1 Introduction

The following discussions derived from the conference article by Shen and Faghih-Naini (2017, accepted for oral presentation) are provided for review process. It is shown that the governing equations of the locally linear 7D-NLM are identical to those in the coupled system with three identical masses and three different springs.

2 Seven-dimensional Non-dissipative Lorenz Model

This section describes the governing equations for the seven-dimensional non-dissipative Lorenz model (7D-NLM) and the corresponding locally linear 7D-NLM. We will then compare the 7D-NLM and a coupled system with three identical masses and three different springs.

The 7D-NLM can be obtained by removing the dissipative terms of the 7D (dissipative) LM (7DLM; Shen, 2016), as follows:

$$\frac{dX}{d\tau} = \cancel{-\sigma X} + \sigma Y, \tag{1}$$

$$\frac{dY}{d\tau} = -XZ + rX \cancel{-Y}, \tag{2}$$

$$\frac{dZ}{d\tau} = XY - XY_1 \cancel{-bZ}, \tag{3}$$

$$\frac{dY_1}{d\tau} = XZ - 2XZ_1 \cancel{-d_0 Y_1}, \tag{4}$$

$$\frac{dZ_1}{d\tau} = 2XY_1 - 2XY_2 \cancel{-4bZ_1}, \tag{5}$$

$$\frac{dY_2}{d\tau} = 2XZ_1 - 3XZ_2 \cancel{-d_0 Y_2}, \tag{6}$$

$$\frac{dZ_2}{d\tau} = 3XY_2 \cancel{-9bZ_2}. \tag{7}$$

The same approach has been used to derive the 3D-NLM and 5D-NLM (e.g., Faghih-Naini and Shen, 2017). The dissipative terms are indicated by the terms with a crossout symbol. As discussed in Shen (2016), $(X, Y, Z, Y_1, Z_1, Y_2, Z_2)$ represent the amplitude of the Fourier modes. We refer to (X, Y, Z) as the primary modes, (Y_1, Z_1) as the secondary modes, and (Y_2, Z_2) as the tertiary modes. τ is dimensionless time. The two parameters $(\sigma,\ r)$ are the Prandtl number and the normalized Rayleigh number (or the heating parameter), respectively. Detailed information regarding these parameters and ignored terms is provided in Shen (2016). On the right-hand side of the above equations, there are the linear heating term (rX) and the nonlinear force terms (e.g., $-XZ$ and XY).

Applying a perturbation method, which represents the total field (A) as a sum of the reference state (A_c) and perturbation (A'), i.e., $A = A_c + A'$, we transform Eqs. (1-7) to the following equations:

$$\frac{dX'}{d\tau} = \sigma Y', \tag{8}$$

$$\frac{dY'}{d\tau} = (r - Z_c)X' - X_c Z' - FN(X'Z'), \tag{9}$$

$$\frac{dZ'}{d\tau} = (Y_c - Y_{1c})\,X' + X_c Y' - X_c Y_1' + FN(X'Y' - X'Y_1'), \tag{10}$$

$$\frac{dY_1'}{d\tau} = (Z_c - 2Z_{1c})X' + X_c Z' - 2X_c Z_1' + FN(X'Z' - 2X'Z_1'), \tag{11}$$

$$\frac{dZ_1'}{d\tau} = (2Y_{1c} - 2Y_{2c})X' + 2X_c Y_1' - 2X_c Y_2' + FN(2X'Y_1' - 2X'Y_2'). \tag{12}$$

$$\frac{dY_2'}{d\tau} = (2Z_{1c} - 3Z_{2c})X' + 2X_c Z_1' - 3X_c Z_2' + FN(2X'Z_1' - 3X'Z_2'), \tag{13}$$

$$\frac{dZ_2'}{d\tau} = 3Y_{2c}X' + 3X_c Y_2' + FN(3X'Y_2'). \tag{14}$$

As discussed in Shen (2014) and Faghih-Naini and Shen (2017), the flag FN is introduced to perform linear simulations ($FN = 0$) or nonlinear simulations ($FN = 1$). Equations (1-7) are referred to the 7D-NLM V1 and Eqs. (8-14) are referred to the 7D-NLM V2. The 7D-NLM V1 and V2 should produce identical results with the same initial conditions except when round-off errors become different in the runs using different models. The V2 with FN=0 is also called the locally linear 7D-NLM, which can be used for the linear stability analysis.

2.1 A comparison with the coupled system with three springs

Choosing $FN = 0$ and $(Y_c, Z_c, Y_{1c}, Z_{1c}, Y_{2c}, Z_{2c}) = (0, r, 0, \frac{r}{2}, 0, \frac{r}{3})$, we can obtain:

$$\frac{d^2Y'}{d\tau^2} = -X_c\frac{d^2Z'}{d\tau^2} = -X_c^2(Y' - Y_1') \tag{15}$$

from Eqs. (9-10),

$$\frac{d^2Y_1'}{d\tau^2} = X_c\frac{d^2Z'}{d\tau^2} - 2X_c\frac{d^2Z_1'}{d\tau^2} = X_c^2(Y' - 5Y_1' + 4Y_2') \tag{16}$$

from Eqs. (10-12),

$$\frac{d^2Y_2'}{d\tau^2} = 2X_c\frac{d^2Z_1'}{d\tau^2} - 3X_c\frac{d^2Z_2'}{d\tau^2} = X_c^2(4Y_1' - 13Y_2') \tag{17}$$

from Eqs. (12-14).

