Dear Editor:

Please consider for publication on *nonlinear processes in geophysics*

the following re-revised and resubmitted manuscript

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On the nonlinear feedback loop and energy cycle of the non-dissipative Lorenz model

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by Bo-Wen Shen

In this study using the 3D-NLM (npg-16-40) and a new manuscript using the 5D-NLM (Faghih-Naini and Shen, 2017, submitted, npg-17-2), we would like to illustrate the fundamental role of the nonlinearity in producing "recurrent'' solutions, including periodic orbits within the 3D-NLM and quasi-periodic orbits within the 5D-NLM. Here, the nonlinearity can be identified as the nonlinear feedback loop and its extension. Since the original manuscript was published as a discussion article in April 2014, our goal has been to illustrate the role of the nonlinear feedback loop (-XZ and XY) in producing a periodic solution using a very simple nonlinear ordinary differential equation (ODE), $X''+X^3/2=0$, where the nonlinear term $X^3$ comes from the nonlinear feedback loop. The abstract in npg-16-40 does indicate this. After the 2014 discussion article being reviewed, we incrementally added other parts, including a comparison of the model with the Duffing equation (e.g., Appendix B) into the npg-16-40. On the other hand, compared to the Duffing equation, the 3D-NLM was used to "indicate" the importance of applying adaptive methods for the numerical integration of the homoclinic orbit in the 3D phase space (e.g., Figs. 7c-d). We also discussed the partitions of the averaged available potential energy from Y and Z modes. Note that the Diffing equation does not include a component that is similar to the Z mode. We believe that our main points in npg-16-40 are valuable and have not been discussed in the literatures. Additionally, results using the 3D-NLM laid the foundation for the recent work using the 5D-NLM in npg-17-2. While detailed responses are given in the sections of "general responses'' and "specific responses'', I would like to provide the following responses below.

We did state that the 3D-NLM with certain types of ICs can be reduced to become the Duffing equation (e.g., Appendix B in npg-16-40 or Appendix C in the revised manuscript), which can help verify the periodicity of solutions and thus build our confidence in the role of nonlinear feedback loop. Additionally, in the newly added Appendix B, we discussed the relationship between the 3D-NLM with r=0 and the 3DLM with a very large r (e.g., Sparrow, 1982), and the real-valued Maxwell-Bloth equations (David and Holm, 1992). All of the three systems can be reduced to become X"+$X^3$/2. Therefore, based on our analysis in npg-16-40 and the 2014 discussion article, two nonlinear terms, -XZ and XY, also acts as a nonlinear restoring forcing in the 2nd system (i.e., the 3DLM with a large r); and two nonlinear terms, XZ and –XY, in the 3rd system (i.e., i.e., the real-valued Maxwell-Bloth equations) play a similar role. Under a certain range of initial conditions, these system and their solutions may be comparable to a simplified Duffing equation (e.g., Appendix C and Roupas, 2012). However, we want to emphasize that our goal is to discuss the role of nonlinear feedback loop in producing periodic (or recurrent) solutions using the 3D-NLM (as well as in producing quasi-periodic solutions using the 5D-NLM).

By comparison, Appendix C of Faghih-Naini and Shen (2017, npg-17-2) has provided an analogy between the locally linear 5D-NLM and a coupled system with two springs that both systems have the same mathematical equations when specific spring constants are selected. This kind of comparison is consistent with our approach of comparing the 3D-NLM, the Duffing equation, the 3DLM with a large r and the real-valued Maxwell-Bloth equations. Through these comparisons, we can build our confidence in the accuracy of mathematical solutions. Then, we could use the results (both solutions and the corresponding terms, including –XZ and XY) to improve our understanding of the underlying physical processes in the specific system. In our papers with the 3D-NLM (npg-16-40), 5D-NLM (npg-17-2) and 7D-NLM (Shen and Faghih-Naini, 2017), we illustrated the role of the nonlinear feedback loop and its extension in producing recurrent (i.e., periodic or quasi-periodic) solutions. As the nonlinear feedback loop appears throughout the spatial mode-mode interactions rooted in the nonlinear temperature advection, we believe that the nonlinear feedback loop can also produce quasi-periodic solutions in nonlinear numerical models based on Navier-Stokes equations.

