the Stokes drift equals to the vorticity of the Gerstner wave with the inverse sign.

Reviewer 2:
Results of the following part, entitled "Rogue waves", describe the evolution of the coefficients of the NLS equation with the structure of vorticity. In each one of the three cases studied, the eventuality of a breather soliton to exist is analyzed. Maybe, the characteristics of these new Peregrine breather could be described, by comparison with classical one. This analysis would provide an idea of whether or not a vertical flow is amenable to increase the probability of occurrence of rogue waves.

Authors: We rewrote the Sec. 4. The differences between potential and vortical wave's solutions are formulated in detail (Sec. 4.3; 4.4, lines 513-592):

In our case $\Omega_1 = 0, \Omega_2 \neq 0$ and assuming the function Ω_2 to be independent of the coordinate a a description of the Gouyon waves could be obtained. The vorticity Ω_2 depends on the coordinate b only and has the following form

$$\Omega_{2Goyuon} = \omega k^2 |A|^2 H(kb), \tag{46}$$

here $H(kb)$ is an arbitrary function. In case of $H(kb) = -2\exp(2kb)$ the vorticity of the square-law Gerstner waves and the Gouyon waves coincide (compare Eqs. (44) and (46)). In the considered approximation the Gouyon wave generalize the Gerstner wave. From Eq. (22) it follows that the function ψ_{t_1} is equal to zero only when the vorticity of the Gouyon waves is equal to the vorticity of the Gerstner wave. Except of this case the average current ψ_{t_1} will be always present in the modulated Gouyon waves.

Substitution of ratio (46) in Eq. (41) yields the NLS equation for the modulated Gouyon wave possessing the square-law in amplitude vorticity:

$$i\frac{\partial A}{\partial a_2} - \frac{k}{\omega^2}\frac{\partial^2 A}{\partial t_1^2} - \beta_G k^3 |A|^2 A = 0;$$
$$\beta_G = 1 + 4\int_{-\infty}^{0} e^{2\tilde{b}} \left(\int_{-\infty}^{\tilde{b}} H(\tilde{b}')\, d\tilde{b}' \right) d\tilde{b}; \quad \tilde{b} = kb, \tag{47}$$

here $\tilde{b}$ is a dimensionless vertical coordinate. The coefficient at the nonlinear term in the NLS equation varies when taking into account the wave's vorticity. For the Gerstner wave it could be equal to zero as well as for the Gouyon waves when satisfying the condition

$$\int_{-\infty}^{0} e^{2\tilde{b}} \left(\int_{-\infty}^{\tilde{b}} H(\tilde{b}')\, d\tilde{b}' \right) d\tilde{b} = -\frac{1}{4}. \tag{48}$$

Obviously an infinite number of distributions of the vorticity $H(\tilde{b})$ meeting this condition are possible. And such distributions represent just a small part of all possible ones. Therefore a realization of one of them seems to be improbable. Most likely that in the natural conditions distributions of the vorticity with a certain sign of β_G are implemented. Its negative values correspond to the defocusing NLS equation and positive ones relate to the focusing NLS equation. In the latter case the maximal value of the increment as well as the width of the modulation instability zone of a uniform train of vortex waves vary depending on the value of β_G.

Eqs. (39) and (47) will be focusing for $\psi_{1t_1} < 0,\ b \le 0$ and defocusing if $\psi_{1t_1} > 0,\ b \le 0$. The case of the sign-variable function ψ_{1t_1} requires an additional research. From the physical viewpoint the sign of this function is defined by a ratio of the velocity of the Stokes drift (45) to the velocity of the current induced by the vorticity (the integral term in Eq. (40)). For $\psi_{1t_1} < 0$ the Stokes drift either dominates over a vortex current or both of them have the same direction. When $\psi_{1t_1} > 0$ the vortex current dominates over the counter Stokes drift. In case of the sign-variable ψ_{1t_1} a ratio between these currents varies at different vertical levels, so requiring a direct calculation of β_G.

4.4 Waves with heterogeneous vorticity distribution in both coordinates

An expression for the vorticity as well as any methods of its definition were not discussed while deriving the NLS equation. In Sections 4.2 and 4.3 for the problems on the Gerstner and the Gouyon waves the vorticity was set proportional to a square modulus of the wave's amplitude. Note that waves can propagate at the background of some vortex current, for example, at the localized vortex. In that case the vorticity could be presented in the form

$$\Omega_2(a_2,b)=\omega\left[\varphi_v(a_2,b)+k^2|A|^2\varphi_w(a_2,b)\right],$$

where the function $\omega\varphi_V$ defines the vorticity of the background vortex current and the function $\omega k^2|A|^2\varphi_W$ defines the vorticity of waves. In the most general case both functions depend on the horizontal Lagrangian coordinate as well. Then Eq.(41) takes a form

$$i\frac{\partial A}{\partial a_2}-\frac{k}{\omega^2}\frac{\partial^2 A}{\partial t_1^2}-k\beta_V(a_2)A-k^3\left(1+\beta_w(a_2)\right)|A|^2A=0,$$
$$\beta_{v,w}(a_2)=4\int_{-\infty}^{0}e^{2\tilde{b}}\left(\int_{-\infty}^{\tilde{b}}\varphi_{v,w}(a_2,\tilde{b}')d\tilde{b}'\right)d\tilde{b}. \qquad (49)$$

By the following substitution

$$A^*=A\exp\left(-ik\int_{-\infty}^{a_2}\beta_v(a_2)da_2\right) \qquad (50)$$

Eq. (49) is reduced to the NLS equation with the non-uniform multiplier for the nonlinear term:

$$i\frac{\partial A^*}{\partial a_2}-\frac{k}{\omega^2}\frac{\partial^2 A^*}{\partial t_1^2}-k^3\left(1+\beta_w(a_2)\right)\left|A^*\right|^2A^*=0. \qquad (51)$$

Let's consider propagation of the Gouyon wave when $\beta_W=const=\beta_G-1$ and Eq.(51) turns into the classical NLS equation Eq. (47). As it is shown in Sec. 4.3 it describes the modulated Gouyon waves. Therefore in view of substitution Eq. (50) one can conclude that the propagation of the Gouyon waves at the background of the non-uniform vortex current yields variation of the wave number of the carrier wave. For $\beta_W=0$ Eq. (51) describes propagation of a packet of potential waves at the background of the non-uniform weakly vortical current. Peculiarities of propagation of waves related to the variable β_W require a special investigation.

The authors would like to express our sincere thanks to the reviewer 2 for all the valuable comments and helpful suggestions.

Dear Editor!

We send the LIST OF CHANGES to the paper "The Lagrange form of the nonlinear Schrödinger equation for low vorticity waves in deep water" ***by Anatoly Abrashkin and Efim Pelinovsky*****:**

1) **We changed the title of the manuscript;**
2) **We rewrote the introduction to exclude a discussion of the problem of rogue waves;**
3) **We drew the new figure;**
4) **We rewrote the Sec. 4;**
5) **We included a new Sec. 5 which contains the comparative analysis of the Lagrangian and the Eulerian approaches;**
6) **We removed all references with the topic of rogue waves.**

Sincerely yours,
Anatoly Abrashkin and Efim Pelinovsky