General responses:

I would like to thank the reviewers and Editor for their valuable comments. One of the major concerns raised by both reviewers is how new modes were selected to derive the 6DLM. Here, I would like to emphasize (1) that based on the analysis of the Jacobian term, $J(\psi, \theta)$, new modes are selected to extend the nonlinear feedback loop that can provide additional nonlinear feedback to stabilize or destabilize solutions; and (2) that our approach, using incremental changes in the number of Fourier modes, is to help trace their individual and/or collective impact on the solution stability as well as the extension of the nonlinear feedback loop. To facilitate discussions, we have (a) created two tables which list the Fourier models used to construct different higher-order Lorenz models and the corresponding critical values of the normalized Rayleigh parameter for the onset of chaos; and (b) finished a pdf file with a brief summary on the mathematical analysis of the nonlinear feedback loop in the 3DLM and its extension in the 5DLM and 6DLM. The tables are included in the end of this response file, while the pdf file will be uploaded separately as supplemental materials. In the following, specific responses are given with the aid of the supplemental materials.

(A) Responses to Reviewer I's comments:

I carefully read the paper several times. The principle question is: Why was this paper written, in principle, and what basic problem[s] is [are] discussed here? The author attempts to discuss a problem of stability of an expanded Lorenz model through the Lyapunov exponent analysis. There are no grammatical errors in the paper, except small ones (e.g., "Model" in capitals in the title, the capture for Figure 7). However there are several problems which should be discussed before the paper is considered for publication

Thanks for your comments. The minor issues have been corrected in the revised manuscript accordingly. Detailed responses to the specific comments are given below.

In this study, the hypothesis to be verified and question to be addressed is stated in the introduction as follows:

``*Shen14 hypothesized that system's stability in the LMs, with a finite number of modes, can be improved with additional modes that provide negative nonlinear feedback associated with additional dissipative terms.*''

`` *However, since new modes can also introduce additional heating term(s), the competing role of the heating term(s) with nonlinear terms and/or with dissipative terms deserves to be examined so that the conditions under which solutions become more stable or chaotic can be better understood.*''

`` *Results obtained from work described here and the work of Shen (2014a) are used to address the following question: for generalized LMs, under which conditions can the increased degree of nonlinearity improve solution stability?*''

In fact, our studies have been performed to help achieve the ultimate goal of determining under which conditions increasing resolutions can improve the predictions in weather/climate models. In our first papers (Shen 2014a and this manuscript), we found that a nonlinear feedback loop in the baseline model (e.g., 3DLM) plays an important role in determining the predictability and its extension may help provide negative nonlinear feedback to improve the predictability. After identifying the impacts of new modes in the 5DLM (which contains the negative nonlinear feedback) and in the 6DLM (which including an additional heating term), we currently extend these studies to examine the role of the extended nonlinear feedback loop and additional heating terms in the solution's stability for 7D, 8D and 9D LMs (Yoo and Shen, 2015, in preparation).

First, I suppose that the model, which was used for the analysis, should be in the form….   For simplicity, coefficients in Eqs (1)-(8) have been omitted to understand the structure of this system. The author used another model. Why? How that model was obtained? It is necessary to explain how that model corresponds to Eqs (1)-(8).

In the following, I will use Table 1 (derived from Roy and Musielak, 2007) and Table 2 to show that our 6DLM is the same as the one in Kennamer (1995), which is not a subset of the the aforementioned 8DLM. More importantly, we want to point out that higher-order LMs display the dependence of rc on the selections of Fourier modes (e.g., Table 1), and that proper selections of Fourier modes, based on the analysis of Jacobian term $J(\Psi, \theta)$, can help extend the nonlinear feedback loop that can provide negative nonlinear feedback to stabilize solutions (e.g., 5D, 6D and 7D LMs).

