Responses to Referee 2

We thank Prof. van Leeuwen for his valuable comments and suggestions. Most importantly, he provided some valuable literature on representation error, which convinced us to reconsider our study in light of these previous works and helped us improve our study in line with the latest advances in studying representation error. Our point-by-point responses to the comments from Prof. van Leeuwen are as follows (please forgive us for not marking revisions in the revised manuscript; many changes were made, and a revised version would make the paper a mess and difficult to read):

1. General reply

In the original version, we considered some results from other researchers, including basic knowledge of scale, scale problem in geophysical processes, and data assimilation with respect to stochastic processes. However, we did not consider the representation error, which was also noted by Prof. van Leeuwen in his interactive comment.

In fact, we also devoted ourselves to studying the representation error. Some results of how to understand and quantify representation error with real world experiments have been published or are being considered for publication:

- Li, X.: Characterization, controlling, and reduction of uncertainties in the modeling and observation of land-surface systems, Sci. China Ser. D, 2014.
- Huang, G., Li, X., et al.: Representativeness errors of point-scale ground-based solar radiation measurements in the validation of remote sensing products, Remote Sens. Environ., 2016.
- Li, X. and Liu, F.: Can Point Measurements of Soil Moisture Be Used to Validate a Footprint-Scale Soil Moisture Product? IEEE Geosci. Remote S., 2016. (Submitted)
- Ran, Y. H., Li, X., et al.: Spatial representativeness and uncertainty of eddy covariance carbon flux measurement for upscaling net ecosystem productivity to field scale, Agric. Forest Meteorol., 2016.

We used "representativeness errors" in our previous studies, and we believe that the representation error is not limited to data assimilation. This factor may remarkably affect many fields that are associated with Earth observations and simulations.

We did not explicitly introduce representation error in the original manuscript because this study mainly focuses on the errors from scale transformation, which is only one component of representation error. However, we accept the advice from Prof. van Leeuwen to consider representation error in our study. According to this comment, some valuable works examined representation error in data assimilation, and considering these works should substantially improve our study.

Changes in the manuscript: In the revised manuscript, the corresponding text was added and the structure of the manuscript was reworked. The main changes are as follows:

(1) The title of the revised manuscript was rewritten as "Formulation of Scale Transformation in a Stochastic Data Assimilation Framework". We made this significant modification for the following reasons. First, defining the scale and scale transformation laid a foundation for our study and makes our work distinct from previous studies. Second, the original title was insufficient because we did not reformulate the framework of a stochastic data assimilation, which was used only to investigate the expression of errors that were determined by scale transformation. Therefore, the new title is more suitable.

(2) We rewrote the introduction. We reduced the original text and added some necessary reviews on representation error. The fundamental causes of representation error, the difference between representation error and the scale-dependent error, the latest advances in studying the representation error in data assimilation, and the room to improve these studies were summarized.

(3) Sect. 2 was also changed. This section's name was changed to "Basic knowledge". We stated the basic knowledge of measure theory in Sect. 2.1, and we introduced the basic knowledge of stochastic calculus in Sect. 2.2. Our study was mainly presented in Sect. 3. In Sect. 3.1, we defined the scale and scale transformation with Lebesgue measure. In Sect. 3.2, we introduced the definition of stochastic variables in data assimilation. The errors from scale transformation in a data assimilation framework were presented in Sect. 3.3. Then, Sect. 4 was divided into three subsections, namely, "Summary", "Discussion" and "Next step". In the last section, we listed the basic notations in measure theory and stochastic calculus, and the new notations in our study. In addition, we removed Sect. 4.4, "An example: the stochastic radiative transfer equation (SRTE)", which was not closely tied to the other sections of this study. We shifted Sect. 4.5 to Sect. 3.4, "Extension to n-dimensional data assimilation".

(4) At the end of Sect. 3.3, we considered a related problem regarding what the error would be if the initial state is not at the scale of the forecasting operator. The results are also based on our framework with respect to scale.