In a brief summary, in the revised manuscript, our main focus is to illustrate that the nonlinear feedback loop serves as the nonlinear restoring force using closed solutions of a very simple nonlinear ODE, $X''+X^3/2=0$, and numerical solutions of the 3D-NLM (e.g., Eqs. 3-5 with r=0 and r≠0). The study in npg-16-40 has been extended to show the role of an extended nonlinear feedback loop in producing recurrent (quasi-periodic) solutions using 5D-NLM (npg-16-40 by Faghih-Naini and Shen, 2017) as well as 7D-NLM (Shen and Faghih-Naini, 2017). These non-dissipative LMs will be compared with the corresponding dissipative LMs to understand the impact of dissipations and their interaction with the non-linear feedback loop on chaotic solutions (e.g., topological transitivity).

Detailed responses are provided below. A pdf file that includes responses to the comments on the 2014 discussion article is also attached. A revised manuscript with changes highlighted in red is uploaded. We really appreciate reviewers' and Editor's comments that have greatly improved the quality of the manuscript. We thank the reviewers and Editor for providing us the opportunity for explaining further the progress We have made and hope that my responses are acceptable. Thank you for your consideration!

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* General Responses:
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The Lorenz model (1963) has been studied extensively and been used to illustrate the sensitive dependence of solutions on initial conditions (i.e., the butterfly effect of the first kind.). *Three types of physical processes in the Lorenz model are: heating, dissipation and nonlinear interactions.* Our studies in npg-16-40 and other papers (e.g., Shen, 2014, 2015, 2016) have been performed to understand their individual and/or collective impact on the following characteristics of a chaotic system defined by Devaney (1989): (1) sensitivity to initial conditions; (2) topological transitivity; and (3) dense periodic points. As the 3$^{rd}$ feature suggests “recurrence”, and our analysis (Shen 2014) indicated a nonlinear feedback loop in the 3D Lorenz model (3DLM) as well as 5DLM, we have analyzed the 3D non-dissipative LM (3D-NLM) and high-dimensional non-dissipative LMs to understand the relationship between recurrent solutions (i.e., periodic of quasi-periodic solutions) and nonlinear feedback loop. [Here, a recurrence is defined when the distance between two states at different times within the phase space is smaller than a threshold ε. Mathematically, the recurrence may be associated with non-zero imaginary parts of the eigenvalues in the locally linear system near a non-trivial critical point.]

The nonlinearity (i.e., the nonlinear feedback loop) in the 3DLM is from the horizontal advection of temperature term (e.g., Shen, 2014), which appears in all of climate and weather models (e.g., Shen et al., 2006, 2012, 2013). Therefore, improving the understanding of the nonlinear terms and the associated (thermodynamic) feedback may help improve the representation of the thermodynamic feedback in numerical models, which remains big uncertainties in climate model simulations. While the role of the nonlinear feedback loop has been discussed using the original and high-dimensional Lorenz models (Shen, 2014, 2015, 2016), it is our belief that the role of nonlinearity can be better understood using non-dissipative versions, as discussed in this study (npg-16-40) and recent studies by Faghih-Naini and Shen (2017, npg-17-2) and Shen and Faghih-Naini (2017).

Using the 3D-NLM, we have two discussion articles posted by NPGD, which are referred to as the npg-14-21 and npg-16-40, respectively. The npg-16-40 discussion article was expanded based on the comments of the reviewers on the npg-14-21 discussion article. Recently, the study with the 3D-NLM has been extended using the 5D-NLM (Faghih-Naini and Shen, 2017, npg-17-2). In the studies with the 3D-NLM and 5D-NLM, we discussed the fundamental role of the nonlinearity (represented by the nonlinear feedback loop in our papers) in producing ``recurrent'' solutions, including periodic orbits within the 3D-NLM and quasi-periodic orbits within the 5D-NLM. The "simplified'' non-dissipative systems are used to reveal the relationship between the nonlinear feedback loop and the "recurrence" (i.e., periodicity or quasi-periodicity, to be specific). [Note that a future study is to compare the results with non-dissipative and dissipative models to analyze topological transitivity.]

