

Responses to Referee 1

We thank the anonymous Referee 1 for taking his/her valuable time to review our manuscript and provide us some very thoughtful and constructive comments. We apologize for our first response letter, which now seems inadequate. We had to update our reply after thoroughly changing this manuscript, especially the introduction of measure theory and definitions of scale and scale transformation. Here are the new point-by-point responses to the comments from Referee 1 (please forgive us for not marking revisions in the revised manuscript; many changes were made, and marking all revisions would make the paper a mess and difficult to read).

1. General reply

In the original version, we only introduced the concept of measure and developed a vector-valued measure to define “scale” (see Sect. 2 in the original manuscript). However, this new measure did not exhibit enough rigorousness. After careful consideration, we decided to abandon the original idea and completely change the definition of “scale” by further introducing Lebesgue measure for the following reasons:

- (1) Lebesgue measure is generally accepted. Its definition is sound and its geometric meaning is similar with scale. Therefore, Lebesgue measure can be potentially applied in mathematical formalism of scale (in the original manuscript, we attempted to develop a measure that belonged to Earth observations and simulations, but the result was not better than that obtained using Lebesgue measure).
- (2) Demonstrating scale transformation is important in addition to the scale. This notion coincides with the concept of change of variable in the Lebesgue integral (in the revised manuscript, this concept was called Lebesgue integration by substitution). In the original manuscript (see Sect. 2.2), we implied the concept of scale transformation with abundant discussion. However, we introduced this concept in the revised manuscript (see Sect. 3.1) following Lebesgue measure for simplicity (see Sect. 3.1 in the revised manuscript).
- (3) Based on the original idea, the explanation and instances of this new measure contained abundant content, which made our presentation redundant and hard to understand. Therefore, we thoroughly

modified this content by introducing Lebesgue measure and other associated concepts. Being similar to our study in terms of scale and scale transformation, this content could provide a more concise and rigorous presentation. Moreover, the length of this article was reduced because no extra explanation was needed.

Changes in the manuscript: Correspondingly, the paper was completely rewritten. The title of the revised manuscript was rewritten as “Formulation of Scale Transformation in a Stochastic Data Assimilation Framework”. We made this significant modification for the following reasons. First, defining the scale and scale transformation laid a foundation for our study and makes our work distinct from the previous studies. Second, the original title was insufficient because we did not reformulate the framework of a stochastic data assimilation, which was used only to investigate the expression of errors that were determined by scale transformation. Therefore, the new title is more suitable.

Sect. 1 was reorganized and Sect. 2 was retitled as “Basic knowledge”, which mainly introduced the basic concepts and theorems of measure theory and stochastic calculus. Sect. 3 was retitled as “Reformulation of scale transformation in a data assimilation framework”, where we first defined some essential concepts, such as the scale, scale transformation and variables. Then, we established a Bayesian description of data assimilation with time- and scale-dependent stochastic processes and formulated the effect of scale transformations on the posterior probability of the state.

In Sect. 2.1, which was retitled “Basic knowledge of measure theory”, we introduced some basic concepts such as σ -algebra, measure, measure space, Lebesgue measure, Lebesgue integral, and so on. Two main references were used: “Billingsley, P.: Probability and Measure, 2nd ed., John Wiley & Sons, New York, 1986.”, and “Bartle, R. G.: The Elements of Integration and Lebesgue Measure, Wiley, New York, 1995.” The latter might be the book that Referee 1 recommended in the interactive comment. Indeed, some terminological incongruences exist between these two books, so we tried our best to make the exposition acceptable and explicit.

In Sect. 3.1, which was retitled “Definition of scale”, we mainly developed the structures of “scale” and “scale transformation” by Lebesgue measure. Scale is logically similar to Lebesgue measure and some technicalities were also included in the previous section, so this section is more concise than that in the original manuscript. In addition, the revised definition of scale is also valid for the following sections of our study.

Please find the detailed information in the revised manuscript.

