

General responses:

I would like to thank the reviewers and Editor for their valuable comments. One of the major concerns raised by both reviewers is how new modes were selected to derive the 6DLM. Here, I would like to emphasize (1) that based on the analysis of the Jacobian term, $J(\psi, \theta)$, new modes are selected to extend the nonlinear feedback loop that can provide additional nonlinear feedback to stabilize or destabilize solutions; and (2) that our approach, using incremental changes in the number of Fourier modes, is to help trace their individual and/or collective impact on the solution stability as well as the extension of the nonlinear feedback loop. To facilitate discussions, we have (a) created two tables which list the Fourier models used to construct different higher-order Lorenz models and the corresponding critical values of the normalized Rayleigh parameter for the onset of chaos; and (b) finished a pdf file with a brief summary on the mathematical analysis of the nonlinear feedback loop in the 3DLM and its extension in the 5DLM and 6DLM. The tables are included in the end of this response file, while the pdf file will be uploaded separately as supplemental materials. In the following, specific responses are given with the aid of the supplemental materials.

(B) Responses to reviewer II's comments

This paper has the merit of studying how changing the structure of feedbacks impact some of the most important properties of a minimal truncated set of equations describing convection. While the paper has indeed merits, I would recommend the authors to improve the discussion on the physical relevance of their results and put them in a broader context of the published literature.

Thanks for your comments. I have done my best to address the concerns and comments in the following.

1) The authors should make clear that the problem was first studied by Salzman, who gave a very extensive treatment of the possibility of constructing reduced order models. Lorenz then studied one of such models and got such an incredible result. Also, in the following paper it is discussed that the L63 model is a member of a class of equivalence: Z.-M. Chen and W. G. Price. Chaos. Solitons Fractals 28. 571 2006 .

Thanks for your suggestions. We have revised the manuscript accordingly and cited the paper of Chen and Price (2006).

Here, we provide a brief discussion on how “symmetry” was introduced in the model of Chen and Price (2006) and will discuss about “symmetry breaking” in the responses to the 2nd question. In the 3DLM, the streamfunction is represented by one Fourier mode, $\psi_1(1,1) = \sqrt{2}\sin(lx)\sin(mz)$, and temperature is represented by two Fourier modes, $\theta_2(1,1) = \sqrt{2}\cos(lx)\sin(mz)$ and $\theta_2(0,2) = \sin(2mz)$. Chen and Price (2006) suggested represent the streamfunction by both $\sqrt{2}\sin(lx)\sin(mz)$ and $\sqrt{2}\cos(lx)$, and temperature by the two modes and $\sin(2mz)$ (e.g., Eq. 12 of Chen and Price, 2006). Their approach produces a model with 5 ordinary differential equations (ODEs) that introduces the “symmetry.” And thus, the attractor in the 3DLM is a cross section of the attractor of the Chen and Price’s model. Note that horizontal and vertical wavenumbers in the 3DLM and the model of Chen and Price are the same. Next, we discuss on how inclusion of higher wavenumber modes may break the symmetry.

2) In presenting their models, the authors should explain more clearly that they have to be derived from the continuum equation by suitable truncation.

It is not clear to me why the authors choose such a truncation, where the first and third modes are used, while the second modes are left out, a bit arbitrarily, from my point of view. Could the authors explain the rationale for this?

In an earlier paper

V. Lucarini and K. Fraedrich, Symmetry breaking, mixing, instability, and low-frequency variability in a minimal Lorenz-like system, PRE 80, 026313 2009

we took a different route, constructing the truncation starting from below, including all the 1- and 2- modes, and adding the same two modes here indicated as M3 and M6. In this way, we also obtain a closed system of equations. Maybe the authors could consider looking into that paper, where it is explained that in order to achieve a complete thermodynamic consistency (see also C. Nicolis, Q. J. R. Meteorol. Soc. 125, 1859 1999 on the Thermodynamic Relevance of the L63 model) some modifications have to be implemented.

Using Table 2 in the response file and supplemental materials, I would like to emphasize the importance in selecting the M5 mode (i.e., $\Theta(1,3)$ in Table 2), based on the analysis of Jacobian term, $J(\Psi, \theta)$. The inclusion of M5 and M6 can extend the original nonlinear feedback loop in the 3DLM to provide negative nonlinear feedback to stabilize the solution in the 5DLM and 6DLM.

