Quasi-periodic solutions associated with the extended nonlinear feedback loop of the five-dimensional nondissipative Lorenz model

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together with Dr. Bo-Wen Shen

Outline

On quasi-periodic solutions associated with the extended nonlinear feedback loop of the five-dimensional nondissipative Lorenz model

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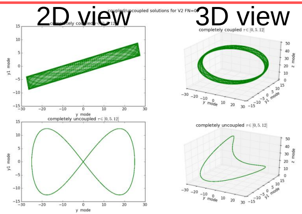
Objective

- Analysis and comparison of 5D NLM
- Fundamental role of nonlinear terms in producing quasi-periodic solutions
- Collective role of the nonlinear feedback loop and heating term in producing quasi-periodic solutions



Y-Y1-plots of periodic/quasiperiodic solutions for tend=0.512 (top) and tend=5.12 (bottom) of coupled (left) and uncoupled (right) LL 5D NLM

Approach

- Performing analytical, symbolic (python scipy) and numerical (Python ODE solver (odeint)) analysis of 5D NLM
- Analyzing quasiperiodic solutions of 5D NLM
- Comparing to solutions of 5D NLM, locally linearized 5D NLM, 3D NLM, …
- Performing frequency analysis using Python fft package
- Visualizing quasiperiodic solutions using Python visualization packages

Key Milestones

- 09/30/2016: solve for eigenvalues of 5D NLM (analytical and symbolic computation)
- 11/01/2016: verify results, plots for comparison of 3D NLM, LL 3D NLM FN=0 and FN=1, 3D-eigenvaluesolution (analytical and symbolic); plots for comparison of 5D NLM, LL 5D NLM FN=0 and FN=1, 5D-eigenvaluesolution (analytical and symbolic)
- 11/20/2016: finish frequency analysis
- 11/22/2016: create slides for presentation
- 11/30/2016: finish oral presentation
- 12/06/2016: draft of paper

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Figure: QuadChart

Introduction: Quasiperiodic Solutions

- Frequency ratio $\frac{\omega_2}{\omega_1}$ of two modes determines behavior of solution:
- If $\frac{\omega_2}{\omega_1}$ is rational, the motion is periodic and has a closed orbit.
- If $\frac{\omega_2}{\omega_1}$ is irrational, these two frequencies are called incommensurate;
- The composite motion is quasiperiodic and its period is infinite;
- The trajectory is dense, that means, it comes arbitrarily close to each point of torus.

Introduction: Quasiperiodic Solutions

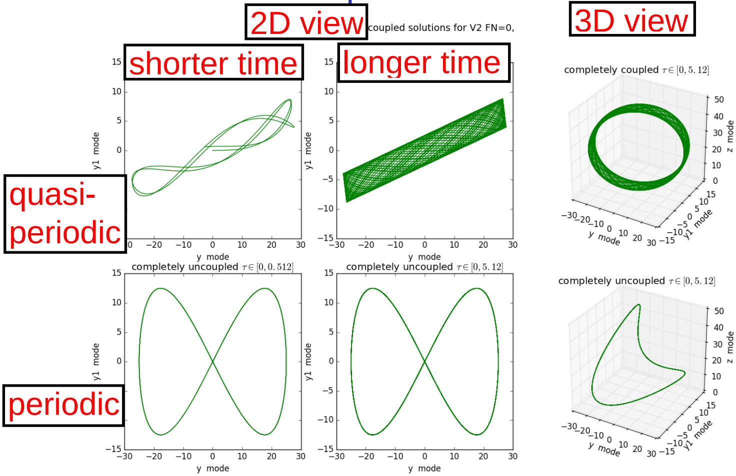


Figure: Quasiperiodic solution of coupled LL 5D NLM with a frequency ratio of $\frac{1}{2}\left(3-\sqrt{5}\right)$ (top) and periodic solution of uncoupled LL 5D NLM with a frequency ratio of 2 (bottom) up to time 0.512 (left) and 5.12 (middle)