Previously, we have shown that the locally linear 3D-NLM and 5D-NLM have the governing equations that are identical to the systems with one spring and two springs, as shown in Figures 1a and 1b, respectively. For a comparison with the 7D-NLM, we present the governing equations for the coupled system with three identical masses and three different springs, as shown in Figure 1c.

$$\frac{d^2x_1}{d\tau^2} = -k_1(x_1 - x_2) \tag{18}$$

$$\frac{d^2x_2}{d\tau^2} = -k_2(x_2 - x_3) - k_1(x_2 - x_1) \tag{19}$$

$$\frac{d^2x_3}{d\tau^2} = -k_3x_3 - k_2(x_3 - x_2) \tag{20}$$

The top, middle, and bottom springs have spring constants of k_3, k_2, k_1, respectively. Here, the top spring is attached to the ceiling on one end and to the top mass on the other end. The upper (low) end of the middle spring is attached to the top (bottom) mass. For the bottom spring, its upper end is attached to the middle mass. $x_1(\tau)$, $x_2(\tau)$ and $x_3(\tau)$ are the displacements of the centers of masses from equilibrium. By choosing $x_1 = Y', k_1 = X_c^2, x_2 = Y_1', k_2 = 4X_c^2, x_3 = Y_2'$ and $k_3 = 9X_c^2$, we show that Eqs. (18-20) are identical to Eqs. (15-17), respectively. In other words, the above coupled system with three springs is identical to the locally linear 7D-NLM. Note that for each of uncoupled one-mass-one-spring systems, the frequency of the oscillatory motion is either X_c, $2X_c$, or $3X_c$. By comparisons, in section 3.1, we will show that the above system have three frequencies, but they are different from the values of X_c, $2X_c$, or $3X_c$. More importantly, these frequencies are incommensurate, leadning a quasi-periodic solution. As the 7DLM (or 7D-NLM) is derived by properly selecting new modes to extend the nonlinear feedback loop of the 5DLM (or 5D-NLM), we will discuss how the extended nonlinear feedback loop introduces two additional pair of downscaling and upscaling processes to produce an additional temporal oscillatory mode that is coupled with existing two temporal oscillatory modes.

References

1. Faghih-Naini, S. and B.-W. Shen, 2017: On quasi-periodic solutions associated with the extended nonlinear feedback loop in the five-dimensional non-dissipative Lorenz model. Nonlin. Processes Geophys. Discuss. (submitted Jan. 16, 2017).
2. Shen, B.-W. and S. Faghih-Naini, 2017: On recurrent solutions in high-dimensional non-dissipative Lorenz models. The 10th Chaos Modeling and Simulation International Conference (CHAOS2017), Barcelona, Spain, 30 May - 2 June, 2017. (accepted, December 30, 2016)

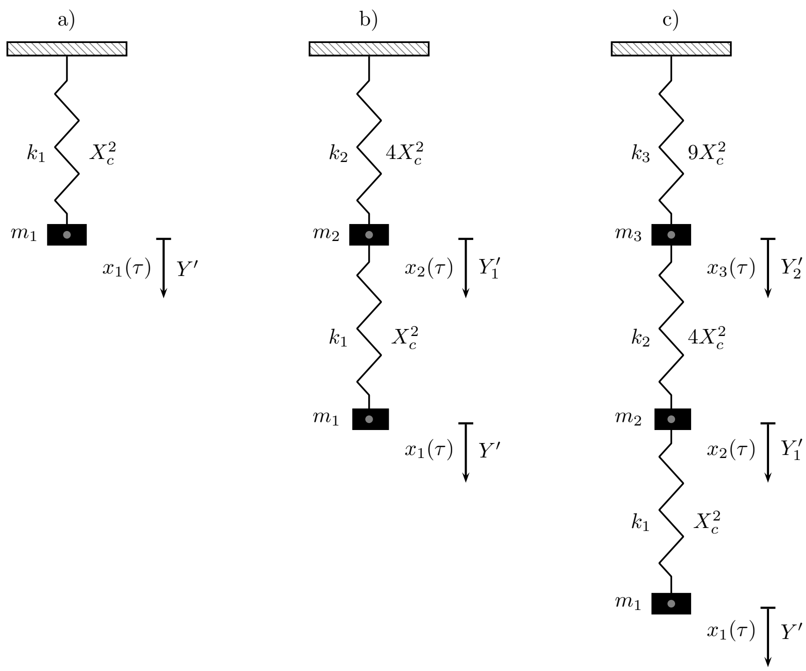

Figure 1: Systems with one mass and one spring (a), two masses and two springs (b) and three masses and three springs (c). Three masses are identical, i.e., $m_1 = m_2 = m_3$. Three spring constants k_1, k_2 and k_3 are selected as X_c^2, $4X_c^2$, and $9X_c^2$, respectively. It is shown that the governing equations for the above systems in panels (a)-(c) are identical to those for the locally linear 3D-NLM, 5D-NLM, and 7D-NLM, respectively. This comparison illustrates how the nonlinear feedback loop and its extension enabled by a proper selection of high wavenumber modes can produce recurrent (i.e., periodic or quasi-periodic) solutions.