I appreciate the reviewer's comments and sharing. Lucarini and K. Fraedrich (2009) is being cited among the studies with the high-order Lorenz models in the revised manuscript. Their study focused more on "symmetry breaking", but their model does not include $\Theta(1,3)$ (i.e., M5) and $\psi(1,3)$ (i.e., M4). In comparison, we have implicitly discussed the "symmetry breaking" using higher wavenumber modes (e.g., M5 mode) that are included to lead to a (new) downscale transfer, which leads to asymmetry as discussed below using Eq. 11-13 and Eq. 17 in the supplemental materials). As the 3DLM does not include the M5 mode, Eq. (13) for $J(M_1, M_3)$ includes only an upscale transfer but neglect a downscale transfer. Therefore, the nonlinear feedback loop of the 3DLM with Eqs. (11) and (13) leads to "spurious" symmetry with respect to the z-axis and invariant for $(X, Y) \rightarrow (-X, -Y)$. Namely, if (X, Y, Z) is a solution, $(-X, -Y, Z)$ is also a solution. This kind of symmetry does not exit in the 5DLM with the inclusion of the M5

mode, which allows the downscale transfer for J (M_1 , M_3), as shown in Eq. 17 in the supplemental materials.

In addition, I have also started paying attention to the “asymmetric characteristics” of the multiscale interactions in weather and climate. In the real world modeling studies (Shen et al., 2012, p25), we made the following statements regarding the asymmetric nature of multiscale interactions:

“

Based on the current study and two previous studies [Shen et al., 2010a; 2010b] using the global mesoscale model, the following view on the predictability of mesoscale tropical cyclone genesis is proposed: (1) Both a downscaling cascade of processes associated with the large-scale systems and an upscaling cascade of processes associated with the small-scale (e.g., precipitation) systems are important. Because of the asymmetry in the spatial and temporal scales and the strengths among these systems, the term “hierarchical multiscale interactions” is used to describe these scale interactions that can lead to TC formation. (2) ...

”

Speaking of the symmetry breaking, I would like to make the following comments as a brief comparison, which may be extended in a future study. By taking a approach which is similar to the one by Chen and Price (2006), Lucarini and Fraedrich (2009) included additional modes with higher horizontal and vertical wave numbers to construct the model for examining the symmetry breaking and other interesting characteristics. The new modes are $\psi_1(2,2)$, $\theta_2(2,2)$ and $\theta_2(0,4)$. Note that the first two modes were used in the 14DLM of Curry (1978), but they are not used in our 5DLM, 6DLM or 7D-9D LMs, or any other models in Tables 1-3. Based on the aforementioned analysis using our higher-order LMs, the modes with higher vertical wavenumbers may break the symmetry. However, as shown in Table 3, new modes with different horizontal wave numbers for streamfunction may introduce new terms via both $J(\psi, \nabla^2 \psi)$ and $J(\psi, \theta)$, which can make it complicated to compare the models with our models. Moreover, $\theta_2(0,4)$ is the same as our M6, but the M5 mode (i.e., $\psi_1(1,3)$), which is selected to extend the nonlinear feedback loop, is not included in their model (see Table 2 for a

comparison). Therefore, it requires efforts to compare the characteristics of solutions from the model of Lucarini and Fraedrich (2009) and our models, which deserves a future study. In this study, we focus on the extension of nonlinear feedback loop associated with $J(\psi, \theta)$, i.e., no nonlinear terms from $J(\psi, \nabla^2 \psi)$ (see Shen 2014a for more details).

For the consistency of dynamics and thermodynamics in the higher-order Lorenz models, we have discussed the conservation laws for both 5DLM and 6DLM in the dissipationless limit (in section 3.3 of Shen 2015). From a thermodynamic perspective, we discussed how additional nonlinear terms and dissipative terms introduced by the M5 and M6 modes can provide negative nonlinear feedback to stabilize solutions in both 5DLM and 6DLM. These processes seem consistent with what has been documented in Lucarini and Fraedrich (2009, p026313-4), as follows:

``

We then conclude that, in spite of introducing a second unstable direction, which is responsible for mixing the phase of the waves, inclusion of the impact of viscous dissipation on the thermal energy balance acts with continuity on the dynamical indicators, by reducing the overall instability, increasing the predictability of the system, and by confining the asymptotic dynamics to a more limited (in terms of dimensionality) set.

''

3) If possible, I would recommend the authors to discuss a bit the fact that the fractal dimension is similar for all their models when they are all in the chaotic regime. What is their take on this?

Also: the first Lyapunov exponent is almost identical in the 3D and one of the 6D model.

Can they find a correspondence also for the other non-zero LE of the 3D model?

Thanks for your comments and suggestions on the calculation of the fractal dimension. We presented the calculation of fractal dimension with 3DLM (as well as 5DLM and 6DLM) in the appendix to provide additional support to the implementation of the GSR scheme. As nonlinearity is viewed as the source of chaos in the 3DLM, some people inferred that systems with more nonlinear terms may become more chaotic. However, using our 5DLM and 6DLM,

we have tried to point out that proper selection of new modes can extend the nonlinear feedback loop to provide negative feedback to stabilize the solutions. Therefore, compared to the 3DLM, a comparable fractal dimension in our 5DLM and 6DLM, which is smaller than those in other high-order LMs, may be consistent with the fact that the nonlinear feedback loop is extended in the 5DLM/6DLM. In each of 3DLM, 5DLM, 6DLM, the first eLE is positive, and the second eLE is close to zero. In higher-order LMs (e.g., 5D and 6 LMs), additional negative eLEs were found, which are indicated by the summation of the eLEs, which is -30.667 (-94.0) for the 5DLM (6DLM) (see Figure A1 in the manuscript).