(5) An extra subsection was added to Sect. 4, which was titled "Discussion". In this section, we made further comparisons between our framework and previous treatments of representation error. After a careful literature survey, we believe that the improvement from our study is a general framework of stochastic data assimilation, which was presented to investigate scale-dependent errors. First, our framework is based on nonlinear forecasting operators and observation operators. The scale transformation, which is similar to the relationship between the model space and observation space, is also nonlinear. For simplicity, we let the scale transformation obey the one-dimensional rule (See Sect. 3.1), which is equal to a linear change in scale. However, our framework does not exclude nonlinear scale transformation, and one can track a more complicated integral path of the scale for this purpose. Second, our framework permits general Gaussian representation error by introducing a non-constant volatility function in Ito-formed states and observations. Finally, we further considered the heterogeneity of geophysical parameters. This improvement highly depended on the first two advantages, i.e., either the nonlinear scale transformation or general Gaussian representation error.

Please find the detailed information in the revised manuscript.

2. Language problem

"The language is also not standard."

Response: Indeed, we are non-native English speakers. Therefore, our manuscript was completely re-edited by a professional native English speaking team. We tried our best to provide a high-quality English expression manuscript that could be understood by average readers in data assimilation.

We updated the data assimilation terminologies by referring to some classic literature, such as

- Talagrand, O.: Assimilation of observations, an introduction, J. Meteorol. Soc. JPN, 75, 191-209, 1997.

- Ide, K., Ghil, M., Lorenc, A. C.: Unified Notation for Data Assimilation: Operational, Sequential and Variational, J. Meteorol. Soc. JPN, 75, 181-189, 1997.

The references that were provided by Prof. van Leeuwen were used to check the language. Correspondingly, some notations were changed as follows:

Previous notation	Current notation
Observation/Measurement region	Observation footprint
Model units	Model unit
System state	State
dynamic model (operator), physical model (operator)	Forecasting model (operator)
System state space	Model space
Instrument error	Measurement error

Additionally, some notations from measure theory and stochastic calculus, as well as the new notations that were defined in our study may seem strange to ordinary readers in data assimilation. These notations were all listed and provided corresponding explanations in Sect. 5, "Notation", in the revised manuscript.

3. Innovation of this study

"There are several recent works that describe the filtering problem for stochastic processes, and I'm not sure the results of the authors are new in this respect."

Response: Indeed, abundant literature exists on data assimilation based on stochastic processes. However, our study is very different from the available works and deduced some remarkable results. The previous works were mostly based on stochastic processes with respect to time. Our study, which introduced measure theory to investigate the natural structure of scale and treated scale variations similarly to time variations, constructed a framework that was based on stochastic calculus with respect to scale. Some components of representation error are associated with scale, so our study provided a more reasonable framework that can further improve our understanding of the structure of representation error in data assimilation.

4. Necessity of introducing measure theory

"I don't see the need for the measure theory developed here."

Response: The scale in representation error is important, but a rigorous mathematical definition of scale is lacking. Simply regarding the resolution as the scale is not reasonable. For example, if two different observation footprints exist, one is a disc field with a diameter of 1 km and the other is a square field with side length of 1 km. Obviously, they have the same resolutions, but their scale is different. We introduced measure theory to further mathematically describe the geometric characteristics of an observation footprint or model unit, as well as the scale transformation. Measure theory laid the foundation of this study in terms of scale-dependent error in data assimilation.

5. Advantages of our study

"My suggestion is that the authors consider these references and reconsider their findings in light of those works. What has been missed by the examples of papers referred above? Why is the new framework needed, using language that the average NPG reader interested in data assimilation can understand?"

Response: The advantages of our study include the general framework of stochastic data assimilation to investigate the scale-dependent errors, i.e., a framework that includes the nonlinear operators and nonlinear scale transformation (such as the relationship from model space to observation space), the general Gaussian representation error, and considering the heterogeneity of geophysical parameters. Although some of these factors were simplified to linear situations for more clear results, nonlinear results could also be deduced from this framework.

Please find the detailed information in the general reply of this response letter and the revised manuscript (main contents in Sect. 1 and Sect. 4.2).