Dear Editor:

Please consider for publication on *Nonlinear Processes in Geophysics*
the following revised manuscript
"
Nonlinear feedback in a six-dimensional Lorenz model: impact of an additional heating term

"
by Bo-Wen Shen

In this study, we investigate the competing impact of the additional heating term and the negative nonlinear feedback on solution's stability using a 6-dimensional Lorenz model (6DLM), which is an extension of the original 3DLM (Lorenz, 1963) and the 5DLM (Shen, 2014a). We thank the reviewers and Editor for providing us the opportunity for explaining further the progress we have made. In the revised manuscript, our major conclusion remains the same, which can be stated as follows:
*While the negative nonlinear feedback associated with two new modes in the 5DLM can stabilize solutions, additional resolved heating processes by a third mode in the 6DLM can destabilize solutions. The findings support the view of Lorenz (1972) on the role of small scale processes: If the flap of a butterfly's wings can be instrumental in generating a tornado, it can equally well be instrumental in preventing a tornado.*

In response to the reviewer's comments, we have

1. provided derivations in a separable file to identify the nonlinear feedback loop in the original 3DLM and to discuss how the nonlinear feedback loop can be extended with proper selections of new modes in the 5DLM (Shen 2014a), 6DLM (Shen 2015, this manuscript), as well as 7D, 8D and 9D LMs (Yoo and Shen, 2015, a manuscript in preparation).
2. provided Tables 2 and 3 in this response file to be compared with Table 1 of Roy and Musielak (2007c), showing the Fourier modes that have been used in different higher-order Lorenz models.

The revised manuscript with tables is uploaded as a Latex file and a PDF file. Note that figures remain the same, so they are not included in the PDF file. In addition, another PDF file with the mathematical analysis of the nonlinear feedback loop and its extension is uploaded as supplemental materials. Thank you for your consideration!

Best Regards,

-Bowen

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General responses:

I would like to thank the reviewers and Editor for their valuable comments. One of the major concerns raised by both reviewers is how new modes were selected to derive the 6DLM. Here, I would like to emphasize (1) that based on the analysis of the Jacobian term, $J(\psi, \theta)$, new modes are selected to extend the nonlinear feedback loop that can provide additional nonlinear feedback to stabilize or destabilize solutions; and (2) that our approach, using incremental changes in the number of Fourier modes, is to help trace their individual and/or collective impact on the solution stability as well as the extension of the nonlinear feedback loop. To facilitate discussions, we have (a) created two tables which list the Fourier models used to construct different higher-order Lorenz models and the corresponding critical values of the normalized Rayleigh parameter for the onset of chaos; and (b) finished a pdf file with a brief summary on the mathematical analysis of the nonlinear feedback loop in the 3DLM and its extension in the 5DLM and 6DLM. The tables are included in the end of this response file, while the pdf file will be uploaded separately as supplemental materials. In the following, specific responses are given with the aid of the supplemental materials.

(A) Responses to Reviewer I's comments:

I carefully read the paper several times. The principle question is: Why was this paper written, in principle, and what basic problem[s] is [are] discussed here? The author attempts to discuss a problem of stability of an expanded Lorenz model through the Lyapunov exponent analysis. There are no grammatical errors in the paper, except small ones (e.g., "Model" in capitals in the title, the capture for Figure 7). However there are several problems which should be discussed before the paper is considered for publication

Thanks for your comments. The minor issues have been corrected in the revised manuscript accordingly. Detailed responses to the specific comments are given below.

In this study, the hypothesis to be verified and question to be addressed is stated in the introduction as follows:

``*Shen14 hypothesized that system's stability in the LMs, with a finite number of modes, can be improved with additional modes that provide negative nonlinear feedback associated with additional dissipative terms.*''

`` *However, since new modes can also introduce additional heating term(s), the competing role of the heating term(s) with nonlinear terms and/or with dissipative terms deserves to be examined so that the conditions under which solutions become more stable or chaotic can be better understood.''*

`` *Results obtained from work described here and the work of Shen (2014a) are used to address the following question: for generalized LMs, under which conditions can the increased degree of nonlinearity improve solution stability?''*

In fact, our studies have been performed to help achieve the ultimate goal of determining under which conditions increasing resolutions can improve the predictions in weather/climate models. In our first papers (Shen 2014a and this manuscript), we found that a nonlinear feedback loop in the baseline model (e.g., 3DLM) plays an important role in determining the predictability and its extension may help provide negative nonlinear feedback to improve the predictability. After identifying the impacts of new modes in the 5DLM (which contains the negative nonlinear feedback) and in the 6DLM (which including an additional heating term), we currently extend these studies to examine the role of the extended nonlinear feedback loop and additional heating terms in the solution's stability for 7D, 8D and 9D LMs (Yoo and Shen, 2015, in preparation).