2. Language problem

“Unfortunately, the exposition is extremely hard to follow, and, after reading the article through carefully, I still don’t understand what the authors are doing. Part of the problem is probably the language. The manuscript would profit considerably if the authors could find a sympathetic native English speaker to read it over.”

Response: Because we are non-native English speakers, our manuscript was thoroughly edited by a manuscript service company, American Journal Experts, before submission (Certificate Verification Key: 53B9-681E-0C9B-BC77-FF4C). However, the exposition could be improved further. The revised manuscript was totally rewritten and was re-edited by a professional native English speaking team. We ensure that we will provide a modified version of this manuscript with higher-quality English expression in the future.

3. Reference problems

“More importantly, I find much of the exposition puzzling. I don’t know the book by Billingsley. In my day students in the USA learned measure theory from the texts by Bartle and Royden, and, relative to my background, much of the material is written very unconventionally.”

“The Bayesian expression of DA in terms of the stochastic calculus appears in many places. The authors should consult the volume by Jazwinski and the recent work of P. J. van Leeuwen and M. Bocquet.”

Response: The literature by Billingsley, Bartle and Royden are all classic works of measure theory (the texts that you mentioned may be *The Elements of Integration and Lebesgue Measure* by Bartle R. G. and *Real analysis* by Royden H. L.) and were all frequently cited according to the search results of Google Scholar. In addition, we received help from the volume “Stochastic Processes and Filtering Theory” by Jazwinski A. H. (1970) during our study, which will be listed in ‘References’ in the revised manuscript. We also appreciate the recommended literature regarding Bayesian data assimilation in terms of

stochastic calculus. This literature introduced the latest advances in this research field, and the related papers will be cited in our present and future works.

4. Why measures were defined as a vector-valued set function in the manuscript

“The usual intuition for the concept of measure is that measure is a generalization of the concepts of length, area and volume, and is thus a scalar valued set function. The authors’ response at the end of section 2.1 to Prof. Talagrand’s comment is inadequate. There is nothing intrinsically wrong with choosing a vector valued measure, but that choice requires more explanation than simply “the measure correspondingly turns to ...” The authors should explain why they want to define measure as a vector valued set function, rather than simply defining the measure of a rectangle in Euclidean space as its area. Again, maybe Billingsley defines it differently, but the Lebesgue measure of a rectangle is its area, not a vector whose components are the lengths of its sides as the authors assert on line 16 of page 6.”

Response: Dr. Talagrand also stressed this problem. In the revised manuscript, we completely changed the definition of scale by introducing Lebesgue measure. Detailed information can be found in “General reply” in this response and in the revised manuscript (main contents are in Sect. 2.1 and Sect. 3.1).

5. Definition of scale

“I do not understand the definition of scale. First, measure is a function whose domain is the sigma field F , as noted at the very beginning of section 2.1. The integrals in the definition of scale, line 16, page 7, don’t make sense to me. A_0 is a specific set. (Pardon my TeX, I don’t know how to make superscripts, subscripts or special characters). In the analysis texts I learned from, $\mu(A_0)$ is the area of the set A_0 . I don’t understand the expression $\mu(A_0)dA_0$. I cannot make sense of the second integral. The domain of the measure μ is the σ -field F , as noted in the beginning of section 2.1. The Lebesgue measure is not a point function”

Response: In the revised manuscript, we completely changed the definition of scale by introducing Lebesgue measure. The scale and scale transformation were formulated by Lebesgue integral and change of variable in the Lebesgue integral. Detailed information can be found in “General reply” in this response letter and in the revised manuscript (main contents are in Sect. 2.1 and Sect. 3.1).

6. Regarding Figure 1

“It would help a great deal if there were more explanation of figure 1. In particular, after reading and rereading the last paragraph on page 8, I can’t understand how C_2 can have the same measure as C_1 and C_3 , and D_1 has the same measure as D_2 , though they have the same “scale.” The problem may be the terminology: As I recall my long-ago analysis classes, the common intuition for measure was that measure corresponds to area, and, in particular, the Lebesgue measure of a geometrical figure in the plane is its area.”

