Reply to referee's comments

22 September 2016

General comments

We thank all reviewers for their comments, leading to an improvement of our manuscript. We collected some general comments, since some issues were addressed by all reviewers.

Restructuring of the manuscript

We restructured the manuscript in the following way:

- We generally revised the text to avoid misunderstanding, following many suggestions of the reviewers.
- We have shortened the derivation of the model and put the details into appendix B.
- We expanded the analysis part as well as the description of the analysis tools.
- We also expanded our comparison of the model results with observations in order to address the topic "subvisible cirrus clouds" adequately.

Dry adiabatic motion, latent heat release, temperature/velocity regime

If latent heat release was taken into account for the rate of RH_i, the temperature equation (18) would read

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}T}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}t} + \left.\frac{\mathrm{d}T}{\mathrm{d}t}\right|_{\mathrm{latheat}} = -\frac{g}{cp}w + \frac{L_{ice}}{M_{air}c_p}\cdot\left.\frac{\mathrm{d}q_c}{\mathrm{d}t}\right|_{\mathrm{growth}}, \tag{1}$$

and the latent heat term would need to be included in the rate equation for RH_i:

$$\frac{\mathrm{d}RH_i}{\mathrm{d}t} = \underbrace{e\cdot w\cdot RH_i}_{\mathrm{adiabatic}} \quad - \left.\frac{\mathrm{d}q_c}{\mathrm{d}t}\right|_{\mathrm{growth}}\left(\underbrace{\frac{p}{\varepsilon p_{si}(T)}\cdot 100\%}_{\mathrm{growth}} - \underbrace{RH_i\frac{L_{ice}^2}{RT^2M_{air}c_p}}_{\text{latent heat}}\right). \tag{2}$$

The contribution to temperature change from latent heat is obviously only important, when there is substantial diffusional growth of ice particles, i.e. when $\frac{\mathrm{d}q_c}{\mathrm{d}t}|_{\mathrm{growth}}$ is considerably large. In that case however, the latent heat term directly competes with the growth term. The growth term is usually at least one order of magnitude larger than the latent heat term. Hence, whenever latent heat release comes into play for temperature changes, the rate of change of RH_i is dominated by the water vapour sink due to growth, anyway. Therefore, we omit latent heating in the RH_i equation and stick with the constant adiabatic forcing term. Including latent heating would produce an additional highly nonlinear term (it contains RH_i as a factor), which we would like to avoid.
In terms of lapse rates we can state that due to phase changes a "moist" (or better "ice") adiabatic lapse rate might be more appropriate. However, since we investigate cirrus clouds in the low temperature regime $T < 235\,\mathrm{K}$, the difference between moist adiabatic lapse rate and dry adiabatic lapse rate is less than 5% and decreases with decreasing temperature. Thus, we can approximate the temperature change by adiabatic lapse rate and omit the additional nonlinear term. This is also noted in the text.
We also describe the relevant regime for SVCs, i.e. low temperatures $T < 235\,\mathrm{K}$ and slow vertical updraughts $w < 0.05\,\mathrm{m\,s^{-1}}$ in the first part of the manuscript.

Terminology

We changed the term "(positive) attractor" to "(positive) point attractor" or "stable focus" for clarity. We use the term "critical point" according to Verhulst [1996], which is equivalent to "steady state" or "equilibrium point". We add some information that the limit cycle is a one-dimensional attractor.

Response to Peter Nevier

Boltzmann equation The evolution equation of the size distribution has a form that is similar to the Boltzmann equation (e.g. in gas dynamics). However, since we have no aggregation terms on the right hand side, we follow the suggestions and omit this name.

Eulerian vs. Lagrangian description Since we are interested in the time evolution of an air parcel, the change from Eulerian to Lagrangian description seems reasonable. We added more details for the transformation of the equations; especially, we describe the reformulation of the evolution equation in advective form, using mass conservation of air (appendix A).

ODE system We write the equation in a simpler way, using constants in order to represent the essential non-linear features of the equations. We added a paragraph for a qualitative description of the different terms (non-linear/linear, etc.) and calculated the divergence for determining the quality of the system (externally forced dissipative system).

Spelling We corrected all typos, especially the names of the physicists and mathematicians Euler and Lagrange.

Response to reviewer #1

Major comments

L252–256 See general comments.

L363–364 As it is a limit cycle, it takes infinitely long for the orbit to reach it. However, in the numerical solution, the oscillation amplitude stays approximately constant after several periods. This is apparent from the results from the Poincaré sections. The sequence of intersection points reaches an accumulation point after ~ 50 cycles. From then on, the points only fluctuate minimally about the asymptotic limit, with no long term trend.