Table 1, which is included near the end of this response file, provides a list of high-order Lorenz models, including two 6D LMs and 8DLM, and the corresponding Fourier modes used to construct the LMs. It is shown that the 6DLM by Kennamer is not a subset of the 8DLM. As mentioned in the manuscript, our 6DLM is the same as the one from Prof. Musielak's group. [Prof. Musielak is Kennamer's advisor. Kennamer published the 6DLM in his/her master thesis, which is not available to the author. The first literature listing the 6DLM of Kennamer is Musielak, Musielak and Kenamer, 2005, which was cited in the manuscript.]. Specifically, the M4, M5 and M6 modes in our 6DLM are exactly the same as $\psi_1(1,3)$, $\Theta_2(1,3)$ and $\Theta_2(0,4)$ in the 6DLM by Kennamer, respectively. However, we derived the 6DLM independently. In addition, we provided the analysis of the Jacobian term, $J(\Psi, \theta)$, to show how the 6DLM is constructed to be an "extension" of the 5DLM. Namely, the former is a superset of the latter. In the supplementary materials, we provide more detailed discussions on the linkage between the 3DLM, 5DLM and 6DLM. In the manuscript, we discuss the impact of additional heating term on solution's stability and the conservation laws for the 6DLM in the dissipationless limit, the latter of which were only partially discussed by Prof. Musielak's group.

Table 2 lists the Fourier modes used to construct our higher-order LMs (e.g., Shen 2014a, 2015), the 14DLM by Curry (1978) and the one by Lucarini and Fraedrich (2009). In addition, it also lists the critical value of the normalized Rayleigh parameter (rc) for the onset of chaos. In the 5DLM, we first showed that the nonlinear feedback loop can be extended through the inclusion of M5 and M6 (which are the same as $\Theta_2(1,3)$ and $\Theta_2(0,4)$, respectively). Compared to the 5DLM, the 6DLM includes an additional mode M4, (i.e., $\psi_1(1,3)$), and has a comparable rc. Currently, based on the analysis of nonlinear feedback loop, we add two modes, $\Theta_2(1,5)$ and $\Theta_2(0,6)$ to extend the nonlinear feedback loop of the 5DLM, which becomes the 7DLM with a much larger rc (e.g., rc~116.9 in Table 2). To be more consistent, additional modes with $\psi_1(1,3)$ and $\psi_1(1,5)$ are being added to derive the 8DLM and 9DLM. All of the three LMs, 7D-9DLMs, have the rc greater than 100. More detailed analyses with the eLE calculation are being performed (e.g., Yoo and Shen, 2015, in preparation).

It is worth noting that the 14DLM, which was shown to be not conservative in the disppiationless limit, is a superset of the 6DLM. However, the vertical wavenumbers in both 6DLM and 14DLM are the same and their critical values of the normalized Raleigh parameter are comparable. In contrast, our 7D-9D LMs include modes with higher wavenumbers, such as 5m and 6m in $\Theta_2(1,5)$ and $\Theta_2(0,6)$, to extend the nonlinear feedback loop, which can stabilize solutions and lead to a larger rc (e.g., Table 2).

> Second, independently from the used model the principle problem for systems like (1)-(8) is not their stability but how different dynamical regimes are realized in such a model. For example, how the regime changes for increasing Ra, where Ra is Rayleigh number. In general, system (1)-(8) was early studied by numerical methods and it has been demonstrated that there are several interesting effects. For example, a 3D attractor does not develop because another attractor with a higher dimensionality exists.

The dependence of the solution's stability over a range of the normalized Rayleigh parameter (r) and the Prandtl number (σ) is discussed in Figure 7 of section 3.4 in the manuscript. We do observe the dependence of rc on the Prandtl number in the LMs. However, given a value of σ, the 6DLM (as well as the 5DLM) is always more predictable than the 3DLM.

From a perspective of weather/climate prediction, our ultimate goal is to apply what we can learn from the idealized high-order LMs to understanding and improving the predictability of the weather and/or climate models. Specifically, it is important to understand if and how the increased resolutions in the weather/climate models can suppress or enhance chaotic responses, because high-resolution global modeling, which is a current trend, requires tremendous computing resources. To achieve our goal, we started examining the impact of increased degree of nonlinearity on solution's stability in the 3DLM and higher-order LMs, and trying to implement suitable methods (e.g., trajectory separation method) into the weather/climate models to perform stability analysis (e.g., calculations of Lyapunov exponent). More details in our modeling approach with the higher-order LMs are provided below.