5D NLM V1

Removing the dissipative terms from the five dimensional Lorenz model (5D LM), the the nondisipative five dimensional Lorenz model (5D NLM) results:

$$\frac{dX}{d\tau} = \cancel{-\sigma X} + \sigma Y, \quad (1)$$

$$\frac{dY}{d\tau} = -XZ + rX - \cancel{Y}, \quad (2)$$

$$\frac{dZ}{d\tau} = XY \boxed{-XY_1} - \cancel{bZ}, \quad (3)$$

$$\frac{dY_1}{d\tau} = \boxed{XZ} - 2XZ_1 - \cancel{d_0 Y_1}, \quad (4)$$

$$\frac{dZ_1}{d\tau} = 2XY_1 - \cancel{4bZ_1}. \quad (5)$$

5D NLM V2

Perturbation method leads to the locally linearized 5D NLM (LL 5D NLM):

$$\frac{dX'}{d\tau} = \sigma Y', \tag{6}$$

$$\frac{dY'}{d\tau} = (r - Z_c)X' - X_c Z' - FN(X'Z'), \tag{7}$$

$$\frac{dZ'}{d\tau} = (Y_c - Y_{1c})X' + X_c Y' - \boxed{X_c Y_1'} + FN(X'Y' - X'Y_1'), \tag{8}$$

$$\frac{dY_1'}{d\tau} = (Z_c - 2Z_{1c})X' + \boxed{X_c Z'} - 2X_c Z_1' + FN(X'Z' - 2X'Z_1'), \tag{9}$$

$$\frac{dZ_1'}{d\tau} = 2Y_{1c}X' + 2X_c Y_1' + 2FN(X'Y_1'). \tag{10}$$

Choosing $FN = 0$ makes the system linear with respect to the critical point and for $FN = 1$ the system is fully nonlinear.

Model Assumptions and List of Simulations

Parameters: $\sigma = 10, r = 25, \Delta\tau = 0.001$

Method	Model	Equations	IC	Python Packages
analytical	V2 FN=0	6-10	$X_0 = \sqrt{\frac{5}{2}\sigma r}$, $X_c = \sqrt{\frac{5}{2}\sigma r + X_0^2}$	-
symbolic	V2 FN=0	6-10	$X_0 = \sqrt{\frac{5}{2}\sigma r}$, $X_c = \sqrt{\frac{5}{2}\sigma r + X_0^2}$	linalg
numerical	V2 FN=0 V2 FN=1 V1	6-10 6-10 1-5	$X_c = \sqrt{\frac{5}{2}\sigma r + X_0^2}$ for $X_0 = 0.25\sqrt{\frac{5}{2}\sigma r}$, $X_0 = 0.5\sqrt{\frac{5}{2}\sigma r}$, $X_0 = \sqrt{\frac{5}{2}\sigma r}$, $X_0 = 4\sqrt{\frac{5}{2}\sigma r}$,	odeint

Impact of ICs

Table: Methods, corresponding model assumptions and computing packages used for verifying results and analyzing 5D NLM

Analytical Solution for Eigenvalues

Setting $FN = 0$, plugging in the basic state values, the Jacobian matrix becomes:

$$A^{5DNLM} = \begin{pmatrix} 0 & \sigma & 0 & 0 & 0 \\ 0 & 0 & -X_c & 0 & 0 \\ 0 & X_c & 0 & \boxed{-X_c} & 0 \\ 0 & 0 & \boxed{X_c} & 0 & -2X_c \\ 0 & 0 & 0 & 2X_c & 0 \end{pmatrix}$$

To obtain the eigenvalues $\lambda_1, \ldots \lambda_5$, with $\lambda = i\beta$, the equation $det(A^{5DNLM} - i\beta\mathbb{I}) = 0$ was solved applying Laplace's formula:

$$\beta_1 = \sqrt{\left(3 + \sqrt{5}\right) X_c}$$

$$\beta_2 = -\sqrt{\left(3 + \sqrt{5}\right) X_c}$$

$$\beta_3 = \sqrt{\left(3 - \sqrt{5}\right) X_c}$$

$$\beta_4 = -\sqrt{\left(3 - \sqrt{5}\right) X_c}$$

$$\beta_5 = 0$$

Analytical Solution for Eigenvectors

For $k = 1, 2, 3, 4$ the form of the corresponding eigenvectors is

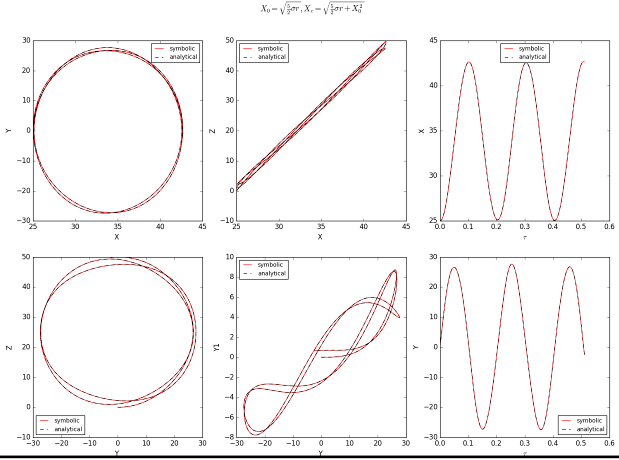
$$v_k = \begin{pmatrix} \sigma\left(\frac{-\beta_k{}^2}{2X_c^3} + \frac{5}{2X_c}\right) \\ i\beta_k\left(\frac{-\beta_k{}^2}{2X_c^3} + \frac{5}{2X_c}\right) \\ \frac{-\beta_k{}^2}{2X_c^2} + 2 \\ \frac{i\beta_k}{2X_c} \\ 1 \end{pmatrix} \text{ and } v_5 = \begin{pmatrix} \sigma \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This leads to the general solution $S(t) = C_1 e^{i\beta_1 t} v_1 + C_2 e^{i\beta_2 t} v_2 + C_3 e^{i\beta_3 t} v_3 + C_4 e^{i\beta_4 t} v_4 + C_5 e^{i\beta_5 t} v_5$, which can be expressed in the sine-cosine representation.

Analysis of Analytical Solution

- $C = C_1, ..., C_5$ can be determined by applying the initial condition $I = (X_0 - X_c, 0, -r, 0, -\frac{r}{2})$.
- Using the analytical solutions of the eigenvectors, which include oscillatory modes with two incommensurate frequencies, as a basis for a solution, the coefficients of the eigenvectors can be determined as $C_1 + C_2 = \frac{-{\beta_3}^2 r}{2\left({\beta_3}^2 - {\beta_1}^2\right)}$, $C_3 + C_4 = \frac{{\beta_1}^2 r}{2\left({\beta_3}^2 - {\beta_1}^2\right)}$ and $C_5 = \frac{X_0 - X_c}{\sigma} + \frac{5r}{4X_c}$ (constant mode) (for more detailed discussions see Appendix)
- The **frequency ratio** $\frac{\beta_3}{\beta_1} = \frac{\sqrt{\left(3-\sqrt{5}\right)X_c}}{\sqrt{\left(3+\sqrt{5}\right)X_c}} = \frac{1}{2}\left(3 - \sqrt{5}\right)$ is **irrational**, so solution is **quasiperiodic**.

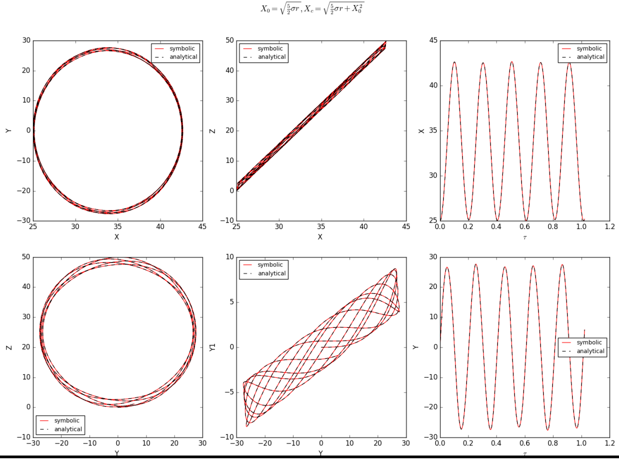
Analytical vs. Symbolic Solution-shorter time



analytical and symbolic solutions are identical

Figure: Analytical vs. Symbolic Solutions for $X_0 = \sqrt{\frac{5}{2}\sigma r}$, $X_c = \sqrt{\frac{5}{2}\sigma r + X_0^2}$ and $\tau \in [0, \mathbf{0.512}]$

Analytical vs. Symbolic Solution-longer time



analytical and symbolic solutions are identical

Figure: Analytical vs. Symbolic Solutions for $X_0 = \sqrt{\frac{5}{2}\sigma r}$, $X_c = \sqrt{\frac{5}{2}\sigma r + X_0^2}$ and $\tau \in [0, \mathbf{1.024}]$