Response: We feel very sorry because the description of this figure was not enough to help you understand. As stated in the above response, we defined scale by Lebesgue measure. Thus, the explanation of Figure 1 was changed accordingly. The conclusion is more reasonable and concise because it was established by the definition of Lebesgue measure.

Changes in the manuscript: The paragraph before Figure 1 was updated as follows:

“Figure 1 shows the relationship between the Lebesgue measure and scale. The measure space $\Omega = [x: 0 \leq x_k \leq 4, k = 1, 2]$ is regularly divided by the unit interval A_0 . Let $m_{C_1}^2$, $m_{C_2}^2$ and $m_{C_3}^2$ be the Lebesgue measures of disc measurements C_1 , C_2 and C_3 , respectively, and let $m_{D_1}^2$ and $m_{D_2}^2$ be the Lebesgue measures of diamond measurements D_1 and D_2 . Then, $m_{C_1}^2 = m_{C_2}^2 = m_{C_3}^2$ because they are the same function. That is, if $\{A_i\}$ is the set with the smallest volume that covers C_1 , then similar sets $\{A_i + 2\}$ and $\{A_i \times 3 + 2\}$ can be used (with the origin located in the upper-left corner) to cover C_3 and C_2 with the smallest volumes, respectively. Here, $A_i + 2 = [x: x_k + 2, x_k \in A_i, k = 1, 2]$ and $A_i \times 3 + 2 = [x: x_k \times 3 + 2, x_k \in A_i, k = 1, 2]$, which proves that $m_{C_1}^2$, $m_{C_2}^2$ and $m_{C_3}^2$ collect the desirable set based on the same scheme, so they are identical. Additionally, $\sum I^2(A_i \times 3 + 2)$ is much larger than $\sum I^2(A_i)$. Therefore, the scale of C_2 is not equal to the two other scales because the volumes of their

subsets are different. However, their scales are governed by one-dimensional rules because their measures are identical and the Jacobian matrices between them are diagonal. Similarly, $m_{D1}^2 = m_{D2}^2$; although their scales are different, they obey a one-dimensional rule.”

7. Inconsistency problems

“Finally, the manuscript seems inconsistent with itself. As examples, consider the abstract. “...measure theory was used to propose [a definition of] spatial scale ...[and the] Jacobian matrix [was used] to describe the change of scale. The Jacobian matrix is introduced on page 7 in the well known change of variables formula, change of scale by the Jacobian matrix is defined on page 8, and the Jacobian is not mentioned again until the summary. No further discussion of the effects of change of scale appears. Again, in the abstract, “...the variation range of this type of error is proportional to the scale gap, ...” I’m sure I’m not the only reader for whom the phrase “scale gap” conjures up ideas of inertial range from turbulence theory and similar notions. The term “scale gap” is never mentioned in the body of the article.”

Response: You mentioned two inconsistency problems. The first is why the Jacobian matrix and the change in scale (the former was included in the introduction of the change of variable in the Lebesgue integral, and the latter was renamed as “scale transformation”) disappeared after Sect. 2. Actually, these concepts were not omitted but simplified by the one-dimensional rule (defined in Sect. 3.1 and Eq. (5)) to suit stochastic calculus. Then, we used this simplified version of scale changes to investigate the uncertainties in data assimilation (we also mentioned this problem in the second paragraph of Sect. 5 in the original manuscript). Although the one-dimensional transformation maybe the simplest case, the results were still complicated and some new components of uncertainty were discovered (Eq. 21~27). However, fully studying the scale transformation using Jacobian matrices is comprehensive and universal, and it will be launched in our following work rather than in this study. The second inconsistency is the phrase “scale gap” in the abstract. We apologize for the trouble with this word. Here, the term “scale gap” refers to the quadratic variation between s_x and s_y (see Eq. 25~27). However, fully explaining this term in the abstract was difficult, so we had to use the term “scale gap”. Replacing this term with “the difference between scales” may be more practical.