Our approach by incrementally changing the number of modes can help examine the individual and combined impact of resolved processes by the new modes. For example, the 5DLM was used to examine the impact of the negative nonlinear feedback (from the additional nonlinear and dissipative terms in association with the two new modes, M5 and M6 modes), while the 6DLM with the inclusion of M4 mode, which is a superset of 5DLM, allows us to examine the competing impact of an additional heating term and dissipative terms on the solution's stability. We then suggest that negative nonlinear feedback associated with new modes (M5 and M6) in the 6DLM as well as 5DLM can stabilize solutions, and that the additional heating term with the M4 mode in the 6DLM can destabilize solutions.

The principle problem is how to use systems like (1)-(8) and of higher dimensionality for the practical analysis. In any case, the dimensionality larger than 6-8 is required to discuss a reality.

My suggestion is that the editor should decide if this paper is suitable for publication at NPG. In my opinion, it needs a major revision before being considered for publication. In its present form, the manuscript does not fit the journal scope because NPG is not a pure mathematical journal.

Our ultimate goal is to examine the impact of increased resolution on the predictability of the real-world weather/climate models (e.g., Shen et al., 2006a). We have been working to implement the trajectory separation (TS) method into our global model for the eLE calculation. In addition, we still continue to improve our understanding of the nonlinear feedback loop in higher-dimensional LMs. For example, since the Spring semester of 2015, I have supervised one master student to derive the 7D, 8D and 9D LMs by analyzing the nonlinear Jacobian term, $J(\psi, \theta)$, and selecting new modes (M7, M8 and M9 modes in Table 2) that can extend the nonlinear feedback loop. With that being said, we believe that the related discussions on the role of different physical processes (i.e., dissipative and heating processes) in solution's stability meet the goal of the NPD journal: *(submissions that) apply nonlinear analysis methods to both models and data.*

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Table 1: Fourier modes selected to construct the 3DLM and higher-order LMs, which is from Table 1 of Roy and Musielak (2007c). The critical values of the normalized Raleigh parameter, shown in red, are derived from Table 2 of Roy and Musielak (2007c).

Table 1
Fourier modes selected to construct the original 3D Lorenz system and other generalized Lorenz systems

Model	Circulation modes	Temperature modes	Temperature modes with $m = 0$		References
3D	$\Psi_1(1,1)$	$\Theta_2(1,1)$	$\Theta_2(0,2)$	rc~24.75	Lorenz [1]
5D	$\Psi_1(1,1)$ $\Psi_1(2,1)$	$\Theta_2(1,1)$ $\Theta_2(2,1)$	$\Theta_2(0,2)$	rc~22.50	Paper II
6D	$\Psi_1(1,1)$ $\Psi_1(2,1)$ $\Psi_1(1,2)$	$\Theta_2(1,1)$ $\Theta_2(2,1)$	$\Theta_2(0,2)$	n/a	Humi [9]
6D	$\Psi_1(1,1)$ $\Psi_1(1,3)$	$\Theta_2(1,1)$ $\Theta_2(1,3)$	$\Theta_2(0,2)$ $\Theta_2(0,4)$	rc~40.15	Kennamer [10]
8D	$\Psi_1(1,1)$ $\Psi_1(2,1)$ $\Psi_1(1,2)$	$\Theta_2(1,1)$ $\Theta_2(2,1)$ $\Theta_2(1,2)$	$\Theta_2(0,2)$ $\Theta_2(0,4)$	rc~35.60	This Paper
9D	$\Psi_1(1,1)$ $\Psi_1(1,2)$ $\Psi_1(1,3)$	$\Theta_2(1,1)$ $\Theta_2(1,2)$ $\Theta_2(1,3)$	$\Theta_2(0,2)$ $\Theta_2(0,4)$ $\Theta_2(0,6)$	rc~40.50	Paper I

Table 2: Fourier modes used in our high-order LMs (e.g., Shen 2014a, 2015; Yoo and Shen, 2015) and the models by Curry (1978) and Lucarini and K. Fraedrich (2009). Note that $M_4 = \psi_1(1,3)$, $M_5 = \Theta_2(1,3)$, and $M_6 = \Theta_2(0,4)$.