In addition, Figure 7a of Shen (2014) provide a better representation for the comparison of LEs between the 3DLM and 5DLM. In both models, LEs are comparable when r is small, but differences appear when r becomes larger (e.g., $r > 95$). As additional nonlinear terms and “dissipative” terms are introduced in the 5DLM, additional negative eLEs are obtained. Again, this is shown in Figure A1 that the sum of eLEs is -13.667 for the 3DLM but becomes -30.667 for the 5DLM.

Compared to the 5DLM, the 6DLM introduces an additional heating term, rX_1 . However, as discussed in the manuscript, the magnitude of X_1 is small, and thus it causes smaller differences, as compared to negative nonlinear feedback that is associated with the additional nonlinear terms and dissipative terms. Therefore, the 6DLM has a comparable but smaller rc than the 5DLM. While this study focuses on the improvement of solution’s stability, we agree with the reviewer that it is important to perform detailed analysis on the each of eLEs, which will be conducted in a future study with the higher-order LMs.

Some additional comments

- a) When discussing the Lyapunov exponents, the authors might consider referring to G. Benettin, L. Galgani, A. Giorgilli, and J. M. Strelcyn, *Meccanica* 15, 9 1980 as they first discussed the benefits of the GS method.
- b) Also: the effect of mode truncation was extensively studied by Franceschini et al. V. Franceschini and C. Tebaldi, *Meccanica* 20, 207 1985; V. Franceschini, C. Giberti, and M. Nicolini, *J. Stat. Phys.* 50, 879 1988
- c) Appendix: Attention: you are citing different definitions of fractal measure. They are not equivalent. See Ruelle 1989. Ruelle, *Chaotic Evolution and Strange Attractors*, 1989
- d) What does it mean that the second Lyapunov exponent is not zero (of course it has to)? Of course it cannot ever be exactly zero. It will converge only asymptotically to that value. The authors might consider adding error bars to their estimates.

- (a) Thanks for your comments. Benettin et al. (1980) has been cited in the revised manuscript.
- (b) We have cited these studies. Once again, I would like to emphasize that the uniqueness of our approach is to incrementally select new modes to extend the nonlinear feedback loop, based on the analysis of $J(\psi, \theta)$, and to examine the individual and combined impact of new terms associated with the new modes on solution's stability.
- (c) Agree. We are aware of the different definitions of fractal dimensions and methods for their calculations. We have revised the manuscript accordingly, and stated that only KY fractal dimension is discussed, because it is calculated using the LEs. Through the calculation of the KY fractal dimension, we provide additional verification for the ensemble LE calculation.
- (d) For the calculation of the (global) LE, it requires the integration of two trajectories (in the control and parallel runs) over an infinite period of time (e.g., the T in Eq. 23 of Shen 2014a should approach infinity). However, in reality, numerical integration (or summation) is applied only over a finite period of time. Therefore, given a specific set of

ICs, a period of $T=1,000$ may not be sufficient to determine the global LE. In addition, in order to minimize the dependence of initial ICs, we calculate LEs with 10,000 different ICs to obtain the ensemble averaged LE (eLE). A non-zero value of LE in any one of 10,000 ensemble runs can contribute to the non-zero eLE. The above reasons may explain why the 2nd eLE is small but is not exactly equal to zero. However, as it is small (compared to the 1st eLE), it should not have significant impact on the calculation of the fractal dimension, which requires the summation of the first two eLEs. Currently, a manuscript regarding the scientific and parallel performance of the implementation for the calculation of eLE is being prepared. For now, we added the following sentences in Italics in the revised manuscript:

Here, the reader should note that the 2nd eLE is very small but not exactly equal to zero, indicating the impact of the 10000 different initial conditions *and/or the "finite" integration time ($T = 1000$) in this study.*

References (which have been included in the revised manuscript)

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- Yoo, E. and B.-W. Shen, 2015: On the extension of the nonlinear feedback loop in 7D, 8D and 9D Lorenz models. (in preparation)

Table 1: Fourier modes selected to construct the 3DLM and higher-order LMs, which is from Table 1 of Roy and Musielak (2007c). The critical values of the normalized Raleigh parameter, shown in red, are derived from Table 2 of Roy and Musielak (2007c).

