Review

Title: Self-breeding: a new method to estimate local Lyapunov structures

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Summary and Recommendation

I found this manuscript to be rather confusing and difficult to follow. Many of the concepts are poorly defined, some of the conclusions seem contradictory, and the references cited are sometimes not germane to the material being discussed. The other two reviewers had similar concerns and the authors posted a reply which clarified some of the material, but didn't significantly resolve the confusion. For a summary, I will discuss what I think the authors have *actually* done and contrast it with that I think they authors *claim* to have done.

The heart of the manuscript is the definition of "self-bred vectors" (SBVs). These appear to be defined as follows: Let $\mathbf{x}(t)$ be a trajectory of a nonlinear model (the "control simulation") and $\mathcal{M}$ be the nonlinear propagator satisfying

$$\mathbf{x}(t_2) = \mathcal{M}(\mathbf{x}(t_1); t_2, t_1), \tag{1}$$

where $t_2 > t_1$ are a final and initial time, respectively.[1] The SBVs are defined via an iterative "breeding cycle" wherein the i^{th} SBV in the n^{th} cycle, $\mathbf{s}_i^n$, is related to the i^{th} SBV in the $(n-1)^{\text{th}}$ cycle, $\mathbf{s}_i^{n-1}$, by

$$\mathbf{s}_i^n = \mathcal{M}\left(\mathbf{x}(t_1) + \alpha \frac{\mathbf{s}_i^{n-1}}{\|\mathbf{s}_i^{n-1}\|}; t_1 + \delta t, t_1\right) - \mathbf{x}(t_1 + \delta t), \tag{2}$$

where δt is the length of the breeding cycle, α is a scale factor controlling the amplitude of the SBV, and $\|\cdot\|$ is a norm. The initial SBVs, $\mathbf{s}_i^0$, are random. The SBVs defined in (2) will generically converge into a one-dimensional subspace; a (vaguely defined) orthogonalization procedure prevents this convergence. The first N SBVs therefore eventually span an N-dimensional subspace—the hope is that this subspace captures also the most important growing disturbances.

If $\alpha \ll 1$ (the authors use $0.005 \leq \alpha \leq 0.1$), then linearization of (2) is appropriate and gives

$$\mathbf{s}_i^n = \alpha \mathbf{M}\left(\mathbf{x}; t_1 + \delta t, t_1\right) \frac{\mathbf{s}_i^{n-1}}{\|\mathbf{s}_i^{n-1}\|}, \tag{3}$$

where $\mathbf{M}\left(\mathbf{x}; t_2, t_1\right)$ is the tangent linear propagator for the trajectory $\mathbf{x}$. Eq. 3 is an implementation of the power method for finding eigenvectors, so the SBVs in the linear limit

[1] My notation is different from that used by the authors, as I find their notation—especially with regard to the nonlinear propagator—somewhat confusing.

are an orthogonalization of leading eigenvectors of the tangent linear propagator. Since the eigenvectors of $\mathbf{M}$ are finite-time normal modes (FTNMs, Frederiksen, 1997), the SBVs are essentially nonlinear generalizations of the FTNMs.

Given their relationship to FTNMs, a comparison of the properties of SBVs to FTNMs would be appropriate. Instead, the authors compare SBVs to what they call local Lyapunov vectors (LLVs), although their definition of LLVs does not correspond to any LLVs I have previously encountered in the literature. The reference given for LLVs in the manuscript, Fujisaka (1983), does not define or even discuss LLVs. Kalnay (2002) defines LLVs as the eigenvectors of

$$\lim_{\tau \to \infty} \mathbf{M}\left(\mathbf{x}; t, t-\tau\right), \tag{4}$$

but this does not appear to the definition the authors use. Instead, they claim to estimate the LLVs as the eigenvectors of the matrix

$$\overline{\mathbf{M}_2} = \frac{1}{51} \sum_{j=0}^{50} \mathbf{M}_j \mathbf{M}_j^{\mathrm{T}}, \tag{5}$$

where

$$\mathbf{M}_j = \mathbf{M}\left(\mathbf{x} + \alpha \frac{\xi_j}{\|\xi_j\|}; t_1 + \delta t, t_1\right) \tag{6}$$

and the ξ_j are random for $j = 1, \ldots, 50$ and $\xi_0 = 0$. For each j, the eigenvectors of $\mathbf{M}_j \mathbf{M}_j^{\mathrm{T}}$ are the final, or evolved, singular vectors (SVs) associated with the trajectory $\mathbf{x} + \alpha \xi_j / \|\xi_j\|$. The evolved SVs converge to an orthogonalization of the LLVs [as defined by Kalnay (2002)] for long optimization intervals, but are not necessarily related to the LLVs for short optimization intervals. The properties of the eigenvectors of $\overline{\mathbf{M}_2}$ are unclear—I have not encountered such a construct before—but given the small values of α, I suspect that $\overline{\mathbf{M}_2} \approx \mathbf{M}_0 \mathbf{M}_0^{\mathrm{T}}$ and its eigenvectors are close approximations to the SVs of $\mathbf{x}$.

Thus, it seems that the authors *claim* to have developed a nonlinear generalization of the LLVs and compared them to approximations of linear LLVs. One the other hand, it appears that the authors have *actually* developed a nonlinear generalization of FNTMs and compared them to an approximation to evolved SVs. It's not clear to me that such a comparison is meaningful; at least in the way that presented in the manuscript. While the SBVs may indeed be a useful quantifier of predictability, the present manuscript does little to demonstrate this. As such, I don't think this manuscript is suitable for publication in NPG.

References

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