> First, I suppose that the model, which was used for the analysis, should be in the form….
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> For simplicity, coefficients in Eqs (1)-(8) have been omitted to understand the structure of this system. The author used another model. Why? How that model was obtained? It is necessary to explain how that model corresponds to Eqs (1)-(8).

In the following, I will use Table 1 (derived from Roy and Musielak, 2007) and Table 2 to show that our 6DLM is the same as the one in Kennamer (1995), which is not a subset of the the aforementioned 8DLM. More importantly, we want to point out that higher-order LMs display the dependence of rc on the selections of Fourier modes (e.g., Table 1), and that proper selections of Fourier modes, based on the analysis of Jacobian term $J(\Psi,\theta)$, can help extend the nonlinear feedback loop that can provide negative nonlinear feedback to stabilize solutions (e.g., 5D, 6D and 7D LMs).

Table 1, which is included near the end of this response file, provides a list of high-order Lorenz models, including two 6D LMs and 8DLM, and the corresponding Fourier modes used to construct the LMs. It is shown that the 6DLM by Kennamer is not a subset of the 8DLM. As mentioned in the manuscript, our 6DLM is the same as the one from Prof. Musielak's group. [Prof. Musielak is Kennamer's advisor. Kennamer published the 6DLM in his/her master thesis, which is not available to the author. The first literature listing the 6DLM of Kennamer is Musielak, Musielak and Kenamer, 2005, which was cited in the manuscript.]. Specifically, the M4, M5 and M6 modes in our 6DLM are exactly the same as $\psi_1(1,3)$, $\Theta_2(1,3)$ and $\Theta_2(0,4)$ in the 6DLM by Kennamer, respectively. However, we derived the 6DLM independently. In addition, we provided the analysis of the Jacobian term, $J(\Psi,\theta)$, to show how the 6DLM is constructed to be an "extension" of the 5DLM. Namely, the former is a superset of the latter. In the supplementary materials, we provide more detailed discussions on the linkage between the 3DLM, 5DLM and 6DLM. In the manuscript, we discuss the impact of additional heating term on solution's stability and the conservation laws for the 6DLM in the dissipationless limit, the latter of which were only partially discussed by Prof. Musielak's group.

Table 2 lists the Fourier modes used to construct our higher-order LMs (e.g., Shen 2014a, 2015), the 14DLM by Curry (1978) and the one by Lucarini and Fraedrich (2009). In addition, it also lists the critical value of the normalized Rayleigh parameter (rc) for the onset of chaos. In the 5DLM, we first showed that the nonlinear feedback loop can be extended through the inclusion of M5 and M6 (which are the same as $\Theta_2(1,3)$ and $\Theta_2(0,4)$, respectively). Compared to the 5DLM, the 6DLM includes an additional mode M4, (i.e., $\psi_1(1,3)$), and has a comparable rc. Currently, based on the analysis of nonlinear feedback loop, we add two modes, $\Theta_2(1,5)$ and $\Theta_2(0,6)$ to extend the nonlinear feedback loop of the 5DLM, which becomes the 7DLM with a much larger rc (e.g., rc~116.9 in Table 2). To be more consistent, additional modes with $\psi_1(1,3)$ and $\psi_1(1,5)$ are being added to derive the 8DLM and 9DLM. All of the three LMs, 7D-9DLMs, have the rc greater than 100. More detailed analyses with the eLE calculation are being performed (e.g., Yoo and Shen, 2015, in preparation).

It is worth noting that the 14DLM, which was shown to be not conservative in the disppiationless limit, is a superset of the 6DLM. However, the vertical wavenumbers in both 6DLM and 14DLM are the same and their critical values of the normalized Raleigh parameter are comparable. In contrast, our 7D-9D LMs include modes with higher wavenumbers, such as 5m and 6m in $\Theta_2(1,5)$ and $\Theta_2(0,6)$, to extend the nonlinear feedback loop, which can stabilize solutions and lead to a larger rc (e.g., Table 2).

> Second, independently from the used model the principle problem for systems like (1)-(8) is not their stability but how different dynamical regimes are realized in such a model. For example, how the regime changes for increasing Ra, where Ra is Rayleigh number. In general, system (1)-(8) was early studied by numerical methods and it has been demonstrated that there are several interesting effects. For example, a 3D attractor does not develop because another attractor with a higher dimensionality exists.