The main focus in both the npg-14-21 and npg-16-40 discussion articles has been on the closed-form solution of the simple nonlinear ODE, $X''+X^3/2 =0$, and the role of the nonlinear term $X^3$. It is our believe the simplicity of the solutions (to $X''+X^3/2=0$) using trigonometric functions can effectively help illustrate the role of the nonlinear term ($X^3$) in producing oscillatory solutions. The cubic nonlinear term comes from the nonlinear feedback loop (-XZ and XY), which is briefly discussed in Appendix B in the revised manuscript, and the nonlinear feedback loop is from the nonlinear advection temperature as first discussed in Shen (2014). To our best knowledge, our closed-form solution to the above equation has never been documented in the literature. Additionally, we discussed the energy cycle associated with the periodic solutions using the 3D-NLM where evolution of kinetic, potential energy and available potential energy can be discussed. [Note that if we simply consider $X''+X^3/2=0$, only kinetic and potential energy can be defined.] The solution (to $X''+X^3/2=0$) can improve our understanding of the nonlinear processes and thus help examine the competing impact between the heating and nonlinearity. In the npg-16-40 discussion article, the other solutions (e.g., solutions using elliptic functions) were added partially in response to reviewers comments and suggestions. The solutions were used to support the view that the nonlinear feedback loop (with or without heating) can produce oscillatory solutions in the 3D-NLM.

When the relationship between the 3D-NLM, $X''+X^3/2=0$, and the Duffing equation was discussed in Appendix B of npg-16-40 article or Appendix C of the revised manuscript. Per reviewer's comments, we added related discussion in the main text of the revised manuscript as well. Additionally, we added a new Appendix B to discuss the relationship between the 3D-MLM with r=0 and the other two systems, including the original 3DLM with a large r (Sparrow, 1982) and the real-valued Maxwell-Bloth equations (David and Holm, 1992). Under a certain range of initial conditions (ICs), all of the three systems can be reduced to become the simple ODE, $X''+X^3/2$, which is a special case of the Duffing equation. Therefore, we may conclude that the nonlinear restoring force that produces periodic solution also appears in the original 3D-NLM with a larger r and the real-valued Maxwell-Bloth equations.

Based on the npg-16-40 discussion article, we have extended our study to examine the impact of the extension of the nonlinear feedback loop on the recurrence of solutions using the 5D-NLM (Faghih-Naini and Shen, 2017, npg-17-2). The extended nonlinear feedback loop produces two incommensurate frequencies, leading to a quasi-periodic solution. The "recurrent'' solution trajectory moves endlessly on a torus but never intersects itself. Additionally, Appendix C of Faghih-Naini and Shen (2017) provides an analogy between a coupled system with two springs and the locally linear 5D-NLM both of which have the same mathematical equation. This analogy helps reveal the role of the extended nonlinear feedback in producing quasi-periodic (recurrent) solutions and the importance of mode selection and coupling. Therefore, using the 3D-NLM and 5D-NLM, we have shown the role of the nonlinear feedback loop in producing periodic and quasi-periodic solutions, respectively. Since the nonlinear feedback loop and its extension appear throughout the spatial mode-mode interactions rooted in the nonlinear temperature advection, we expect that recurrent solutions such as quasi-periodic solutions may appear in the system based on the (full) Navier-Stokes equations. As "folding" is crucial for the occurrence of chaotic solution, future work will examine the relationship between the folding and recurrence in the high dimensional dissipative and non-dissipative Lorenz models.

In the npg-16-40 discussion article, we also illustrated how the collective impact of the nonlinear feedback loop and the heating may produce three types of solutions, including nonlinear periodic solutions with a small or large cycle and the homoclinic orbit solution. Additionally, we discussed the energy cycle. Note that kinetic, available potential energy (APE) and potential energy (PE), can be identified as $X^2$, $Y^2+Z^2$ and Z, respectively, in the 3D-NLM (e.g., Eqs. 10-12). However, simply considering

$X''+M^2X=0$ (Eq. 15) that is a the "special" Duffing equation, we can only define kinetic and potential energy, as follows: $KE_d= (1/2)\sigma^2Y^2$ and $PE_d=(1/8)X^4-(1/2)(\sigma r+C_1/C_0) X^2$, where $C_1/C_0$ is defined in Eq. 13. From a perspective of the 3D-NLM, "Z" is missing in the "special" Duffing equation. It is known that Z is crucial for introducing nonlinear terms in the 3DLM or 3D-NLM. Therefore, 3D-NLM can provide more detailed information about the energy transfer among the KE, PE and APE, as compared to the 2$^{nd}$ order ODE, $X''+M^2X=0$ or $X''+X^3/2=0$.