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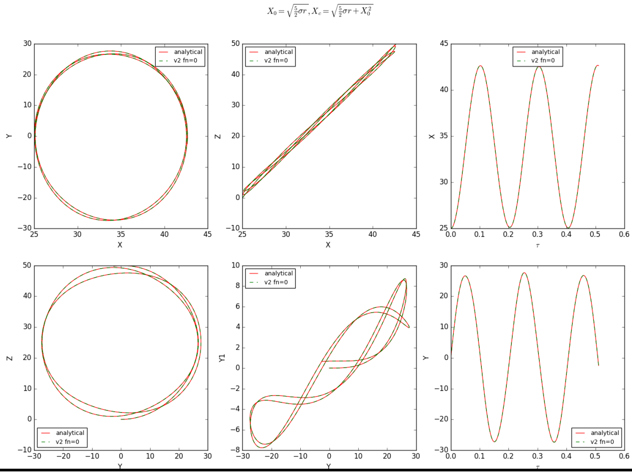
Model Verification



analytical and numerical solutions are identical

Figure: Analytical vs. Numerical solution of LL 5D NLM (V2, FN=0) for $X_0 = \sqrt{\frac{5}{2}\sigma r}$, $X_c = \sqrt{\frac{5}{2}\sigma r + X_0^2}$ and $\tau \in [0, 0.512]$

Impact of Nonlinearity

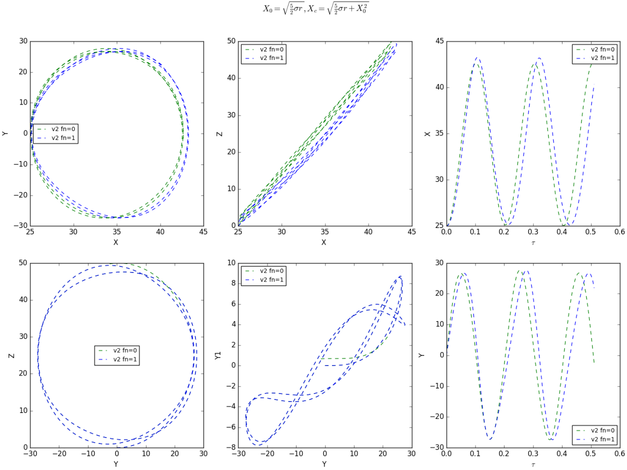


Figure: Numerical solutions of LL 5D NLM FN=0 vs. FN=1 for $X_0 = \sqrt{\frac{5}{2}\sigma r}$, $X_c = \sqrt{\frac{5}{2}\sigma r + X_0^2}$ and $\tau \in [0, 0.512]$

Impact of Initial Conditions

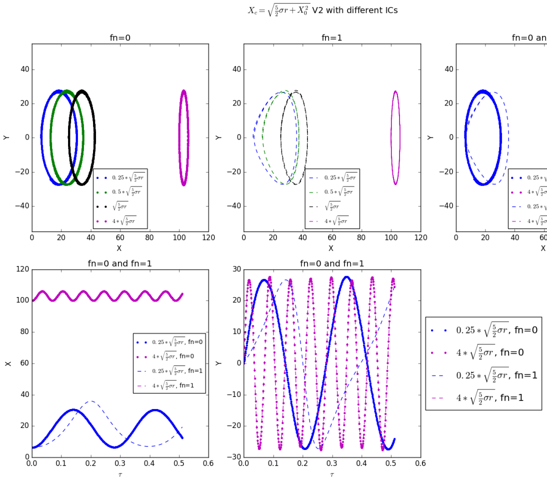


Figure: Comparison of different initial conditions in LL 5D NLM FN=0 and FN=1 for $t \in [0, 0.512]$

Examine coupling terms using 5D NLM V2

$$\frac{dX'}{d\tau} = \sigma Y',$$

$$\frac{dY'}{d\tau} = (r - Z_c)X' - X_c Z' - FN(X'Z'),$$

1

$$\frac{dZ'}{d\tau} = (Y_c - Y_{1c})X' + X_c Y' - \boxed{X_c Y_1'} + FN(X'Y' - X'Y_1'),$$