The dependence of the solution's stability over a range of the normalized Rayleigh parameter (r) and the Prandtl number (σ) is discussed in Figure 7 of section 3.4 in the manuscript. We do observe the dependence of rc on the Prandtl number in the LMs. However, given a value of σ, the 6DLM (as well as the 5DLM) is always more predictable than the 3DLM.

From a perspective of weather/climate prediction, our ultimate goal is to apply what we can learn from the idealized high-order LMs to understanding and improving the predictability of the weather and/or climate models. Specifically, it is important to understand if and how the increased resolutions in the weather/climate models can suppress or enhance chaotic responses, because high-resolution global modeling, which is a current trend, requires tremendous computing resources. To achieve our goal, we started examining the impact of increased degree of nonlinearity on solution's stability in the 3DLM and higher-order LMs, and trying to implement suitable methods (e.g., trajectory separation method) into the weather/climate models to perform stability analysis (e.g., calculations of Lyapunov exponent). More details in our modeling approach with the higher-order LMs are provided below.

Our approach by incrementally changing the number of modes can help examine the individual and combined impact of resolved processes by the new modes. For example, the 5DLM was used to examine the impact of the negative nonlinear feedback (from the additional nonlinear and dissipative terms in association with the two new modes, M5 and M6 modes), while the 6DLM with the inclusion of M4 mode, which is a superset of 5DLM, allows us to examine the competing impact of an additional heating term and dissipative terms on the solution's stability. We then suggest that negative nonlinear feedback associated with new modes (M5 and M6) in the 6DLM as well as 5DLM can stabilize solutions, and that the additional heating term with the M4 mode in the 6DLM can destabilize solutions.

The principle problem is how to use systems like (1)-(8) and of higher dimensionality for the practical analysis. In any case, the dimensionality larger than 6-8 is required to discuss a reality.

My suggestion is that the editor should decide if this paper is suitable for publication at NPG. In my opinion, it needs a major revision before being considered for publication. In its present form, the manuscript does not fit the journal scope because NPG is not a pure mathematical journal.

Our ultimate goal is to examine the impact of increased resolution on the predictability of the real-world weather/climate models (e.g., Shen et al., 2006a). We have been working to implement the trajectory separation (TS) method into our global model for the eLE calculation. In addition, we still continue to improve our understanding of the nonlinear feedback loop in higher-dimensional LMs. For example, since the Spring semester of 2015, I have supervised one master student to derive the 7D, 8D and 9D LMs by analyzing the nonlinear Jacobian term, $J(\psi, \theta)$, and selecting new modes (M7, M8 and M9 modes in Table 2) that can extend the nonlinear feedback loop. With that being said, we believe that the related discussions on the role of different physical processes (i.e., dissipative and heating processes) in solution's stability meet the goal of the NPD journal: *(submissions that) apply nonlinear analysis methods to both models and data.*

**Responses to reviewer II's comments**

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| This paper has the merit of studying how changing the structure of feedbacks impact some of the most important properties of a minimal truncated set of equations describing convection. While the paper has indeed merits, I would recommend the authors to improve the discussion on the physical relevance of their results and put them in a broader context of the published literature. |

Thanks for your comments. I have done my best to address the concerns and comments in the following.

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|---|
| 1) The authors should make clear that the problem was first studied by Salzman, who gave a very extensive treatment of the possibility of constructing reduced order models. Lorenz then studied one of such models and got such an incredible result. Also, in the following paper it is discussed that the L63 model is a member of a class of equivalence: Z.-M. Chen and W. G. Price. Chaos. Solitons Fractals 28. 571 2006 . |

Thanks for your suggestions. We have revised the manuscript accordingly and cited the paper of Chen and Price (2006).

Here, we provide a brief discussion on how "symmetry" was introduced in the model of Chen and Price (2006) and will discuss about "symmetry breaking" in the responses to the 2nd question. In the 3DLM, the streamfunction is represented by one Fourier mode, $\psi_1(1,1) = \sqrt{2}\sin(lx)\sin(mz)$, and temperature is represented by two Fourier modes, $\Theta_2(1,1) = \sqrt{2}\cos(lx)\sin(mz)$ and $\Theta_2(0,2) = \sin(2mz)$. Chen and Price (2006) suggested represent the streamfunction by both $\sqrt{2}\sin(lx)\sin(mz)$ and $\sqrt{2}\cos(lx)$, and temperature by the two modes and $\sin(2mz)$ (e.g., Eq. 12 of Chen and Price, 2006). Their approach produces a model with 5 ordinary differential equations (ODEs) that introduces the ``symmetry." And thus, the attractor in the 3DLM is a cross section of the attractor of the Chen and Price's model. Note that horizontal and vertical wavenumbers in the 3DLM and the model of Chen and Price are the same. Next, we discuss on how inclusion of higher wavenumber modes may break the symmetry.