While specific responses are provided below, a summary on what has been discussed in re-revised manuscript is given as follows:

- Using a nondissipative Lorenz model, we present a closed-form solution using trigonometric functions to show that the nonlinear feedback loop (consisting of the nonlinear terms –XZ and XY) acts as a restoring forcing and the heating term alone can produce a saddle point.
- Using closed-form and numerical solutions, we showed that the nonlinear feedback loop and heating term collectively lead to three critical points and three types of solutions.
- Based on the energy analysis, a small energy cycle with four different regimes, which is half of a big energy cycle, is identified in one type of oscillatory solutions. We illustrated that the relative impact of the nonlinear restoring force and linear (heating) force determines the partitions of the averaged available potential energy associated with the Y and Z modes at different stages (i.e., linear and nonlinear stages). A big energy cycle appears in another type of oscillatory solutions. [Note that if we simply consider $X''+M^2X=0$ or $X''+X^3/2=0$, we can only define kinetic and potential energy.]
- The existence of the homoclinic orbit and two types of oscillatory solutions indicates the importance of the nonlinearity (i.e., the nonlinear feedback loop) and suggests the appearance of diverged trajectories. This type of solution dependence on ICs is different from the one associated with a chaotic attractor in the 3DLM.
- As suggested by one reviewer (for the npg-14-21 discussion article), Appendix A was added to discuss the derivations of Eq. (3) that includes the σY term;
- As suggested by one reviewer (for the npg-14-21 discussion article), Appendix C and Figure C1 were added to present the closed-form solution to the special Duffing

equation using elliptic functions and compare it with the closed-form solution represented by the elementary trigonometric functions in section 3.

- To respond to the comments on the npg-16-40 discussion article, Appendix B is added as a brief summary on what has been discussed in Shen (2014) to identify the nonlinear feedback loop. Additionally, we provided a comparison between the 3D-NLM, the 3DLM with a very large r and the real-valued Maxwell-Bloch equations.

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* Responses to the comments by Reviewer I
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> The paper concerns itself with a set of equations that have its origins on the standard quasigeostrophic equations for atmospheric flows. The analysis is competent but not novel, the outcomes are not remarkable. Unless the Author could suggest in very specific and complete terms what relevance these equations have to a physical problem of interest it would be difficult for this paper to have a readership.

As discussed in the general responses, our main goal is to illustrate the role of the nonlinear feedback loop (i.e., -XZ and XY) and its extension in producing recurrent solutions, including periodic solutions in the 3D-NLM and quasi-periodic solutions in the 5D-NLM. In the following, We provided brief discussions on the origins of various Lorenz models, including the Lorenz84 model that is more applicable to the quasi-geostrophic (QG) system.

The ODEs for the original Lorenz-63 model (i.e., 3DLM) were derived from the partial differential equations (PDEs) for Rayleigh Benard convection. The nondissipative version (i.e., 3D-NLM) was derived using the same PDEs but ignoring dissipative terms (as discussed in Appendix A of npg-16-40). The original PDEs are non-hydrostatic. The nonlinear advection of temperature in the PDEs produces the nonlinear terms (e.g., XY and –XZ) in the 3DLM. As the current trend is to develop global non-hydrostatic models at a resolution of 1km or finer, it is my belief that improving our understanding of the feedback by small processes within the original and high-dimensional Lorenz models can help understand the thermodynamic feedback associated the explicitly resolving convective processes in the next generation global models.

In 1980, using shallow water equations (with a constant of “f”, the Coriolis parameter), Lorenz applied the Galerkin method to derive a model and transformed it into the 3D Lorenz model (i.e., the 3DLM) using a complicated transformation. Later, Lorenz published a paper with different ODEs for the QG system in 1984, which has been referred to as the Lorenz-84 model. However, to our best knowledge, we could not find any paper by Lorenz regarding related derivations. Here, we would like to provide the following notes for reviewers’ information. In 2015, I personally checked with two researchers (who jointly published a paper using the Lorenz-84 model) and learned that they could not find references by Lorenz about the derivations of the Lorenz84 model. By searching for literatures later, derivations of the Lorenz84 model can be found in Veen (2002), as shown in the extracted image below. Veen (2002) made the following comments (on page 11 of Veen, 2002), which are similar to what I learned:

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*To the author's knowledge, no derivation of the Lorenz-84 model from atmospheric flow equations has been presented before. A rather ad hoc link was established by Wiin-Nielsen [1992, 1994], but in his work the reduction to three degrees of freedom is not based on physical or mathematical arguments. The link established here enables us to calculate the parameters in the Lorenz-84 model from the physical parameters in the filtered equations. As it turns out, one of the parameters comes out significantly different from its traditional, yet unmotivated, value. A continuation in this parameter relates the bifurcation diagram found at the traditional parameter value, presented in Shilnikov et al. [1995], to the one found at the physical value. The latter still bears resemblance to the bifurcation diagram of the six dimensional model, but the neutral saddle-focus transition is no longer there. Hence the route to chaos through a Shilnikov type bifurcation is absent. It is shown, that chaos through period doubling cascades, the Ruelle-Takens scenario and intermittency does occur in the Lorenz-84 model.*

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### 2. The Lorenz-84 general circulation model

Like the Lorenz-63 model, the Lorenz-84 model is related to a Galerkin truncation of the Navier-Stokes equations. Where the '63 model describes convection, the '84 model gives the simplest approximation to the general atmospheric circulation at midlatitude. The approximation is applicable on an $f$-plane, placed over the North Atlantic ocean.

We can give a physical interpretation of the variables of the Lorenz-84 model: $x$ is the intensity of the westerly circulation, $y$ and $z$ are the sine and cosine components of a large traveling wave. The time derivatives are given by

$$\dot{x} = -y^2 - z^2 - ax + aF \tag{1.1}$$

$$\dot{y} = xy - bxz - y + G \tag{1.2}$$

$$\dot{z} = bxy + xz - z \tag{1.3}$$

where $F$ and $G$ are forcing terms due to the average north-south temperature contrast and the earth-sea temperature contrast, respectively. Conventionally we take $a = 1/4$ and $b = 4$.

*Figure: the extracted image shows the Lorenz-84 model (see Veen, 2002 for details).*

**References:**


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* Responses to the comments by reviewer II
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> This manuscript discusses a model called by the author the non-dissipative Lorenz model. The model is related to the Lorenz 1963 model, but with several terms missing. It is apparently derived from the Boussinesq equations by a method similar to that of Lorenz. The model has a conserved quantity, and for a particular value of the conserved quantity, the model is equivalent to the Duffing equation with no damping and no forcing, i.e., frictionless motion in a double-well potential given by a symmetric 4th-degree polynomial. The analysis of periodic and homoclinic solutions of this model is not novel, and it is not clear to me that it offers significant insight into the physics of the Boussinesq equations.

Thanks for your comments. My responses are provided as follows:

(1) The ODEs were derived from the PDEs for Rayleigh-Benard convection. The Boussinesq approximation was used in the PDEs, which may be called the Boussinesq-type equations.

(2) Our goal is to reveal the role of nonlinear feedback loop in producing periodic solutions within the 3D-NLM (as well as quasi-periodic solutions using the 5D-NLM, e.g., Faghih-Naini and Shen, 2017). We discussed how the nonlinear feedback loop, consisting of -XZ and XY, may lead to the nonlinear restoring forcing term (i.e., $X^3$). Previously, we have shown that the nonlinear feedback loop is from the nonlinear advection of temperature and can be extended using additional high wavenumber modes.

(3) While it has been shown that the homoclinic orbit may appear under certain conditions within the 3DLM, we did not find any published papers discussing the homoclinic orbit using the non-dissipative Lorenz model (3D-NLM).

(4) We are aware that a further simplified 3D-NLM (e.g., with a particular set of initial conditions) and the Duffing equation (as discussed in the Appendix) (or double-well potential system) may be dynamically equivalent. However, while the former (the 3D-NLM) has three ODEs and the latter (with or without an external forcing) is a second-order ODE. [In general, a second-order ODE can be reduced to a set of two first-order ODEs.] Therefore, without providing a proof regarding a homeomorphism, we avoided a detailed comparison between the two systems. On the other hand, using the 3D-NLM (e.g., Eqs. 4-5), we were able to

discuss the energy cycle (e.g., the partitions of averaged available potential energy from Y and Z modes) and the challenge of performing the numerical integration for obtaining the homoclinic orbit solution (see also discussions in (5) below).

(5) As shown in Figure 7 for the solutions of homoclinic orbits as well as oscillatory solutions with different temporal scales, adaptive methods is effective to perform numerical integrations of a homoclinic orbit, implying that it may be challenging to determine the Lyapunov exponent numerically when an initial condition is on the homoclinic orbit.

(6) We have already extended the current study using the 3D-NLM to examine the impact of the extended nonlinear feedback loop on the quasi-periodicity of solutions in the 5D-NLM (Faghih-Naini and Shen, 2017, npg-17-2).