Model	Circulation modes	Temperature modes	Temperature modes with $m = 0$	References
3D	$\Psi_1(1, 1)$	$\Theta_2(1, 1)$	$\Theta_2(0, 2)$ rc~24.75	Lorenz [1]
5D	$\Psi_1(1, 1)$ $\Psi_1(2, 1)$	$\Theta_2(1, 1)$ $\Theta_2(2, 1)$	$\Theta_2(0, 2)$ rc~22.50	Paper II
6D	$\Psi_1(1, 1)$ $\Psi_1(2, 1)$ $\Psi_1(1, 2)$	$\Theta_2(1, 1)$ $\Theta_2(2, 1)$	$\Theta_2(0, 2)$ n/a	Humi [9]
6D	$\Psi_1(1, 1)$ $\Psi_1(1, 3)$	$\Theta_2(1, 1)$ $\Theta_2(1, 3)$	$\Theta_2(0, 2)$ $\Theta_2(0, 4)$ rc~40.15	Kenamer [10]
8D	$\Psi_1(1, 1)$ $\Psi_1(2, 1)$ $\Psi_1(1, 2)$	$\Theta_2(1, 1)$ $\Theta_2(2, 1)$ $\Theta_2(1, 2)$	$\Theta_2(0, 2)$ $\Theta_2(0, 4)$ rc~35.60	This Paper
9D	$\Psi_1(1, 1)$ $\Psi_1(1, 2)$ $\Psi_1(1, 3)$	$\Theta_2(1, 1)$ $\Theta_2(1, 2)$ $\Theta_2(1, 3)$	$\Theta_2(0, 2)$ $\Theta_2(0, 4)$ $\Theta_2(0, 6)$ rc~40.50	Paper I

Table 2: Fourier modes used in our high-order LMs (e.g., Shen 2014a, 2015; Yoo and Shen, 2015) and the models by Curry (1978) and Lucarini and K. Fraedrich (2009). Note that $M_4 = \psi_1(1,3)$, $M_5 = \theta_2(1,3)$, and $M_6 = \theta_2(0,4)$.

model	Ψ	Θ	Θ	rc	References
5DLM	$\psi_1(1,1)$	$\theta_2(1,1),$ $\theta_2(1,3)$	$\theta_2(0,2),$ $\theta_2(0,4)$	42.9	Shen (2014)
6DLM	$\psi_1(1,1),$ $\psi_1(1,3),$	$\theta_2(1,1),$ $\theta_2(1,3)$	$\theta_2(0,2),$ $\theta_2(0,4)$	41.1	Shen(2015)
7DLM	$\psi_1(1,1)$	$\theta_2(1,1),$ $\theta_2(1,3),$ $\theta_2(1,5)$	$\theta_2(0,2),$ $\theta_2(0,4),$ $\theta_2(0,6),$	~ 116.9	Yoo and Shen (2015, in preparation)
8DLM	$\psi_1(1,1),$ $\psi_1(1,3)$	$\theta_2(1,1),$ $\theta_2(1,3),$ $\theta_2(1,5)$	$\theta_2(0,2),$ $\theta_2(0,4),$ $\theta_2(0,6)$	~ 105 (TBD with the eLE analysis)	Yoo and Shen (2015)
9DLM	$\psi_1(1,1),$ $\psi_1(1,3),$ $\psi_1(1,5)$	$\theta_2(1,1),$ $\theta_2(1,3),$ $\theta_2(1,5)$	$\theta_2(0,2),$ $\theta_2(0,4),$ $\theta_2(0,6)$	~ 105 (TBD with the eLE analysis)	Yoo and Shen (2015)
14DLM	$\psi_1(1,1),$ $\psi_1(1,3),$ $\psi_1(2,2),$ $\psi_1(2,4),$ $\psi_1(3,1),$ $\psi_1(3,3)$	$\theta_2(1,1),$ $\theta_2(1,3),$ $\theta_2(2,2),$ $\theta_2(2,4),$ $\theta_2(3,1),$ $\theta_2(3,3)$	$\theta_2(0,2),$ $\theta_2(0,4)$	rc ~ 43	Curry (1978)
10EQs	$\psi_1(1,1)$ $\psi_1(2,2)$	$\theta_2(1,1),$ $\theta_2(2,2)$	$\theta_2(0,2),$ $\theta_2(0,4)$	n/a	Lucarini and K. Fraedrich (2009)