2

$$\frac{dY_1'}{d\tau} = (Z_c - 2Z_{1c})X' + \boxed{X_c Z'} - 2X_c Z_1' + FN(X'Z' - 2X'Z_1'),$$

$$\frac{dZ_1'}{d\tau} = 2Y_{1c}X' + 2X_c Y_1' + 2FN(X'Y_1').$$

Impact of coupling terms

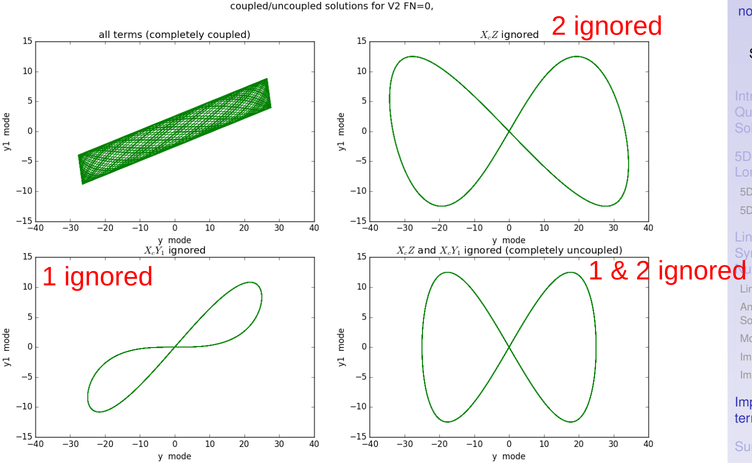


Figure: Solution of LL 5D NLM FN=0 becomes peridoic, if coupling terms X_cY_1 in $\frac{dZ'}{d\tau}$ and/or X_cZ in $\frac{dY_1'}{d\tau}$ are ignored

Summary

- For the locally linear 5D NLM, its analytical solution with two incommensurate frequencies, whose ratio is irrational, was obtained.
- A comparison between the 3D NLM and 5D NLM suggests that the incommensurate frequencies in the 5D NLM are produced by the extension of the nonlinear feedback loop.
- While the 3D NLM includes periodic solutions, the 5D NLM produces a quasi-periodic solution. The coupling terms that are associated with the extension of the nonlinear feedback loop are crucial for the appearance of incommensurate frequencies in the 5D NLM.
- Linear and nonlinear solutions for the 5D NLM (i.e., version 2) are closer for greater values for X_c, i.e. values being further away from the origin.

Appendix
Analytical Solution - General Form

$C = C_1, \ldots, C_5$ can be determined by setting $t = 0$ and equating S with the initial condition $I = (X_0 - X_c, 0, -r, 0, -\frac{r}{2})$. The the system has the following form:

$$S(t) = \frac{-{\beta_3}^2 r}{2({\beta_3}^2 - {\beta_1}^2)} \begin{pmatrix} \sigma\left(\frac{-{\beta_1}^2}{2X_c^3} + \frac{5}{2X_c}\right) cos(\beta_1 t) \\ -\beta_1 \left(\frac{-{\beta_1}^2}{2X_c^3} + \frac{5}{2X_c}\right) sin(\beta_1 t) \\ \left(\frac{-{\beta_1}^2}{2X_c^2} + 2\right) cos(\beta_1 t) \\ \frac{-\beta_1}{2X_c} sin(\beta_1 t) \\ cos(\beta_1 t) \end{pmatrix} +$$

$$\frac{{\beta_1}^2 r}{2({\beta_3}^2 - {\beta_1}^2)} \begin{pmatrix} \sigma\left(\frac{-{\beta_3}^2}{2X_c^3} + \frac{5}{2X_c}\right) cos(\beta_3 t) \\ -\beta_3 \left(\frac{-{\beta_3}^2}{2X_c^3} + \frac{5}{2X_c}\right) sin(\beta_3 t) \\ \left(\frac{-{\beta_3}^2}{2X_c^2} + 2\right) cos(\beta_3 t) \\ \frac{-\beta_3}{2X_c} sin(\beta_3 t) \\ cos(\beta_3 t) \end{pmatrix} + \left(\frac{X_0 - X_c}{\sigma} + \frac{5r}{4X_c}\right) \begin{pmatrix} \sigma \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Symbolic Solution

```
from sympy import Symbol, Matrix

X_c = Symbol('X_c')
sig = Symbol('sig')
la= Symbol('la')
A5D=Matrix([[0,sig,0,0,0],
»             [0,0,-X_c,0,0],
»             [0,X_c,0,-X_c,0],
»             [0,0,X_c,0,-2*X_c],
»             [0,0,0,2*X_c,0]])
eigenvalues=A5D.eigenvals()
eigenvectors=A5D.eigenvects()
```

