

1 **Possible explanation of the values of Hack's drainage basin, river length scaling exponent**

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12 **Abstract**

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14 Percolation theory can be used to find water flow paths of least resistance. Application of
15 percolation theory to drainage networks allows identification of the range of exponent values that
16 describe the tortuosity of rivers in real river networks, which is then used to generate the
17 observed scaling between drainage basin area and channel length, a relationship known as
18 Hack's law. Such a theoretical basis for Hack's law may allow interpretation of the range of
19 exponent values based on an assessment of the heterogeneity of the substrate.

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22 **1. Introduction**

23 River networks display complex organization as documented in numerous studies. This
24 work addresses but one of them. In particular, the relationship between drainage basin area and
25 river length is non-trivial. In Euclidean geometry the basin area, A , would be proportional to the
26 square of the river length, l , i.e., l should be proportional to $A^{1/2}$. In actuality, as determined by
27 Hack (1957), this relationship is

$$l = CA^\beta$$

28 with the value of β approximately 0.6. Later investigations did not always return the identical
29 value of β . Nevertheless, Maritan et al., (1996) consider the value of β to be well constrained and
30 refer to a study of Gray (1961) as having established that “the accepted values for the exponent
31 $[\beta]$ are in the range 0.57 to 0.6.”

32 Hack (1957) (page 65) asserts that the relationship was a consequence of the lengthening
33 of drainage basins with increasing size. But Montgomery and Dietrich (1992) compare straight-
34 line basin length, L , to A over seven orders of magnitude of length scale and find precisely $L =$
35 $A^{0.5}$. That result allows stream length to be expressed in terms of the straight line basin
36 dimension, $l = A^\beta = L^\gamma = L^{2\beta}$, so that $\gamma = 2\beta$. The exponent $\gamma > 1$ then defines the tortuosity
37 (sometimes known as sinuosity) of the stream path through the drainage basin.

38 Hack’s law explanations have been sought in fractal (Tarboton et al., 1988; Maritan et al.,
39 1996), constructal (Reis, 2006) and “feasible optimality” (Rigon et al., 1998) theories. Fractal
40 theories produce the required self-similar drainage basins (Peckham, 1995) as well as increasing
41 stream sinuosity downstream. I suggest that Hack’s law can be understood using percolation
42 theory (Stauffer and Aharony, 1994) because 1) the fractal structure of the percolation cluster

43 generates values for γ that constrain the data for β appropriately, and 2) it exploits the concept
44 that water flows along paths of minimal resistance, as in the subsurface also (Hunt et al., 2014).

45 **2. Theory**

46 There are two distinct applications of percolation theory to flow or conduction problems,
47 and these two applications are those that provide the bounds to Hack's exponent values. The
48 more familiar application is to a binary system, where, e.g., bonds either connect neighboring
49 sites (which in the simplest case are located on a lattice, or grid), or they don't. If enough such
50 neighboring sites are connected, a continuous path of interconnected bonds spans the system.
51 This is denoted the percolation threshold. The shortest distance across the system within this
52 connected cluster is called the chemical path length (Porto et al., 1997). Since all bonds have
53 equal resistance, the shortest flow path has also the lowest resistance and optimal dissipation.

54 The second possibility is a system, in which bonds of varying resistance connect each
55 pair of neighboring sites. When the system is strongly heterogeneous, i.e., when the distribution
56 of the natural logarithm of the resistances has variance, $\sigma^2 \gg 1$, the proper application of
57 percolation theory is to find the path of least cumulative resistance. Quantification of this process
58 equates an integral over the local conductance distribution, from a "critical" value, to the largest
59 value, with the percolation threshold (Pollak, 1972). This particular method became known as
60 "critical path analysis," or CPA. The subnetwork so defined is precisely at the percolation
61 threshold (Stauffer and Aharony, 1994). However, the most interesting path across this system is
62 not the shortest, but the optimal path, which provides the least energetic cost in a strongly
63 heterogeneous network (Lopez et al., 2005), i.e., with variance tending to infinity. Since the
64 optimization is not for length, but for energy costs, the path is longer than in a homogeneous
65 system, meaning that its tortuosity exponent is larger. Nevertheless, the chemical path length in a

66 homogeneous system is the analogue of the optimal path length in a heterogeneous system,
67 because the shortest path in a system with identical links also represents the path of minimum
68 energy dissipation.