Table 3: Lorenz models with different Fourier modes. 3DLM and 5DLM are discussed in the manuscript, while the 6DLM will be discussed in a companion paper. 6DLM_HK is referred to as the 6DLM proposed by Howard and Krishnamurti (1986). 7DLM_TH and 7DLM_Hetal are referred as the 7DLMs proposed by Thiffeault and Horton (1995) and Hermiz et al. (1995), respectively. The one denoted as ‘8DLM (suggested)’ was suggested by Thiffeault and Horton (1995) who did not derive the 8DLM nor discuss its characteristics. Only one horizontal wave number was used in the first several Lorenz models. $\cos(2lx)$ was used in the 8DLM by Roy and Musielak (2007c), denoted as 8DLM_RM. M_1 - M_6 are defined in the manuscript. M_a - M_d are defined as $\sin(mz)$, $\cos(lx)\sin(2mz)$, $\sin(lx)\sin(2mz)$, and $\sin(3mz)$, respectively. In the studies by Howard and Krishnamurti (1986) and Hermiz et al. (1995), the symbol ‘ α ’ is equivalent to ‘ a ’ in our study, which is equal ‘ l/m ’, namely $\alpha=a=l/m$.

1	3DLM -- ψ	$\sin(lx)\sin(mz)$				
	θ	$\cos(lx)\sin(mz)$	$\sin(2mz)$			
2	5DLM -- ψ	$\sin(lx)\sin(mz)$				
	θ	$\cos(lx)\sin(mz)$	$\sin(2mz)$	$\cos(lx)\sin(3mz)$	$\sin(4mz)$	
3	6DLM -- ψ	$\sin(lx)\sin(mz)$	$\sin(lx)\sin(3mz)$			
	θ	$\cos(lx)\sin(mz)$	$\sin(2mz)$	$\cos(lx)\sin(3mz)$	$\sin(4mz)$	
4	6DLM_HK -- ψ	$\sin(lx)\sin(mz)$	$\sin(mz)$	$\cos(lx)\sin(2mz)$		
	θ	$\cos(lx)\sin(mz)$	$\sin(2mz)$	$\sin(lx)\sin(2mz)$		
5	7DLM_TH-- ψ	$\sin(lx)\sin(mz)$	$\sin(mz)$	$\cos(lx)\sin(2mz)$		
	θ	$\cos(lx)\sin(mz)$	$\sin(2mz)$	$\sin(lx)\sin(2mz)$	$\sin(4mz)$	
6	7DLM_Hetal -- ψ	$\sin(lx)\sin(mz)$	$\sin(mz)$	$\cos(lx)\sin(2mz)$	$\sin(3mz)$	
	θ	$\cos(lx)\sin(mz)$	$\sin(2mz)$	$\sin(lx)\sin(2mz)$		
7	8DLM (suggested)	$\sin(lx)\sin(mz)$	$\sin(mz)$	$\cos(lx)\sin(2mz)$	$\sin(3mz)$	
	θ	$\cos(lx)\sin(mz)$	$\sin(2mz)$	$\sin(lx)\sin(2mz)$	$\sin(6mz)$	
8	8DLM_RM---- ψ	$\sin(lx)\sin(mz)$	$\sin(lx)\sin(2mz)$			$\sin(2lx)\sin(mz)$
		$\cos(lx)\sin(mz)$	$\sin(2mz)$	$\cos(lx)\sin(2mz)$	$\sin(4mz)$	$\cos(2lx)\sin(mz)$

1	3DLM -- ψ	M_1				
	θ	M_2	M_3			
2	5DLM -- ψ	M_1				
	θ	M_2	M_3	M_5	M_6	
3	6DLM -- ψ	M_1	M_4			
	θ	M_2	M_3	M_5	M_6	
4	6DLM_HK -- ψ	M_1	M_a	M_b		
	θ	M_2	M_3	M_c		
5	7DLM_TH-- ψ	M_1	M_a	M_b		
	θ	M_2	M_3	M_c	M_6	
6	7DLM_Hetal -- ψ	M_1	M_a	M_b	M_d	
	θ	M_2	M_3	M_c		
7	8DLM (suggested)	M_1	M_a	M_b	M_d	
	θ	M_2	M_3	M_c	$\sin(6mz)$	
8	8DLM_RM---- ψ	M_1	$\sin(lx)\sin(2mz)$			$\sin(2lx)\sin(mz)$
		M_2	M_3	$\cos(lx)\sin(2mz)$	M_6	$\cos(2lx)\sin(mz)$

Manuscript prepared for Nonlin. Processes Geophys.
with version 5.0 of the L^AT_EX class copernicus.cls.
Date: 29 September 2015

**Supplemental Materials for the Paper entitled “ Nonlinear
feedback in the six-dimensional Lorenz model: impact of an
additional heating term. By Bo-Wen Shen”**

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1 Introduction

This report, which documents the mathematical analysis on the extensions of the nonlinear feedback loop in the 5DLM and 6DLM as well as higher-dimensional Lorenz models, is provided as supplementary materials to the manuscript entitled “Nonlinear feedback in the six-dimensional Lorenz model: impact of an additional heating term. by Shen (2015).” In the following, we briefly introduce the three-dimensional (3D) Lorenz model (3DLM, Lorenz, 1963) and its Fourier modes, and identify the nonlinear feedback loop of the 3DLM by analyzing the nonlinear Jacobian term $J(\psi, \theta)$. We then discuss how the analysis of $J(\psi, \theta)$ can help select new modes to extend the nonlinear feedback loop in higher-dimensional LMs. Our approach, using incremental changes in the number of Fourier modes, can help trace their individual and/or collective impact on the solution stability as well as the extension of the nonlinear feedback loop. To avoid repeated definitions, we use the same symbols as those in Shen (2014) and Shen (2015).