2) In presenting their models, the authors should explain more clearly that they have to be derived from the continuum equation by suitable truncation.

It is not clear to me why the authors choose such a truncation, where the first and third modes are used, while the second modes are left out, a bit arbitrarily, from my point of view. Could the authors explain the rationale for this?

In an earlier paper

*V. Lucarini and K. Fraedrich, Symmetry breaking, mixing, instability, and low-frequency variability in a minimal Lorenz-like system, PRE 80, 026313 2009*

we took a different route, constructing the truncation starting from below, including all the 1- and 2- modes, and adding the same two modes here indicated as M3 and M6. In this way, we also obtain a closed system of equations. Maybe the authors could consider looking into that paper, where it is explained that in order to achieve a complete thermodynamic consistency (see also C. Nicolis, Q. J. R. Meteorol. Soc. 125, 1859 1999 on the Thermodynamic Relevance of the L63 model) some modifications have to be implemented.

Using Table 2 in the response file and supplemental materials, I would like to emphasize the importance in selecting the M5 mode (i.e., $\Theta(1,3)$ in Table 2), based on the analysis of Jacobian term, $J(\Psi, \theta)$. The inclusion of M5 and M6 can extend the original nonlinear feedback loop in the 3DLM to provide negative nonlinear feedback to stabilize the solution in the 5DLM and 6DLM.

I appreciate the reviewer's comments and sharing. Lucarini and K. Fraedrich (2009) is being cited among the studies with the high-order Lorenz models in the revised manuscript. Their study focused more on "symmetry breaking", but their model does not include $\Theta(1,3)$ (i.e., M5) and $\psi(1,3)$ (i.e., M4). In comparison, we have implicitly discussed the "symmetry breaking" using higher wavenumber modes (e.g., M5 mode) that are included to lead to a (new) downscale transfer, which leads to asymmetry as discussed below using Eq. 11-13 and Eq. 17 in the supplemental materials). As the 3DLM does not include the M5 mode, Eq. (13) for J ($M_1$, $M_3$) includes only an upscale transfer but neglect a downscale transfer. Therefore, the nonlinear feedback loop of the 3DLM with Eqs. (11) and (13) leads to "spurious" symmetry with respect to the z-axis and invariant for (X, Y) → (-X, -Y). Namely, if (X, Y, Z) is a solution, (-X, -Y, Z) is also a solution. This kind of symmetry does not exit in the 5DLM with the inclusion of the M5 mode, which allows the downscale transfer for J ($M_1$, $M_3$), as shown in Eq. 17 in the supplemental materials.

In addition, I have also started paying attention to the "asymmetric characteristics" of the multiscale interactions in weather and climate. In the real world modeling studies (Shen et al., 2012, p25), we made the following statements regarding the asymmetric nature of multiscale interactions:

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*Based on the current study and two previous studies [Shen et al., 2010a; 2010b] using the global mesoscale model, the following view on the predictability of mesoscale tropical cyclone genesis is proposed: (1) Both a downscaling cascade of processes associated with the large-scale systems and an upscaling cascade of processes associated with the small-scale (e.g., precipitation) systems are important. Because of the asymmetry in the spatial and temporal scales and the strengths among these systems, the term "hierarchical multiscale interactions" is used to describe these scale interactions that can lead to TC formation. (2) ...*

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Speaking of the symmetry breaking, I would like to make the following comments as a brief comparison, which may be extended in a future study. By taking a approach which is similar to the one by Chen and Price (2006), Lucarini and Fraedrich (2009) included additional modes with higher horizontal and vertical wave numbers to construct the model for examining the symmetry breaking and other interesting characteristics. The new modes are $\psi_1(2,2)$, $\Theta_2(2,2)$ and $\Theta_2(0,4)$. Note that the first two modes were used in the 14DLM of Curry (1978), but they are not used in our 5DLM, 6DLM or 7D-9D LMs, or any other models in Tables 1-3. Based on the aforementioned analysis using our higher-order LMs, the modes with higher vertical wavenumbers may break the symmetry. However, as shown in Table 3, new modes with different horizontal wave numbers for streamfunction may introduce new terms via both $J(\psi, \nabla^2\psi)$ and $J(\psi, \theta)$), which can make it complicated to compare the models with our models. Moreover, $\Theta_2(0,4)$ is the same as our M6, but the M5 mode (i.e., $\psi_1(1,3)$), which is selected to extend the nonlinear feedback loop, is not included in their model (see Table 2 for a comparison). Therefore, it requires efforts to compare the characteristics of solutions from the model of Lucarini and Fraedrich (2009) and our models, which deserves a future study. In this study, we focus on the extension of nonlinear feedback loop associated with $J(\psi, \theta)$, i.e., no nonlinear terms from $J(\psi, \nabla^2\psi)$ (see Shen 2014a for more details).