L377–380 We thank the reviewer for the careful recalculation. Equation (42) was indeed wrong. The error occurred when the equation was rearranged for better readability and transferred to LaTeX-Code. In the actual calculations for the analysis, the correct equation was used. The correct equation is now provided (eq. 28) as well as a short derivation in appendix C. The equation was solved using Newton's method to obtain the critical point for different parameter values (w,T), and the corresponding Jacobians and eigenvalues thereof.

L384–406 The dependence of the eigenvalues on w and T is shown in figures 3 and 4.

L405–413 The paragraph describing the construction of the Poincaré section is now improved and accompanied by a more detailed explanation in appendix D. In fact, the numerical results are needed for the determination of the limit cycle. The periods of the limit cycle are shown in figure 11.

Minor comments

Comments regarding wording are not specifically addressed, since we rewrote the manuscript considering these suggestions.

L11 See General comments.

L24 We changed the text in this respect. In fact, the exact net effect of liquid clouds is not known yet. However, it is known that the albedo effect is usually dominant, thus liquid clouds usually have a cooling impact on Earth's energy budget. We cite now the latest IPCC report for this qualitative behaviour of warm clouds.

L28 In the revised version, the low temperature regime is now mentioned in the first sentence of the introduction.

L41 We added more text, to make clear that they warm the Earth-Atmosphere system.

L50 Sedimentation is important for large crystals. In this work it plays a vital roll for the described mechanism. In the revised version, this is explained in several parts of the manuscript, e.g. in the first paragraph of section 3.1. In fact, the oscillations can only take place because of sedimentation; otherwise, the system would fast reach an equilibrium state.

L79 In the revised version, ∇_x is defined in the passage after equation 1.

L83 We skip this term since it leads to confusion. We investigate an air parcel, i.e. follow the time evolution of the parcel in a Lagrangian way; sedimentation leads to motion of ice crystals relative to the motion of the air parcel.

L90 Splitting up the velocity into different components allows for separation of the sedimentation term in (new) equation 3 and for changing to the Lagrangian perspective.

L94 and L101 For the microphysical parameterisations, non-integer moments arise from integrating power-law relationships with non-integer exponents, as used for mass-length-relations or representations of terminal velocities.

L113 A reference for the two-moment scheme approach is now provided Seifert and Beheng [2006].

L115 J does not depend on r and is therefore just a factor in the integration w.r.t. r. In the revised version, we left out the sentence because stating that "$\partial J/\partial r = 0$" is sufficient.

L159 The geometric standard deviation of the lognormal distribution is dimensionless, it represents the width of the distribution. This has been clarified in the text.

L158–161 We added a reference for the choice of the settings for solution droplet distribution, motivated by observations.

L208 We use a simple power law for representing the columnar shape of the ice crystals in a simple way. For this purpose we "fit" this power law (coefficients C_i and α_i) to a more sophisticated relationship from the Spichtinger and Gierens [2009] model.

L216 We define $S_i = p_v/p_{si} = \mathrm{RH}_i/100\,\%$. Since $q_v/q_{v,si} \approx p_v/p_{si}$, both definitions are almost the same.

L246 See General comments.

L261 We now adjusted the spaces between the units.

L262 For studying the long-term behaviour of SVCs, we have to assure that constant vertical motion is persistent. Slow vertical upward motions $w < 0.05\,\mathrm{m\,s^{-1}}$ can be maintained for quite a long time; examples are motions along warm fronts in the extra-tropics or Kelvin waves in the tropics, leading to almost constant vertical velocities over long time. We provided references for these situation. However, the temperature and pressure change in such situations is quite small; this provides a justification for the assumption of constant temperature and pressure, whereas temperature and pressure changes are used in the evolution equation of relative humidity over ice.

L270-271 The partial derivative term with $\partial/\partial z$ is a hyperbolic term. See also section 2.2.4, point 3.

L292–294 We rewrote the section and added some more details.

L296 We referenced the wrong equation, it was supposed to be (37), now (19).

L299-300 The term $\mathrm{DEP_{RH}}$ is wrong, it has to be $\mathrm{DEP_q}$, since

$$\left.\frac{\mathrm{d}q_c}{\mathrm{d}t}\right|_{Dep} = \mathrm{DEP_q}. \tag{3}$$

The factor ρ was also wrong. However, we changed notation so the terms are not called NUC, DEP, SED anymore. Also, we got rid of ρ in (new) equation (3) using the continuity equation, to avoid confusion.