2 The Nonlinear Feedback Loop and its Extensions in the Lorenz Models

To derive the 3DLM, we use the following three Fourier modes:

$$M_1 = \sqrt{2}\sin(lx)\sin(mz), \quad M_2 = \sqrt{2}\cos(lx)\sin(mz), \quad M_3 = \sin(2mz), \quad (1)$$

here l and m are defined as $\pi a/H$ and π/H , representing the horizontal and vertical wavenumbers, respectively. And, a is a ratio of the vertical scale of the convection cell to its horizontal scale, i.e., $a = l/m$. H is the domain height, and $2H/a$ represents the domain width. With the three modes in Eq. (1), the streamfunction ψ and the temperature perturbation θ can be represented as:

$$\psi = C_1 \begin{pmatrix} X M_1 \end{pmatrix}, \quad (2)$$

$$\theta = C_2 \begin{pmatrix} Y M_2 - Z M_3 \end{pmatrix}, \quad (3)$$

here, C_1 and C_2 are constants (Shen 2014). (X, Y, Z) represent the amplitudes of (M_1, M_2, M_3) , respectively. The modes in the 3DLM include one horizontal wavenumber (i.e., l) and two vertical wavenumbers (i.e., m and $2m$). After the derivations, the 3DLM is written as:

$$\frac{dX}{d\tau} = -\sigma X + \sigma Y, \quad (4)$$

$$\frac{dY}{d\tau} = -XZ + rX - Y, \quad (5)$$

$$\frac{dZ}{d\tau} = XY - bZ. \quad (6)$$

In the following, we will show that the two nonlinear terms, $-XZ$ and XY , appear in association with the nonlinear advection of temperature ($J(\psi, \theta)$), and illustrate that these two terms form a nonlinear feedback loop in the 3DLM. Then, we discuss how new modes are selected to extend the nonlinear feedback loop in the higher-dimensional LMs. To facilitate discussions below, the additional modes that have been used in the higher-dimensional LMs (Shen, 2014, 2015; Yoo and Shen, 2015) are defined as follows:

$$M_4 = \sqrt{2}\sin(lx)\sin(3mz), \quad M_7 = \sqrt{2}\sin(lx)\sin(5mz), \quad (7)$$

$$M_5 = \sqrt{2}\cos(lx)\sin(3mz), \quad M_6 = \sin(4mz), \quad (8)$$

$$M_8 = \sqrt{2}\cos(lx)\sin(5mz), \quad M_9 = \sin(6mz). \quad (9)$$

2.1 The nonlinear feedback loop in the 3DLM

In this section, we first discuss the characteristics of nonlinearity associated with the Jacobian term represented by a finite number of Fourier modes. With Eqs. (2-3), we have

$$J(\psi, \theta) = C_1 C_2 \left(XY J(M_1, M_2) - XZ J(M_1, M_3) \right). \quad (10)$$

$J(\psi, \theta)$ is now expressed in terms of the summation of two nonlinear terms, $J(M_1, M_2)$ and $J(M_1, M_3)$ whose coefficients are XY and $-XZ$, respectively. Through straightforward derivations, we obtain

$$J(M_1, M_2) \approx 2ml\sin(mz)\cos(mz) = mlM_3, \quad (11)$$

and

$$J(M_1, M_3) \approx \sqrt{2}ml\cos(lx) \left(\sin(3mz) + \sin(-mz) \right). \quad (12)$$

The vertical wave number of $3m$ is not used in the 3DLM, so the $\sin(3mz)$ is neglected. Thus, Eq. (12) becomes

$$J(M_1, M_3) \approx \sqrt{2}ml\cos(lx)\sin(-mz) = -mlM_2. \quad (13)$$

From Eqs. (11) and (13), a loop can be identified as follows. As Eq. (13) gives $M_2 \approx -J(M_1, M_3)/(ml)$, we can plug the M_2 into Eq. (11) to have