For the consistency of dynamics and thermodynamics in the higher-order Lorenz models, we have discussed the conservation laws for both 5DLM and 6DLM in the dissipationless limit (in section 3.3 of Shen 2015). From a thermodynamic perspective, we discussed how additional nonlinear terms and dissipative terms introduced by the M5 and M6 modes can provide negative nonlinear feedback to stabilize solutions in both 5DLM and 6DLM. These processes seem consistent with what has been documented in Lucarini and Fraedrich (2009, p026313-4), as follows:

``

*We then conclude that, in spite of introducing a second unstable direction, which is responsible for mixing the phase of the waves, inclusion of the impact of viscous dissipation on the thermal energy balance acts with continuity on the dynamical indicators, by reducing the overall instability, increasing the predictability of the system, and by confining the asymptotic dynamics to a more limited (in terms of dimensionality) set.*

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> 3) If possible, I would recommend the authors to discuss a bit the fact that the fractal dimension is similar for all their models when they are all in the chaotic regime. What is their take on this?
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> Also: the first Lyapunov exponent is almost identical in the 3D and one of the 6D model.
>
> Can they find a correspondence also for the other non-zero LE of the 3D model?

Thanks for your comments and suggestions on the calculation of the fractal dimension. We presented the calculation of fractal dimension with 3DLM (as well as 5DLM and 6DLM) in the appendix to provide additional support to the implementation of the GSR scheme. As nonlinearity is viewed as the source of chaos in the 3DLM, some people inferred that systems with more nonlinear terms may become more chaotic. However, using our 5DLM and 6DLM, we have tried to point out that proper selection of new modes can extend the nonlinear feedback loop to provide negative feedback to stabilize the solutions. Therefore, compared to the 3DLM, a comparable fractal dimension in our 5DLM and 6DLM, which is smaller than those in other high-order LMs, may be consistent with the fact that the nonlinear feedback loop is extended in the 5DLM/6DLM. In each of 3DLM, 5DLM, 6DLM, the first eLE is positive, and the second eLE is close to zero. In higher-order LMs (e.g., 5D and 6 LMs), additional negative eLEs were found, which are indicated by the summation of the eLEs, which is -30.667 (-94.0) for the 5DLM (6DLM) (see Figure A1 in the manuscript).

In addition, Figure 7a of Shen (2014) provide a better representation for the comparison of LEs between the 3DLM and 5DLM. In both models, LEs are comparable when r is small, but differences appear when r becomes larger (e.g., r>95). As additional nonlinear terms and ``dissipative'' terms are introduced in the 5DLM, additional negative eLEs are obtained. Again, this is shown in Figure A1 that the sum of eLEs is -13.667 for the 3DLM but becomes -30.667 for the 5DLM.

Compared to the 5DLM, the 6DLM introduces an additional heating term, $rX_1$. However, as discussed in the manuscript, the magnitude of $X_1$ is small, and thus it causes smaller differences, as compared to negative nonlinear feedback that is associated with the additional nonlinear terms and dissipative terms. Therefore, the 6DLM has a comparable but smaller rc than the 5DLM. While this study focuses on the improvement of solution's stability, we agree with the reviewer that it is important to perform detailed analysis on the each of eLEs, which will be conducted in a future study with the higher-order LMs.

Some additional comments

a) When discussing the Lyapunov exponents, the authors might consider referring to G. Benettin, L. Galgani, A. Giorgilli, and J. M. Strelcyn, Meccanica 15, 9 1980 as they first discussed the benefits of the GS method.

b) Also: the effect of mode truncation was extensively studied by Franceschini et al. V. Franceschini and C. Tebaldi, Meccanica 20, 207 1985; V. Franceschini, C. Giberti, and M. Nicolini, J. Stat. Phys. 50, 879 1988

c) Appendix: Attention: you are citing different definitions of fractal measure. They are not equivalent. See Ruelle 1989. Ruelle, Chaotic Evolution and Strange Attractors, 1989

d) What does it mean that the second Lyapunov exponent is not zero (of course it has to)? Of course it cannot ever be exactly zero. It will converge only asymptotically to that value. The authors might consider adding error bars to their estimates.