L315–318 In the following sections, the abbreviation $\mathbf{F}$ is used for the right hand side of the system, therefore we decided to already introduce it here.

L337 Yes, we mean that relative humidity increases again. Sedimentation removes ice crystals, which constitute a sink for relative humidity due to growth.

L340 State 1 only occurs if either temperature is "rather high" (i.e. right side of the interval $190 \leq T \leq 230\,\mathrm{K}$) or the vertical motion is slow (i.e. left side of the interval $0 < w \leq 0.05\,\mathrm{m\,s^{-1}}$). For the qualitative overview we stay at these vague statements, in the bifurcation diagram the quantitative values are given.

L392 See General comments. The critical point is a stable focus, i.e. a positive (point) attractor.

L422 The critical point was found by first computing the roots wrt RH_i of (new) equation (28), former equation (42). From that, the values for N_c and q_c were derived analytically using (new) equations (C3) and (C4). See appendix C.

L439–441 In the revised version, we state that the bifurcation point is a function of w and T.

L455 Mean sizes are calculated from the mass-size relationship $L = C_i m^{\alpha_i}$, as provided in appendix B. Here, we use the mean mass, as given by $\overline{m} = q_c/N_c$.

L462–464 More details on the Spichtinger and Gierens [2009] model are now given in the first paragraph of section 3.6.

L487 Here we referred to comparison with the model by Spichtinger and Gierens [2009]. We clarified this in the new text.

L502 and L516 "Theoretical" refers to investigations with our simple "analytical" model.

L537 Crystal sizes in this work are to be interpreted as crystal length.

L610 A more detailed outlook on how minimal models could be useful for cloud parameterisations is given in the conclusions of the revised version.

Figure 7 The median is now indicated by a dotted-dashed line.

Figure 9 The correct unit is m s^{-1}.

Response to Ulrike Wacker (reviewer #2)

Theory of dynamical systems

- We reduced some text about description of SVCs. We added text on the mathematical methods and the conceptual model. We included all indicated references.
- We simplified the system introducing summarising coefficients. The relative humidity is not changed by nucleation, since this constitutes a phase transition from liquid to solid, whereas the gas phase is not changed.

  The behaviour of pure ice clouds is completely different than for mixed-phase or pure liquid clouds, which usually exist in a thermodynamic state close to water saturation. Ice clouds are usually not in thermodynamic equilibrium since ice nucleation takes place at high supersaturations and diffusional growth/evaporation is quite slow. Therefore, relative humidity is an important control variable, which is more natural for ice clouds than specific humidity. Since the system does not fulfil mass conservation, there is no additional benefit from changing the variable from RH_i to q_v but the formulation of the nucleation rate would be more complicated. Therefore we decided not to transform the system into different coordinates.

- We added a paragraph describing the quality of the different terms in the ODE system.
- We expanded the discussion of the mathematical analysis, including figures of eigenvalues etc. .

Terminology We used many terms from theory of dynamical systems from the books by Verhulst [1996] and Argyris et al. [2010]. We tried to change and explain the terms in different ways.

Dry adiabatic lapse rate See general comment.

Derivation of system of equations We have shortened the derivation of equations; especially the description of the cloud processes is now partly transferred to appendix B.

Dimensions We checked all terms again and corrected the inconsistencies.

Minor comments

- We skip the term “Boltzmann equation”.
- Equation (9): corrected
- Since the nucleation rate as described by Koop et al. [2000] is formulated as a volume rate, we used this approach. The nucleation rate does not depend on the size of droplets.
- We deleted the term “aerosol particles”.
- In contrast to liquid droplet formation the activity is not equal 1 for ice nucleation. The nucleation rate J as described by Koop et al. [2000] can be formulated in terms of differences in water activity (appendix B).
- We changed the layout of equation (16).
- Equation (17) was wrong, now corrected.
- “Length” indicates the length of an ice crystal, i.e. the long side of a columnar-shaped ice crystal.