$$J(M_1, J(M_1, M_3)) = -(ml)^2 M_3.$$

Similarly, we can derive

$$J(M_1, J(M_1, M_2)) = -(ml)^2 M_2.$$

Therefore, with the inclusion of the M_3 , a loop with $M_2 \rightarrow M_3 \rightarrow M_2$ is introduced in the 3DLM. More importantly, downscale and upscale transfer processes can be identified using Eqs. (11) and (13). M_2 and M_3 have vertical wave numbers of m and $2m$, respectively. Eq. (11) suggests that the nonlinear interaction between M_1 and M_2 leads to a downscale transfer (to the M_3 mode), while Eq. (13) suggests that the nonlinear interaction between M_1 and M_3 leads to an upscale transfer (to the M_2). However, as $\sin(3mz)$ is not included, the approximation using Eq. (13) neglects a downscale transfer (from the M_5 mode with $\sin(2mz)$ to the mode with $\sin(3mz)$, which will be discussed in detail in section 2.2.

Next, we illustrate the role of the nonlinear feedback loop in the “nonlinear” 3DLM. Without the inclusion of the nonlinear terms $-XZ$ and XY , Eqs. (4-6) of the 3DLM reduce to

$$\frac{dX}{d\tau} = -\sigma X + \sigma Y, \quad (14)$$

$$\frac{dY}{d\tau} = rX - Y, \quad (15)$$

$$\frac{dZ}{d\tau} = -bZ. \quad (16)$$

Equations (14-15), which are decoupled with Eq. (16), form a forced dissipative system with only linear terms. The system has only a trivial critical point ($X = Y = 0$) and produces unstable normal-mode solutions (i.e., exponentially growing with time) as $r > 1$. Therefore, our analysis indicates that the inclusion of M_3 introduces Eq. (16) and the enabled feedback loop (i.e., Eqs. 11 and 13) couples Eq. (16) with Eqs. (14-15) to form the (nonlinear) 3DLM (Eqs. 4-6) which enables the appearance of convection solutions. From a perspective of total energy conservation, the inclusion of the M_3 mode can help conserve the total energy in the dissipationless limit, which is discussed in Appendix A of Shen (2014). Mathematically, the feedback loop with the nonlinear terms in Eqs. 5 and 6 (i.e., $-XZ$ and XY) leads to the change in the behavior of the system’s solutions; the (nonlinear) 3DLM system produces non-trivial critical points, which may be stable (e.g., for $1 < r < 24.74$) or “unstable” (chaotic) (e.g., for $r > 25$). In the next sections, we discuss how the nonlinear feedback loop in the 3DLM can be extended through proper selections of new modes.

2.2 An extension of the nonlinear feedback loop in the 5DLM

The increased degree of nonlinearity in the 5DLM, which has been discussed in Fig. 1 of Shen (2014), is briefly summarized below. In the derivation of the 3DLM, the mode with $\sin(3mz)$ in Eq. (12) was neglected. Therefore, it is natural to include $\sqrt{2}\cos(lx)\sin(3mz)$

as the M_5 mode (Eq. 8). Thus, Eq. (12) can be written as

$$J(M_1, M_3) \approx \sqrt{2}ml\cos(lx) \left(\sin(3mz) + \sin(-mz) \right) = ml(M_5 - M_2). \quad (17)$$

From a perspective of nonlinear interaction, the above mode-mode interaction in Eq. (17) indicates the route of the downscale and upscale energy transfer to the M_5 and M_2 modes, respectively. The M_5 mode can further interact with the M_1 mode to provide feedback to the M_3 mode through

$$J(M_1, M_5) \approx ml \left(2\sin(4mz) - \sin(2mz) \right) = 2mlM_6 - mlM_3. \quad (18)$$

The processes in Eqs. (17-18) add a new loop (e.g., $M_3 \rightarrow M_5 \rightarrow M_3$) which is connected to the (existing) feedback loop (e.g., $M_2 \rightarrow M_3 \rightarrow M_2$) of the 3DLM. Therefore, the feedback loop in the 3DLM is extended with the inclusion of the M_5 mode in the 5DLM. The original feedback loop and new feedback loop may be viewed as the main trunk and branch, respectively. *Note that the term "extension of the nonlinear feedback loop" indicates the linkage between the existing loop and the new loop.* It was reported that inclusion of new modes could produce additional equations that are not coupled with the 3DLM, leading to a generalized LM with the same stability as the 3DLM (e.g., Eqs. 11-16 of Roy and Musielak (2007a)). In this case, the original nonlinear feedback loop (of the 3DLM) is not extended with the new modes.

With the inclusion of M_5 , $J(M_1, M_5)$ provides not only upscaling feedback to the M_3 mode but also a downscale energy transfer to a smaller-scale wave mode that, in turn, requires the inclusion of the $\sin(4mz)$ mode (i.e., M_6 mode) (Eq. 18). As discussed in Appendix A of Shen (2014), the M_6 mode is required to conserve the total energy in the dissipationless limit. The feedback loop is further extended to $M_5 \rightarrow M_6 \rightarrow M_5$ through $J(M_1, M_5)$ and $J(M_1, M_6)$, as shown in Table 2 of Shen (2014) and discussed in section 3.1 of Shen (2015).