(a) Thanks for your comments. Benettin et al. (1980) has been cited in the revised manuscript.

(b) We have cited these studies. Once again, I would like to emphasize that the uniqueness of our approach is to incrementally select new modes to extend the nonlinear feedback loop, based on the analysis of $J(\psi, \theta)$, and to examine the individual and combined impact of new terms associated with the new modes on solution's stability.

(c) Agree. We are aware of the different definitions of fractal dimensions and methods for their calculations. We have revised the manuscript accordingly, and stated that only KY fractal dimension is discussed, because it is calculated using the LEs. Through the calculation of the KY fractal dimension, we provide additional verification for the ensemble LE calculation.

(d) For the calculation of the (global) LE, it requires the integration of two trajectories (in the control and parallel runs) over an infinite period of time (e.g., the T in Eq. 23 of Shen 2014a should approach infinity). However, in reality, numerical integration (or summation) is applied only over a finite period of time. Therefore, given a specific set of

ICs, a period of T=1,000 may not be sufficient to determine the global LE. In addition, in order to minimize the dependence of initial ICs, we calculate LEs with 10,000 different ICs to obtain the ensemble averaged LE (eLE). A non-zero value of LE in any one of 10,000 ensemble runs can contribute to the non-zero eLE. The above reasons may explain why the $2^{nd}$ eLE is small but is not exactly equal to zero. However, as it is small (compared to the $1^{st}$ eLE), it should not have significant impact on the calculation of the fractal dimension, which requires the summation of the first two eLEs. Currently, a manuscript regarding the scientific and parallel performance of the implementation for the calculation of eLE is being prepared. For now, we added the following sentences in Italics in the revised manuscript:

Here, the reader should note that the 2nd eLE is very small but not exactly equal to zero, indicating the impact of the 10000 different initial conditions *and/or the "finite" integration time (T = 1000) in this study*.

**References** (which have been included in the revised manuscript)


Benettin, G., L. Galgani, A. Giorgilli, and J. M. Strelcyn, 1980: Lyapunov Characteristic Exponents fro Smooth Dynamical Systems and for Hamiltonian Systems; A method for computing all of them. Part 1: Theory. Meccanica 15, 9-20.

Chen, Z.-M. and W. G. Price, 2006: On the relation between Raleigh-Benard convection and Lorenz system. *Chaos, Solitons Fractals*, 28, 571-578.

Franceschini, V. and C. Tebaldi, 1985: Truncations to 12, 14 and 18 Modes of the Navier-Stokes Equations on a Two-Dimensional Torus. Meccanica 20, 207-230.

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Nicolis, C., 1999: Entropy production and dynamical complexity in a low-order atmospheric model. Q. J. R. Meteorol. SOC., 125, pp. 1859-1 878

Pelino, V., F, Maimone, A. Pasini, 2004: Energy cycle for the Lorenz attractor, Chaos, Solitons & Fractals 64 (2014), 67–77.

Ruelle. D., 1989: *Chaotic Evolution and Strange Attractors*. [Online]. Lezioni Lincee. Cambridge: Cambridge University Press. Available from: Cambridge Books Online <**http://dx.doi.org/10.1017/CBO9780511608773**> [Accessed 27 September 2015].

Yoo, E. and B.-W. Shen, 2015: On the extension of the nonlinear feedback loop in 7D, 8D and 9D Lorenz models. (in preparation)

Table 1: Fourier modes selected to construct the 3DLM and higher-order LMs, which is from Table 1 of Roy and Musielak (2007c). The critical values of the normalized Raleigh parameter, shown in red, are derived from Table 2 of Roy and Musielak (2007c).