model	Ψ	Θ	Θ	rc	References
5DLM	$\psi_1(1,1)$	$\Theta_2(1,1)$, $\Theta_2(1,3)$	$\Theta_2(0,2)$, $\Theta_2(0,4)$	42.9	Shen (2014)
6DLM	$\psi_1(1,1)$, $\psi_1(1,3)$,	$\Theta_2(1,1)$ $\Theta_2(1,3)$	$\Theta_2(0,2)$ $\Theta_2(0,4)$	41.1	Shen(2015)
7DLM	$\psi_1(1,1)$	$\Theta_2(1,1)$, $\Theta_2(1,3)$, $\Theta_2(1,5)$	$\Theta_2(0,2)$, $\Theta_2(0,4)$, $\Theta_2(0,6)$,	~116.9	Yoo and Shen (2015, in preparation)
8DLM	$\psi_1(1,1)$, $\psi_1(1,3)$	$\Theta_2(1,1)$, $\Theta_2(1,3)$, $\Theta_2(1,5)$	$\Theta_2(0,2)$, $\Theta_2(0,4)$, $\Theta_2(0,6)$	~105 (TBD with the eLE analysis)	Yoo and Shen (2015)
9DLM	$\psi_1(1,1)$, $\psi_1(1,3)$, $\psi_1(1,5)$	$\Theta_2(1,1)$, $\Theta_2(1,3)$, $\Theta_2(1,5)$	$\Theta_2(0,2)$, $\Theta_2(0,4)$, $\Theta_2(0,6)$	~105 (TBD with the eLE analysis)	Yoo and Shen (2015)
14DLM	$\psi_1(1,1)$, $\psi_1(1,3)$, $\psi_1(2,2)$, $\psi_1(2,4)$, $\psi_1(3,1)$, $\psi_1(3,3)$	$\Theta_2(1,1)$, $\Theta_2(1,3)$, $\Theta_2(2,2)$, $\Theta_2(2,4)$, $\Theta_2(3,1)$, $\Theta_2(3,3)$	$\Theta_2(0,2)$, $\Theta_2(0,4)$	rc~43	Curry (1978)
10EQs	$\psi_1(1,1)$ $\psi_1(2,2)$	$\Theta_2(1,1)$, $\Theta_2(2,2)$	$\Theta_2(0,2)$, $\Theta_2(0,4)$	n/a	Lucarini and K. Fraedrich (2009)

Table 3: Lorenz models with different Fourier modes. 3DLM and 5DLM are discussed in the manuscript, while the 6DLM will be discussed in a companion paper. 6DLM_HK is referred to as the 6DLM proposed by Howard and Krishnamurti (1986). 7DLM_TH and 7DLM_Hetal are referred as the 7DLMs proposed by Thiffeault and Horton (1995) and Hermiz et al. (1995), respectively. The one denoted as '8DLM (suggested)' was suggested by Thiffeault and Horton (1995) who did not derive the 8DLM nor discuss its characteristics. Only one horizontal wave number was used in the first several Lorenz models. cos(2lx) was used in the 8DLM by Roy and Musielak (2007c), denoted as 8DLM_RM. M_1-M_6 are defined in the manuscript. M_a-M_d are defined as sin(mz), cos(lx)sin(2mz), sin(lx)sin(2mz), and sin(3mz), respectively. In the studies by Howard and Krishnamurti (1986) and Hermiz et al. (1995), the symbol 'α' is equivalent to 'a' in our study, which is equal 'l/m', namely α=a=l/m.