(7) Based on the above discussions, it is our belief that the original and high-dimensional non-dissipative Lorenz models are unique as compared to the double-well potential system or the Duffing equation. On the other hand, the similarity between the 3D-NLM and the Duffing equation can help verify the our results (including numerical solutions) and build our confidence in the validity of analytical and numerical methods for studying the role of the extended nonlinear feedback loop in producing quasi-periodic solutions in high-dimensional non-dissipative Lorenz models.

To be more specific, equation (15) in the manuscript is the Duffing equation discussed above; I find it peculiar that the manuscript does not mention Duffing until Appendix B. The Duffing equation has been studied extensively, and in the particular case of no damping and no forcing, its solutions are particularly well understood. The closed-form solution in the special case presented in Appendix B is also derived, for example, in the discussion below equation (23) of the article at: http://www.sciencedirect.com/science/article/pii/S0094114X14002079 Closed-form solutions for more general equations are derived in the articles at: http://www.sciencedirect.com/science/article/pii/S0307904X12002302 http://isidl.com/downloadfile/106810 The present manuscript focuses primarily on the analysis of the Duffing model, which I already well understand, and I don't feel that I learned anything from it about the Lorenz or Boussinesq equations.

Thanks for sharing the two references. We have cited the references in the main text of the revised manuscript. We wanted to point out that in addition to a citation to the Duffing oscillator equation in Appendix B/C, we also cited the paper by Roupas (2012) who provided more detailed discussions regarding the impact of parameters on solutions and their relationship with the Duffing oscillator equation. For example, we did state that two types of solutions associated various $C_1/C_o$ were discussed by Roupas (2012). [Here the constant $C_1$ represents the conservation of KE and PE, i.e., $C_1/C_o = X^2/2 - Z$ (Eg. 13). The Duffing oscillator equation does not include a component which is similar to "Z" in the 3D-NLM.] Specifically, depending on the sign of ($\sigma r + C_1/C_o$), Eq. 15 is called the Duffing equation or two-well potential system.

For the suggested references, which have been cited in the revised manuscript, I would like to provide additional responses as follows:

1. In the newly added references (i.e., Elias-Zuniga, 2013; Starossek, 2015), we could not find discussions regarding the homoclinic orbit.
2. As compared to the analytical solutions in Elias-Zuniga (2013) and Starossek, (2015), our closed-form solutions in the main text have a simpler representation. It is our belief that the simplicity of our closed-form solutions (to the simple nonlinear ODE $X'' + X^3/2$) using trigonometric functions can help illustrate the periodicity of the solutions. Additionally, discussions on the solution and Duffing equation in Appendix B/C using elliptic functions were used for verification. Numerical simulations using the 3D-NLM are computed and inter-compared with our closed-form solutions. The 3D-NLM can help discuss the partitions of

averaged potential energy from Y and Z modes. We are not sure if this kind of analysis for the energy cycle can be done using the Duffing equation.

3. The study in the npg-16-40 discussion article has been extended to reveal the role of the extended nonlinear feedback loop in producing quasi-periodic solutions within the 5D-NLM (Faghih-Naini and Shen 2017, npg-17-2) as well as the 7D-NLM (Shen and Faghih-Naini, 2017).

Finally, I would like to add the following to support our approach and conclusion. In the newly-added Appendix B, we listed a reference showing that the 3D-NLM with r=0 can represent the original 3DLM with a very large r. Additionally, we documented the similarity between the 3D-NLM with r=0 and the real-valued Maxwell-Bloch equations, which can be written as follows (Eq. 1.20 of David and Holm, 1992):

$X' = Y,$

$Y' = XZ,$

$Z' = -XY.$

The above can be reduced to become

$X'' - CX + (1/2)X^3 = 0,$

here C (=(1/2) $X^2$ + Z) is a constant, representing a conservation law. Note that replacing (XZ) and (–XY) by (-XZ) and (XY), respectively, we can obtain a set of equations that are the same as the 3D-NLM with r=0. Therefore, based on our study, it is suggested that the pair of (XZ) and (−XY) serves as a restoring force in the Maxwell-Bloch equations.

**References:**

David, D. and D. D. Holm, 1992: Multiple Lie-Poisson Structures, Reductions, and Geometric Phases for the Maxwell-Bloch Travelling Wave Equations. *J. Nonlinear Sci.* Vol. 2: pp. 241-262 (1992).