Table 1
Fourier modes selected to construct the original 3D Lorenz system and other generalized Lorenz systems

| Model | Circulation modes | Temperature modes | Temperature modes with $m = 0$ | | References |
|---|---|---|---|---|---|
| 3D | $\Psi_1(1,1)$ | $\Theta_2(1,1)$ | $\Theta_2(0,2)$ | rc~24.75 | Lorenz [1] |
| 5D | $\Psi_1(1,1)$<br>$\Psi_1(2,1)$ | $\Theta_2(1,1)$<br>$\Theta_2(2,1)$ | $\Theta_2(0,2)$ | rc~22.50 | Paper II |
| 6D | $\Psi_1(1,1)$<br>$\Psi_1(2,1)$<br>$\Psi_1(1,2)$ | $\Theta_2(1,1)$<br>$\Theta_2(2,1)$ | $\Theta_2(0,2)$ | n/a | Humi [9] |
| 6D | $\Psi_1(1,1)$<br>$\Psi_1(1,3)$ | $\Theta_2(1,1)$<br>$\Theta_2(1,3)$ | $\Theta_2(0,2)$<br>$\Theta_2(0,4)$ | rc~40.15 | Kennamer [10] |
| 8D | $\Psi_1(1,1)$<br>$\Psi_1(2,1)$<br>$\Psi_1(1,2)$ | $\Theta_2(1,1)$<br>$\Theta_2(2,1)$<br>$\Theta_2(1,2)$ | $\Theta_2(0,2)$<br>$\Theta_2(0,4)$ | rc~35.60 | This Paper |
| 9D | $\Psi_1(1,1)$<br>$\Psi_1(1,2)$<br>$\Psi_1(1,3)$ | $\Theta_2(1,1)$<br>$\Theta_2(1,2)$<br>$\Theta_2(1,3)$ | $\Theta_2(0,2)$<br>$\Theta_2(0,4)$<br>$\Theta_2(0,6)$ | rc~40.50 | Paper I |

Table 2: Fourier modes used in our high-order LMs (e.g., Shen 2014a, 2015; Yoo and Shen, 2015) and the models by Curry (1978) and Lucarini and K. Fraedrich (2009). Note that $M_4 = \psi_1(1,3)$, $M_5 = \Theta_2(1,3)$, and $M_6 = \Theta_2(0,4)$.

| model | $\Psi$ | $\Theta$ | $\Theta$ | rc | References |
|---|---|---|---|---|---|
| 5DLM | $\psi_1(1,1)$ | $\Theta_2(1,1)$, $\Theta_2(1,3)$ | $\Theta_2(0,2)$, $\Theta_2(0,4)$ | 42.9 | Shen (2014) |
| 6DLM | $\psi_1(1,1)$, $\psi_1(1,3)$, | $\Theta_2(1,1)$ $\Theta_2(1,3)$ | $\Theta_2(0,2)$ $\Theta_2(0,4)$ | 41.1 | Shen(2015) |
| 7DLM | $\psi_1(1,1)$ | $\Theta_2(1,1)$, $\Theta_2(1,3)$, $\Theta_2(1,5)$ | $\Theta_2(0,2)$, $\Theta_2(0,4)$, $\Theta_2(0,6)$, | ~116.9 | Yoo and Shen (2015, in preparation) |
| 8DLM | $\psi_1(1,1)$, $\psi_1(1,3)$ | $\Theta_2(1,1)$, $\Theta_2(1,3)$, $\Theta_2(1,5)$ | $\Theta_2(0,2)$, $\Theta_2(0,4)$, $\Theta_2(0,6)$ | ~105 (TBD with the eLE analysis) | Yoo and Shen (2015) |
| 9DLM | $\psi_1(1,1)$, $\psi_1(1,3)$, $\psi_1(1,5)$ | $\Theta_2(1,1)$, $\Theta_2(1,3)$, $\Theta_2(1,5)$ | $\Theta_2(0,2)$, $\Theta_2(0,4)$, $\Theta_2(0,6)$ | ~105 (TBD with the eLE analysis) | Yoo and Shen (2015) |
| 14DLM | $\psi_1(1,1)$, $\psi_1(1,3)$, $\psi_1(2,2)$, $\psi_1(2,4)$, $\psi_1(3,1)$, $\psi_1(3,3)$ | $\Theta_2(1,1)$, $\Theta_2(1,3)$, $\Theta_2(2,2)$, $\Theta_2(2,4)$, $\Theta_2(3,1)$, $\Theta_2(3,3)$ | $\Theta_2(0,2)$, $\Theta_2(0,4)$ | rc~43 | Curry (1978) |
| 10EQs | $\psi_1(1,1)$ $\psi_1(2,2)$ | $\Theta_2(1,1)$, $\Theta_2(2,2)$ | $\Theta_2(0,2)$, $\Theta_2(0,4)$ | n/a | Lucarini and K. Fraedrich (2009) |