In summary, the two modes (M_5 and M_6) with higher vertical wavenumbers are added to improve the presentation of vertical temperature, and, therefore, the accuracy of the vertical advection of temperature, as shown:

$$\theta = C_2 \left(Y M_2 - Z M_3 + Y_1 M_5 - Z_1 M_6 \right), \quad (19)$$

$$J(\psi, \theta) = C_1 C_2 \left(XY J(M_1, M_2) - XZ J(M_1, M_3) + XY_1 J(M_1, M_5) - XZ_1 J(M_1, M_6) \right). \quad (20)$$

While the inclusion of M_3 forms a feedback loop in the 3DLM, the inclusion of M_5 and M_6 in the 5DLM extends the original feedback loop.

2.3 An extended nonlinear feedback loop in the 6DLM

As discussed in the previous sections, the inclusion of M_5 and M_6 modes is not only to improve the representations of the temperature perturbation and the nonlinear advection of temperature, but also to extend the original nonlinear feedback loop. In this section, we discuss the selection of M_4 that is in association with the M_5 mode. The appearance of $\partial M_5/\partial x$ associated with the linear term $\partial\theta/\partial x$ of Eq. (1) of Shen (2014,2015) requires the inclusion of an M_4 mode and the $\partial M_4/\partial x$ associated with $\Delta T\partial\psi/\partial x$ of Eq. (2) of Shen (2014,2015) provides feedback to the M_5 mode (in Table 1 of Shen, 2014). The M_4 mode shares the same horizontal and vertical wave numbers as the M_5 but has a different phase (i.e., $\sin(lx)$ vs. $\cos(lx)$ in Eqs. 7-8 or in Eq. 4 of Shen 2015). Alternatively, via the $\partial\theta/\partial x$ and $\Delta T\partial\psi/\partial x$, the M_4 and M_5 modes are linked, as discussed in section 3.1 in the submitted manuscript (Shen 2015).

When M_4 is included, it improves the representation of the streamfunction and thus the advection of temperature, as shown:

$$\psi = C_1 \left(X M_1 + X_1 M_4 \right), \quad (21)$$

$$J(\psi, \theta) = C_1 C_2 \left(J(X M_1 + X_1 M_4, Y M_2 + Y_1 M_5 - Z M_3 - Z_1 M_6) \right), \quad (22)$$

here X_1 represents the amplitude of the mode M_4 . Now, the Jacobian term includes $J(X M_1, Y M_2 + Y_1 M_5 - Z M_3 - Z_1 M_6)$ and $J(X_1 M_4, Y M_2 + Y_1 M_5 - Z M_3 - Z_1 M_6)$. The former was first discussed in the 5DLM by Shen (2014), while the latter is discussed using the 6DLM in this study. While the M_4 mode introduces linear forcing term (e.g., $r X_1$), it also extends the nonlinear feedback loop with $J(X_1 M_4, Y M_2)$, $J(X_1 M_4, Y_1 M_5)$, $J(X_1 M_4, Z M_3)$, and $J(X_1 M_4, Z_1 M_6)$. The outcome of each of these Jacobian terms can be found in the Table 2 of Shen (2014), and the impact of M_4 is discussed in Shen (2015).

2.4 Further extensions of the nonlinear feedback loop in Higher-order LMs

To examine the role of the nonlinear feedback loop in the solution stability of higher-order LMs, we have derived the following higher-dimensional Lorenz models, including 7D, 8D and 9D LMs. These models give a larger critical value of the normalized Rayleigh parameter for the onset of chaos, as compared to the 3D, 5D and 6D Lorenz models. A manuscript is being prepared for publication (Yoo and Shen, 2015). Here, a brief description for the higher-order LMs is given as follows:

1. 7DLM includes all modes in the 5DLM and the M_8 and M_9 modes (Eq. 9) that can improve the representation of θ and $J(\psi, \theta)$ and to extend the nonlinear feedback loop to provide negative nonlinear feedback;

2. 8DLM contains all modes in the 7DLM and the M_4 mode (Eq. 7) that can improve the representation of ψ and $J(\psi, \theta)$;
3. 9DLM includes all modes in the 8DLM and an additional mode M_7 (Eq. 7) to improve the representation of ψ and $J(\psi, \theta)$.

Note that M_8 with $\sin(5mz)$ is selected based on the analysis of $J(M_1, M_6)$ as shown in the Table 2 of Shen (2014). M_9 is added to enable the downscale transfer from $J(M_1, M_8)$. Similar to the inclusion of M_4 , M_7 is introduced to have a different phase to that of M_8 .

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