Figure: Python code for symbolic calculation of eigenvalues and eigenvectors

Numerical Solution - 5D NLM V1

```
from scipy.integrate import odeint
import numpy as np

def Lorenz5D_nondissipative(state,t, sigma, r, beta, d0):

  x,y,z,y1,z1 = state

  dx = sigma * (y)
  dy = -x*z    + r*x
  dz =  x*y    - x*y1
  dy1 = x*z    - 2*x*z1
  dz1 = 2*x*y1

  return [dx, dy, dz, dy1, dz1]

sigma = 10.0
r = 25.0
beta = 8.0/3.0
d0 = 19.0/3.0

X0=np.sqrt(5./2*sigma*r)
xc=np.sqrt(5./2*sigma*r+X0**2)
zc=r
z1c = r/2
yc= 0.
y1c= 0.

dt=0.001
L=512*dt
t = np.arange(0.0,L, dt)
state = [X0_1, 0, 0, 0, 0]

out = odeint(Lorenz5D_nondissipative, state, t,  args=(sigma, r, beta, d0))
```

Figure: Python code for numerical solution of the 5D NLM using odeint

Numerical Solution - 5D NLM V2

```
from scipy.integrate import odeint
import numpy as np

def LL_Lorenz5D(state,t, parameters):

  x,y,z,y1,z1 = state
  sigma, r, beta, d0, FN, X0, xc, zc, yc, z1c, y1c = parameters

  dx = sigma * (y)
  dy = (r-zc)*x  - xc*z - FN*(x*z)
  dz = (yc-y1c)* x + xc*y - xc*y1 + FN*(x*y-x*y1)
  dy1 = (zc-2*z1c)*x + xc * z  - 2*xc*z1 + FN*(x*z-2*x*z1)
  dz1 = 2*y1c*x + 2*xc*y1 + 2*FN*(x*y1)

  return [dx, dy, dz, dy1, dz1]

sigma = 10.0
r = 25.0
beta = 8.0/3.0
d0 = 19.0/3.0

X0=np.sqrt(5./2*sigma*r)
xc=np.sqrt(5./2*sigma*r+X0**2)
zc=r
z1c = r/2
yc= 0.
y1c= 0.

dt=0.001
L=512*dt
t = np.arange(0.0,L, dt)
state = [X0_1-xc_1, 0, -r, 0, -r/2] #=[X', Y', Z', Y1', Y2']

FN=0
parameters = (sigma, r, beta, d0, FN, X0, xc, zc, yc, z1c, y1c)
out0 = odeint(LL_Lorenz5D, state, t, args=(parameters,))
FN=1
parameters = (sigma, r, beta, d0, FN, X0, xc, zc, yc, z1c, y1c)
out1 = odeint(LL_Lorenz5D, state, t, args=(parameters,))
```

Figure: Python code for numerical solution of the LL 5D NLM using odeint

Frequency analysis for LL 5D NLM

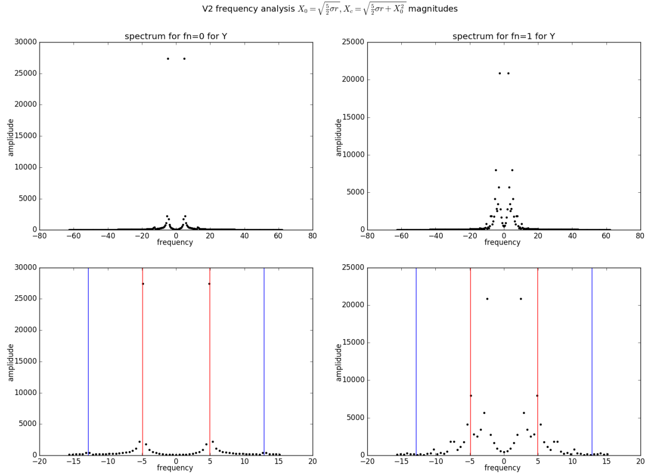


Figure: Magnitudes of fourier modes of LL 5D NLM FN=0 and FN=1 and zoomed-in presentation for comparison with eigenvalue solution for $\tau \in [0, 0.512]$