Table 3: Lorenz models with different Fourier modes. 3DLM and 5DLM are discussed in the manuscript, while the 6DLM will be discussed in a companion paper. 6DLM_HK is referred to as the 6DLM proposed by Howard and Krishnamurti (1986). 7DLM_TH and 7DLM_Hetal are referred as the 7DLMs proposed by Thiffeault and Horton (1995) and Hermiz et al. (1995), respectively. The one denoted as '8DLM (suggested)' was suggested by Thiffeault and Horton (1995) who did not derive the 8DLM nor discuss its characteristics. Only one horizontal wave number was used in the first several Lorenz models. cos(2lx) was used in the 8DLM by Roy and Musielak (2007c), denoted as 8DLM_RM. $M_1$-$M_6$ are defined in the manuscript. $M_a$-$M_d$ are defined as sin(mz), cos(lx)sin(2mz), sin(lx)sin(2mz), and sin(3mz), respectively. In the studies by Howard and Krishnamurti (1986) and Hermiz et al. (1995), the symbol 'α' is equivalent to 'a' in our study, which is equal 'l/m', namely α=a=l/m.

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 3DLM --ψ | sin(lx)sin(mz) | | | | |
| | θ | cos(lx)sin(mz) | sin(2mz) | | | |
| 2 | 5DLM --ψ | sin(lx)sin(mz) | | | | |
| | θ | cos(lx)sin(mz) | sin(2mz) | cos(lx)sin(3mz) | sin(4mz) | |
| 3 | 6DLM --ψ | sin(lx)sin(mz) | sin(lx)sin(3mz) | | | |
| | θ | cos(lx)sin(mz) | sin(2mz) | cos(lx)sin(3mz) | sin(4mz) | |
| 4 | 6DLM_HK --ψ | sin(lx)sin(mz) | sin(mz) | cos(lx)sin(2mz) | | |
| | θ | cos(lx)sin(mz) | sin(2mz) | sin(lx)sin(2mz) | | |
| 5 | 7DLM_TH-- ψ | sin(lx)sin(mz) | sin(mz) | cos(lx)sin(2mz) | | |
| | θ | cos(lx)sin(mz) | sin(2mz) | sin(lx)sin(2mz) | sin(4mz) | |
| 6 | 7DLM_Hetal -- ψ | sin(lx)sin(mz) | sin(mz) | cos(lx)sin(2mz) | sin(3mz) | |
| | θ | cos(lx)sin(mz) | sin(2mz) | sin(lx)sin(2mz) | | |
| 7 | 8DLM (suggested) | sin(lx)sin(mz) | sin(mz) | cos(lx)sin(2mz) | sin(3mz) | |
| | θ | cos(lx)sin(mz) | sin(2mz) | sin(lx)sin(2mz) | sin(6mz) | |
| 8 | 8DLM_RM---- ψ | sin(lx)sin(mz) | sin(lx)sin(2mz) | | | sin(2lx)sin(mz) |
| | | cos(lx)sin(mz) | sin(2mz) | cos(lx)sin(2mz) | sin(4mz) | cos(2lx)sin(mz) |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 3DLM --ψ | $M_1$ | | | | |
| | θ | $M_2$ | $M_3$ | | | |
| 2 | 5DLM --ψ | $M_1$ | | | | |
| | θ | $M_2$ | $M_3$ | $M_5$ | $M_6$ | |
| 3 | 6DLM --ψ | $M_1$ | $M_4$ | | | |
| | θ | $M_2$ | $M_3$ | $M_5$ | $M_6$ | |
| 4 | 6DLM_HK --ψ | $M_1$ | $\mathbf{M_a}$ | $\mathbf{M_b}$ | | |
| | θ | $M_2$ | $M_3$ | $\mathbf{M_c}$ | | |
| 5 | 7DLM_TH-- ψ | $M_1$ | $\mathbf{M_a}$ | $\mathbf{M_b}$ | | |
| | θ | $M_2$ | $M_3$ | $\mathbf{M_c}$ | $M_6$ | |
| 6 | 7DLM_Hetal -- ψ | $M_1$ | $\mathbf{M_a}$ | $\mathbf{M_b}$ | $\mathbf{M_d}$ | |
| | θ | $M_2$ | $M_3$ | $\mathbf{M_c}$ | | |
| 7 | 8DLM (suggested) | $M_1$ | $\mathbf{M_a}$ | $\mathbf{M_b}$ | $\mathbf{M_d}$ | |
| | θ | $M_2$ | $M_3$ | $\mathbf{M_c}$ | sin(6mz) | |
| 8 | 8DLM_RM---- ψ | $M_1$ | sin(lx)sin(2mz) | | | sin(2lx)sin(mz) |
| | | $M_2$ | $M_3$ | cos(lx)sin(2mz) | $M_6$ | cos(2lx)sin(mz) |