1	3DLM --ψ	sin(lx)sin(mz)				
	θ	cos(lx)sin(mz)	sin(2mz)			
2	5DLM --ψ	sin(lx)sin(mz)				
	θ	cos(lx)sin(mz)	sin(2mz)	cos(lx)sin(3mz)	sin(4mz)	
3	6DLM --ψ	sin(lx)sin(mz)	sin(lx)sin(3mz)			
	θ	cos(lx)sin(mz)	sin(2mz)	cos(lx)sin(3mz)	sin(4mz)	
4	6DLM_HK --ψ	sin(lx)sin(mz)	sin(mz)	cos(lx)sin(2mz)		
	θ	cos(lx)sin(mz)	sin(2mz)	sin(lx)sin(2mz)		
5	7DLM_TH-- ψ	sin(lx)sin(mz)	sin(mz)	cos(lx)sin(2mz)		
	θ	cos(lx)sin(mz)	sin(2mz)	sin(lx)sin(2mz)	sin(4mz)	
6	7DLM_Hetal -- ψ	sin(lx)sin(mz)	sin(mz)	cos(lx)sin(2mz)	sin(3mz)	
	θ	cos(lx)sin(mz)	sin(2mz)	sin(lx)sin(2mz)		
7	8DLM (suggested)	sin(lx)sin(mz)	sin(mz)	cos(lx)sin(2mz)	sin(3mz)	
	θ	cos(lx)sin(mz)	sin(2mz)	sin(lx)sin(2mz)	sin(6mz)	
8	8DLM_RM---- ψ	sin(lx)sin(mz)	sin(lx)sin(2mz)			sin(2lx)sin(mz)
		cos(lx)sin(mz)	sin(2mz)	cos(lx)sin(2mz)	sin(4mz)	cos(2lx)sin(mz)

1	3DLM --ψ	M_1				
	θ	M_2	M_3			
2	5DLM --ψ	M_1				
	θ	M_2	M_3	M_5	M_6	
3	6DLM --ψ	M_1	M_4			
	θ	M_2	M_3	M_5	M_6	
4	6DLM_HK --ψ	M_1	$\mathbf{M_a}$	$\mathbf{M_b}$		
	θ	M_2	M_3	$\mathbf{M_c}$		
5	7DLM_TH-- ψ	M_1	$\mathbf{M_a}$	$\mathbf{M_b}$		
	θ	M_2	M_3	$\mathbf{M_c}$	M_6	
6	7DLM_Hetal -- ψ	M_1	$\mathbf{M_a}$	$\mathbf{M_b}$	$\mathbf{M_d}$	
	θ	M_2	M_3	$\mathbf{M_c}$		
7	8DLM (suggested)	M_1	$\mathbf{M_a}$	$\mathbf{M_b}$	$\mathbf{M_d}$	
	θ	M_2	M_3	$\mathbf{M_c}$	sin(6mz)	
8	8DLM_RM---- ψ	M_1	sin(lx)sin(2mz)			sin(2lx)sin(mz)
		M_2	M_3	cos(lx)sin(2mz)	M_6	cos(2lx)sin(mz)

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Supplemental Materials for the Paper entitled " Nonlinear feedback in the six-dimensional Lorenz model: impact of an additional heating term. By Bo-Wen Shen"

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1 Introduction

This report, which documents the mathematical analysis on the extensions of the nonlinear feedback loop in the 5DLM and 6DLM as well as higher-dimensional Lorenz models, is provided as supplementary materials to the manuscript entitled " Nonlinear feedback in the six-dimensional Lorenz model: impact of an additional heating term. by Shen (2015)." In the following, we briefly introduce the three-dimensional (3D) Lorenz model (3DLM, Lorenz, 1963) and its Fourier modes, and identify the nonlinear feedback loop of the 3DLM by analyzing the nonlinear Jacobian term $J(\psi,\theta)$. We then discuss how the analysis of $J(\psi,\theta)$ can help select new modes to extend the nonlinear feedback loop in higher-dimensional LMs. Our approach, using incremental changes in the number of Fourier modes, can help trace their individual and/or collective impact on the solution stability as well as the extension of the nonlinear feedback loop. To avoid repeated definitions, we use the same symbols as those in Shen (2014) and Shen (2015).

2 The Nonlinear Feedback Loop and its Extensions in the Lorenz Models

To derive the 3DLM, we use the following three Fourier modes:

$$M_1 = \sqrt{2}sin(lx)sin(mz), \quad M_2 = \sqrt{2}cos(lx)sin(mz), \quad M_3 = sin(2mz), \tag{1}$$

here l and m are defined as $\pi a/H$ and π/H, representing the horizontal and vertical wavenumbers, respectively. And, a is a ratio of the vertical scale of the convection cell to its horizontal scale, i.e., $a = l/m$. H is the domain height, and $2H/a$ represents the domain width. With the three modes in Eq. (1), the streamfunction ψ and the temperature perturbation θ can be represented as:

$$\psi = C_1\left(XM_1\right), \tag{2}$$

$$\theta = C_2\left(YM_2 - ZM_3\right), \tag{3}$$

here, C_1 and C_2 are constants (Shen 2014). (X,Y,Z) represent the amplitudes of (M_1, M_2, M_3), respectively. The modes in the 3DLM include one horizontal wavenumber (i.e., l) and two vertical wavenumbers (i.e., m and $2m$). After the derivations, the 3DLM is written as:

$$\frac{dX}{d\tau} = -\sigma X + \sigma Y, \tag{4}$$

$$\frac{dY}{d\tau} = -XZ + rX - Y, \tag{5}$$

$$\frac{dZ}{d\tau} = XY - bZ. \tag{6}$$

In the following, we will show that the two nonlinear terms, $-XZ$ and XY, appear in association with the nonlinear advection of temperature ($J(\psi,\theta)$), and illustrate that these two terms form a nonlinear feedback loop in the 3DLM. Then, we discuss how new modes are selected to extend the nonlinear feedback loop in the higher-dimensional LMs. To facilitate discussions below, the additional modes that have been used in the higher-dimensional LMs (Shen, 2014, 2015; Yoo and Shen, 2015) are defined as follows:

$$M_4 = \sqrt{2}sin(lx)sin(3mz), \quad M_7 = \sqrt{2}sin(lx)sin(5mz), \tag{7}$$

$$M_5 = \sqrt{2}cos(lx)sin(3mz), \quad M_6 = sin(4mz), \tag{8}$$

$$M_8 = \sqrt{2}cos(lx)sin(5mz), \quad M_9 = sin(6mz). \tag{9}$$

2.1 The nonlinear feedback loop in the 3DLM

In this section, we first discuss the characteristics of nonlinearity associated with the Jacobian term represented by a finite number of Fourier modes. With Eqs. (2-3), we have

$$J(\psi,\theta) = C_1C_2\Bigg(XYJ(M_1,M_2) - XZJ(M_1,M_3)\Bigg). \tag{10}$$

$J(\psi,\theta)$ is now expressed in terms of the summation of two nonlinear terms, $J(M_1,M_2)$ and $J(M_1,M_3)$ whose coefficients are XY and $-XZ$, respectively. Through straightforward derivations, we obtain

$$J(M_1,M_2) \approx 2mlsin(mz)cos(mz) = mlM_3, \tag{11}$$

and

$$J(M_1,M_3) \approx \sqrt{2}mlcos(lx)\Bigg(sin(3mz) + sin(-mz)\Bigg). \tag{12}$$

The vertical wave number of 3m is not used in the 3DLM, so the sin(3mz) is neglected. Thus, Eq. (12) becomes

$$J(M_1,M_3) \approx \sqrt{2}mlcos(lx)sin(-mz) = -mlM_2. \tag{13}$$

From Eqs. (11) and (13), a loop can be identified as follows. As Eq. (13) gives $M_2 \approx -J(M_1,M_3)/(ml)$, we can plug the M_2 into Eq. (11) to have

$$J(M_1,J(M_1,M_3)) = -(ml)^2M_3.$$

Similarly, we can derive

$$J(M_1,J(M_1,M_2) = -(ml)^2M_2.$$

Therefore, with the inclusion of the M_3, a loop with $M_2 \rightarrow M_3 \rightarrow M_2$ is introduced in the 3DLM. More importantly, downscale and upscale transfer processes can be identified using Eqs. (11) and (13). M_2 and M_3 have vertical wave numbers of m and $2m$, respectively. Eq. (11) suggests that the nonlinear interaction between M_1 and M_2 leads to a downscale transfer (to the M_3 mode), while Eq. (13) suggests that the nonlinear interaction between M_1 and M_3 leads to a upscale transfer (to the M_2). However, as $sin(3mz)$ is not included, the approximation using Eq. (13) neglects a downscale transfer (from the M_5 mode with sin(2mz) to the mode with sin(3mz), which will discussed in details in section 2.2.