- Droxtal is the term for very small and almost spherical ice crystals.
- We corrected (former) equation (24)
- We corrected (former) equation (30)
- We use latent heat and heat capacity in molar units, i.e. L_{ice} is actually the molar heat of sublimation; in addition with the molar mass M_{air} and the molar isobaric heat capacity c_p the units in the equations are correct. Please note that we also corrected the equation for adiabatic temperature change.
- We moved the respective text.
- Yes, equation (37) is the correct reference.
- N_a is the number concentration of solution droplets per mass dry air. We skip n_a for clarification.
- We corrected the equation.
- We do not apply $w < 0$ in our model, only upward vertical motions are considered.
- This is correct, the nucleation rate J is positive for $RH_i > 100\%$. Due to the exponential behaviour of J no significant amount of ice crystals is formed at low supersaturations. Only if the supersaturation exceeds certain values (i.e. a "threshold") then a significant amount of ice crystals is produced. We added some text and a reference about the concept of freezing probability and its description by a differential equation, including the nucleation rate.
- We added more text about the use of the Poincaré section for the numerical determination of the limit cycle.
- We rewrote the complete section 3.
- l. 433: Decreased growth rates lead to slower reduction of supersaturation, thus RH_i stays longer above the "freezing thresholds" and more ice crystals can be produced.
- We changed the text from "exponential behaviour" to "power law".
- We restructured section 3.
- We added some more text for describing the features of the models by Spichtinger and Gierens [2009] and Kärcher [2002].
- "Analytical model" refers to the model developed in this study.

- Within the first 4 hours of integrations the oscillation is damped and the amplitude in variables decreases. later, the amplitude of the variables increases again. Admittedly, the system does not reach a "constant" limit cycle, since the properties of the limit cycle are continuously changed by cooling of the temperature. We added some text for clarification.

- We added some text about sources, sinks and forcing terms in the beginning of section 3.

- The microphysical properties of the attractors can be similar, since in the limit cycle case the variables changes within a quite large range that includes also the values of the point attractor.

- We arrange the figure captions in a way that they might be understandable without reading the whole text. Thus, some repetition might be possible.

- The dependence of the mean mass on w is only marginal. We added some text in the manuscript to clarify this issue.

Response to reviewer #3

Discussion of SVCs We enhanced the discussion of SVCs; in fact, we added comparison with remote sensing observations and report more qualitative results of the simulations.

How realistic are model simulations?

- Both states can develop, if previous heterogeneous nucleation takes place. In fact, the system will just start at a different point in phase space.

- We have not investigated the system in details for variations in vertical velocity. In fact, it seems from numerical simulations that for weak perturbations, the system approaches similar states as in the undisturbed scenarios. Theoretical analysis would be much more difficult, since we have no longer an autonomous system. The investigation of this system, externally forced by time-dependent vertical velocities is beyond the scope of our study and is dedicated to future work.

Specific comments

1. Abstract: We changed the text following reviewer's suggestions.

2. Introduction: We skip the word "usually".

3. Introduction: We added more basic references about heterogeneous nucleation. We also added the suggested references by Immler et al., 2008.

4. Section 2, additional settings: We rewrote section 2, changing also the titles of the subsections.

5. We added some text about the change in $n_c = N_c \cdot \rho$ due to temperature/pressure change.
6. We defined ODE in the text.
7. We rewrote the sentences.
8. We added some description of the attractor states, which occur also in the definitions.
9. We added the processes.
10. We changed the text.
11. Former figure 5 (now figure 7) is now described in more details.
12. We changed the wording.
13. We discussed former figure 6 (now figure 8) in more details. The mean mass is shown in figure 10, thus we do not add another panel to figure 6. We added comparison with remote sensing observations considering the issue of subvisible cirrus (or not).
14. In former figure 6 the stable or unstable point attractor is shown. The lines, both solid and dashed, are valid at the respective critical point of the system. The critical point is well defined for both states, it only has different implications, depending on the nature of the state. For the point attractor regime (solid lines), it is a stable focus and the long-term solution approaches it. For the limit cycle regime (dashed lines), the critical point is approximately at the centre of the periodic orbit and provides a good estimate of the mean (or median) properties of the cloud, in terms of mass and number concentration and relative humidity. So the parallel lines in former figure 6, now figure 8, are valid for the critical point which is either a stable focus (solid lines) or an unstable focus (dashed lines).
15. We tried to keep the same wording.
16. We added some text, see also reply to reviewer #2.

Comments to figures

1. We changed the caption.
2. We changed the caption.
3. We changed the figure.
4. It is not possible to add other units, since we refer to different temperature, and thus density, regimes. However, for figure 9 showing the number concentrations for point attractor and limit cycle we added units per litre.

5. Figure 7 (now figure 9): The solid line represents values at the critical point, regardless of the nature of the critical point. It is well defined for both states, see also comment 14.

6. We added some text and we changed the line for the median in the figure. However, the median in the limit cycle case is close to the value at the critical point in the limit cycle regime, not the point attractor. See also comment 14 and comment to figure 6. Limit cycle and point attractor do not coexist, as the environmental conditions determine which state is present.

References

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