Responses to the editor's comments

We appreciate the kind considerations from Dr. Olivier Talagrand on our manuscript in NPG Discussion. We made major revisions to our original manuscript based on all the comments from the editor and referees. The main amendments are as follows (please forgive us for not marking revisions in the revised manuscript; many changes were made, and marking all revisions would make the paper a mess and difficult to read):

1. Structural changes

We thoroughly revised the structure to make our manuscript more understandable. The title of the revised manuscript was rewritten as "Formulation of Scale Transformation in a Stochastic Data Assimilation Framework". We made this significant modification for the following reasons. First, defining the scale and scale transformation laid a foundation for our study and makes our work distinct from the previous studies. Second, the original title was insufficient because we did not reformulate the framework of a stochastic data assimilation, which was used only to investigate the expression of errors that are determined by scale transformation. Therefore, the new title is more suitable.

First, we rewrote Sect. 1. We reduced the original text and added some necessary reviews on representativeness error. Second, Sect. 2 was also changed to "Basic knowledge"; we stated the basic knowledge of measure theory in Sect. 2.1, and the basic knowledge of stochastic calculus in Sect. 2.2. Our study was mainly presented in Sect. 3. In Sect. 3.1, we defined the scale and scale transformation with Lebesgue measure. In Sect. 3.2, we introduced the definition of stochastic variables in data assimilation. The errors from scale transformation in a data assimilation framework were presented in Sect. 3.3. Then, Sect. 4 was divided into three subsections, namely, "Summary", "Discussion" and "Next Step". In the last section, we listed the basic notations in measure theory and stochastic calculus and the new notations in our study.

In addition, we removed Sect. 4.4, "An example: the stochastic radiative transfer equation (SRTE)", which was not closely tied to the other sections of this study. We shifted Sect. 4.5 to Sect. 3.4, "Extension to n-dimensional data assimilation".

2. Full introduction of Lebesgue measure to define "scale"

In the original version, we developed a vector-valued measure to define the "scale" (see Sect. 2 in the original manuscript). Referee 1 argued that this measure mismatches with his/her basic knowledge of measure theory and leads to a poor understanding of the manuscript. Dr. Talagrand also stressed this problem. We accept this advice. After careful consideration, we decided to abandon the original idea and completely change the definition of the "scale" by further introducing Lebesgue measure.

Changes in the manuscript: Correspondingly, Sect. 2 was completely rewritten. In Sect. 2.1, which was retitled "Measure theory", we introduced some basic concepts such as σ-algebra, measure, measure space, Lebesgue measure, Lebesgue integral, and so on. In Sect. 3.1, "Definition of scale", we used Lebesgue measures to investigate the structures of scales and scale transformation.

3. Previous studies on representativeness error were included and compared to our study

Prof. van Leeuwen noted that no reviews of representativeness error exist and that this topic requires a comparison between our study and previous work on representativeness error. We added the corresponding content in the revised manuscript.

Changes in the manuscript:

(1) The introduction mainly considered previous work on representation error.

(2) Sect. 4.2 was added to compare our study with the available works on representation error.

Please find this information in the revised manuscript (main contents are in Sect. 1 and Sect. 4.2).

4. Equation recompilation

We reedited all the equations with Microsoft equation editor. The typesetting of the mathematical equations in the revised manuscript is better than that in the original manuscript.

5. Notation consistency

Both the referees and Dr. Talagrand emphasized that the notations in the original manuscript were inconsistent. We seriously addressed this issue. The corresponding changes in the new version are as follows:

Previous notation	Current notation
Scale change, change of scale	Scale transformation

Observation/measurement region	Observation footprint
Model units	Model unit
System state	State
dynamic model (operator), physical model (operator)	Forecasting model (operator)
System state space	Model space
Instrument error	Measurement error

6. New Notations

Dr. Talagrand advised us to clearly explain the new notations that were introduced in our manuscript. We accept this advice. In the revised manuscript, we expanded the "Notation" section and classified all the notations into two types. One was the basic notations in measure theory and stochastic calculus, which were given only a full name, and their definitions can be found in Sect. 2.1 and Sect. 2.2. The other type was the new notations. We offered their full names and their detailed explanations and indices in the manuscript. Please find more information in the revised manuscript (Sect. 5).

7. Other changes

Many other small presentation changes were made in the current manuscript. Some of these changes involved our re-examination, and some were provided by a professional native English speaking team. We hope that these changes improved the manuscript substantially to meet the quality standards of NPG. However, the changes were not presented as revisions in our revised manuscript because so many revisions that presenting them all would make the paper a mess.