69 The restriction of river networks to the surface of the earth, and the measurement of
70 stream lengths on 2D maps, makes the topology of stream connections and the application of
71 percolation theory two dimensional. In two dimensions, the chemical path length scales with the
72 system size (Sheppard et al., 1999), L as $L^{1.13}$, but the optimal path length scales with system size
73 as $L^{1.21}$, and does not depend on whether a percolation process is classed as random, or invasion
74 (Sheppard et al., 1999). However, not all possible underlying correlations in the local
75 conductance distribution have been investigated. It is known that certain fractal correlation
76 structures in the local conductances can reduce the exponent associated with the conductivity, or
77 reduce the fractal dimensionality of the percolation backbone, (Sahimi and
78 Mukhopadhyay, 1996), but there was no corresponding effect noted on the tortuosity or optimal
79 paths exponent. Physical arguments suggest that positive correlations will tend to shorten paths,
80 reducing tortuosity, in accord with the general result that making all conductance magnitudes
81 equal reduces the tortuosity of connected paths. Consequently, one might ask whether negative
82 correlations could lengthen paths. In any case, calling the scaling exponent, γ , as above, we
83 therefore find that known results from percolation theory constrain its values to be, $1.13 < \gamma <$
84 1.21 . It should be emphasized here that the precision of the numerical calculations of these
85 exponent values by Sheppard et al., (1999) exceeded all other attempts by at least an order of
86 magnitude. These it appears that two possible endpoints for the application of percolation theory
87 to the formation of river networks, generated from homogeneous and heterogeneous systems,

88 respectively, constrain observed values of Hack's exponents reasonably well. How could they
89 be realized in nature, or in landscape models?

90 One can start from an initially homogeneous landscape, and allow stream incision
91 through random headward erosion, analogous to the processes treated in early landscape
92 evolution models (Willgoose et al., 1991), which generate hierarchical structures from random
93 chance associated with rainfall magnitude variability. A connected path with the lowest
94 dissipation (shortest length) will soon acquire the highest flow, through channel erosion
95 feedbacks. Thus, once a river makes a random choice, the enhanced erosion power from the
96 stream reinforces the initial random choice.

97 The optimal path exponent describes the tortuosity of a channel, when the channel is
98 determined by a global optimization of the flow path in a heterogeneous substrate, and could not
99 be a simple product of headward erosion, which might produce only a local optimization. In such
100 a case geological constraints from varying erodibility can dominate as channels extend either
101 upward, by headward erosion, or downward, (e.g., by overtopping of sills).

102 Using the above result that $\gamma = 2\beta$, we find for Hack's (1957) original result, $l = L^{1.2}$. The
103 range quoted by Maritan et al. (1996), $0.57 < \beta < 0.6$ generates $1.14 < \gamma < 1.20$. The predicted
104 range of tortuosity exponent values, γ , $1.13 < \gamma < 1.21$, generated by percolation theory appears
105 in accord with the observed range of values, and to be slightly larger, consistent with interpreting
106 this range as bounds on observed values.

107 Note that, while e.g., Willemin (2000) found a wider range of β (0.5 to 0.7) than did Gray
108 (1961), these values were for limited statistics (as small as 11 data points). Although this range is
109 wide, compared with our predictions, when all statistics were put together (Willemin's figure 11)
110 the resulting value of β was 0.58. Further, individual values did increase monotonically with

111 increasing geologic heterogeneity. Northwestern Iowa, in the middle of the North American
112 craton produced 0.5, New York, 0.64, and coastal Oregon, in a region of active tectonics, 0.7.
113 Finally, the range of values for β quoted by Gray (1961) arose from his consideration of studies
114 over different regions with distinct terrain; uncertainty in a given region was reported in the
115 variation of the numerical prefactor, rather than the exponent.

116 **4. Conclusions**

- 117 • Percolation predictions generate a range of exponents consistent with those
118 reported in Hack's law, including the tendency for the largest exponent values to
119 occur in geologically heterogeneous environments.
- 120 • The statistical nature of percolation theory is in accord with the tendency of the
121 spread in Hack's exponent values to diminish with increasing sample size,
- 122 • The source of the tortuosity in the "optimal paths" of lowest energy dissipation is
123 in general accord with the "feasible optimality" (Rigon et al., 1998) proposed to
124 explain Hack's law.

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