Next, we illustrate the role of the nonlinear feedback loop in the "nonlinear" 3DLM. Without the inclusion of the nonlinear terms $-XZ$ and XY, Eqs. (4-6) of the 3DLM reduce to

$$\frac{dX}{d\tau} = -\sigma X + \sigma Y, \tag{14}$$

$$\frac{dY}{d\tau} = rX - Y, \tag{15}$$

$$\frac{dZ}{d\tau} = -bZ. \tag{16}$$

Equations (14-15), which are decoupled with Eq. (16), form a forced dissipative system with only linear terms. The system has only a trivial critical point ($X = Y = 0$) and produces unstable normal-mode solutions (i.e., exponentially growing with time) as $r > 1$. Therefore, our analysis indicates that the inclusion of M_3 introduces Eq. (16) and the enabled feedback loop (i.e., Eqs. 11 and 13) couples Eq. (16) with Eqs. (14-15) to form the (nonlinear) 3DLM (Eqs. 4-6) which enables the appearance of convection solutions. From a perspective of total energy conservation, the inclusion of the M_3 mode can help conserve the total energy in the dissipationless limit, which is discussed in Appendix A of Shen (2014). Mathematically, the feedback loop with the nonlinear terms in Eqs. 5 and 6 (i.e., $-XZ$ and XY) leads to the change in the behavior of the system's solutions; the (nonlinear) 3DLM system produces non-trivial critical points, which may be stable (e.g., for $1 < r < 24.74$) or "unstable" (chaotic) (e.g., for $r > 25$). In the next sections, we discuss how the nonlinear feedback loop in the 3DLM can be extended through proper selections of new modes.

2.2 An extension of the nonlinear feedback loop in the 5DLM

The increased degree of nonlinearity in the 5DLM, which has been discussed in Fig. 1 of Shen (2014), is briefly summarized below. In the derivation of the 3DLM, the mode with $sin(3mz)$ in Eq. (12) was neglected. Therefore, it is natural to include $\sqrt{2}cos(lx)sin(3mz)$

as the M_5 mode (Eq. 8). Thus, Eq. (12) can be written as

$$J(M_1, M_3) \approx \sqrt{2} mlcos(lx)\Bigg(sin(3mz) + sin(-mz)\Bigg) = ml(M_5 - M_2). \tag{17}$$

From a perspective of nonlinear interaction, the above mode-mode interaction in Eq. (17) indicates the route of the downscale and upscale energy transfer to the M_5 and M_2 modes, respectively. The M_5 mode can further interact with the M_1 mode to provide feedback to the M_3 mode through

$$J(M_1, M_5) \approx ml\Bigg(2sim(4mz) - sin(2mz)\Bigg) = 2mlM_6 - mlM_3. \tag{18}$$

The processes in Eqs. (17-18) add a new loop (e.g., $M_3 \to M_5 \to M_3$) which is connected to the (existing) feedback loop (e.g., $M_2 \to M_3 \to M_2$) of the 3DLM. Therefore, the feedback loop in the 3DLM is extended with the inclusion of the M_5 mode in the 5DLM. The original feedback loop and new feedback loop may be viewed as the main trunk and branch, respectively. *Note that the term "extension of the nonlinear feedback loop" indicates the linkage between the existing loop and the new loop.* It was reported that inclusion of new modes could produce additional equations that are not coupled with the 3DLM, leading to a generalized LM with the same stability as the 3DLM (e.g., Eqs. 11-16 of Roy and Musielak (2007a)). In this case, the original nonlinear feedback loop (of the 3DLM) is not extended with the new modes.

With the inclusion of M_5, $J(M_1, M_5)$ provides not only upscaling feedback to the M_3 mode but also a downscale energy transfer to a smaller-scale wave mode that, in turn, requires the inclusion of the $sin(4mz)$ mode (i.e., M_6 mode) (Eq. 18). As discussed in Appendix A of Shen (2014), the M_6 mode is required to conserve the total energy in the dissipationless limit. The feedback loop is further extended to $M_5 \to M_6 \to M_5$ through $J(M_1, M_5)$ and $J(M_1, M_6)$, as shown in Table 2 of Shen (2014) and discussed in section 3.1 of Shen (2015).

In summary, the two modes (M_5 and M_6) with higher vertical wavenumbers are added to improve the presentation of vertical temperature, and, therefore, the accuracy of the vertical advection of temperature, as shown:

$$\theta = C_2\Bigg(YM_2 - ZM_3 + Y_1M_5 - Z_1M_6\Bigg), \tag{19}$$

$$J(\psi, \theta) = C_1C_2\Bigg(XYJ(M_1, M_2) - XZJ(M_1, M_3) + XY_1J(M_1, M_5) - XZ_1J(M_1, M_6)\Bigg). \tag{20}$$

While the inclusion of M_3 forms a feedback loop in the 3DLM, the inclusion of M_5 and M_6 in the 5DLM extends the original feedback loop.

2.3 An extended nonlinear feedback loop in the 6DLM

As discussed in the previous sections, the inclusion of M_5 and M_6 modes is not only to improve the representations of the temperature perturbation and the nonlinear advection of temperature, but also to to extend the original nonlinear feedback loop. In this section, we discuss the selection of M_4 that is in association with the M_5 mode. The appearance of $\partial M_5/\partial x$ associated with the linear term $\partial\theta/\partial x$ of Eq. (1) of Shen (2014,2015) requires the inclusion of an M_4 mode and the $\partial M_4/\partial x$ associated with $\triangle T\partial\psi/\partial x$ of Eq. (2) of Shen (2014,2015) provides feedback to the M_5 mode (in Table 1 of Shen, 2014). The M_4 mode shares the same horizontal and vertical wave numbers as the M_5 but has a different phase (i.e., $sin(lx)$ vs. $cos(lx)$ in Eqs. 7-8 or in Eq. 4 of Shen 2015). Alternatively, via the $\partial\theta/\partial x$ and $\triangle T\partial\psi/\partial x$, the M_4 and M_5 modes are linked, as discussed in section 3.1 in the submitted manuscript (Shen 2015).

When M_4 is included, it improves the representation of the streamfunction and thus the advection of temperature, as shown:

$$\psi = C_1\left(XM_1 + X_1M_4\right), \tag{21}$$

$$J(\psi,\theta) = C_1C_2\left(J(XM_1 + X_1M_4,\ YM_2 + Y_1M_5 - ZM_3 - Z_1M_6)\right), \tag{22}$$

here X_1 represents the amplitude of the mode M_4. Now, the Jacobian term includes $J(XM_1, YM_2 + Y_1M_5 - ZM_3 - Z_1M_6)$ and $J(X_1M_4, YM_2 + Y_1M_5 - ZM_3 - Z_1M_6)$. The former was first discussed in the 5DLM by Shen (2014), while the latter is discussed using the 6DLM in this study. While the M_4 mode introduces linear forcing term (e.g., rX_1), it also extends the nonlinear feedback loop with $J(X_1M_4, YM_2)$, $J(X_1M_4, Y_1M_5)$, $J(X_1M_4, ZM_3)$, and $J(X_1M_4, Z_1M_6)$. The outcome of each of these Jacobian terms can be found in the Table 2 of Shen (2014), and the impact of M_4 is discussed in Shen (2015).

2.4 Further extensions of the nonlinear feedback loop in Higher-order LMs

To examine the role of the nonlinear feedback loop in the solution stability of higher-order LMs, we have derived the following higher-dimensional Lorenz models, including 7D, 8D and 9D LMs. These models give a larger critical value of the normalized Rayleigh parameter for the onset of chaos, as compared to the 3D, 5D and 6D Lorenz models. A manuscript is being prepared for publication (Yoo and Shen, 2015). Here, a brief description for the higher-order LMs is given as follows:

1. 7DLM includes all modes in the 5DLM and the M_8 and M_9 modes (Eq. 9) that can improve the representation of θ and $J(\psi,\theta)$ and to extend the nonlinear feedback loop to provide negative nonlinear feedback;

2. 8DLM contains all modes in the 7DLM and the M_4 mode (Eq. 7) that can improve the representation of ψ and $J(\psi,\theta)$;

3. 9DLM includes all modes in the 8DLM and an additional mode M_7 (Eq. 7) to improve the representation of ψ and $J(\psi,\theta)$.

Note that M_8 with $sin(5mz)$ is selected based on the analysis of $J(M_1, M_6)$ as shown in the Table 2 of Shen (2014). M_9 is added to enable the downscale transfer from $J(M_1, M_8)$. Similar to the inclusion of M_4, M_7 is introduced to have a different phase to that of M_8.

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