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Horton laws for Hydraulic-Geometric variables and their scaling exponents in self-similar Tokunaga river networks

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Abstract. An analytical theory is developed that obtains Horton laws for five Hydraulic-Geometric (H-G) variables (stream discharge Q, width W, depth D, velocity U, slope S, and friction n') in self-similar Tokunaga in the limit of large network order. The theory uses several disjoint theoretical concepts like, Horton laws of stream numbers and areas in self-similar Tokunaga networks as asymptotic relations, dimensional analysis, Buckingham-Pi theorem, asymptotic self-similarity of the first kind, or SS-1, and asymptotic self-similarity of the second kind, or SS-2. A self-contained review of these concepts with examples is given as 'methods'. The theory builds on six independent *dimensionless River-Basin numbers*. A mass conservation equation in terms of Horton bifurcation and discharge ratios in self-similar Tokunaga networks is derived. Assuming that the H-G variables are homogeneous and self-similar functions of stream discharge, it is shown that the functions are of a power law form. SS-1 is applied to predict the Horton laws for width, depth and velocity as asymptotic relationships. Exponents of width and the Reynold's number are predicted. Assuming that SS-1 is valid for slope, depth and velocity, corresponding Horton laws and the H-G exponents are derived. Our predictions of the exponents are the same as those previously predicted for the optimal channel network (OCN) model. In direct contrast to our work, the OCN model does not consider Horton laws for the H-G variables, and uses optimality assumptions. The predicted exponents deviate substantially from the values obtained from three field studies, which suggests that H-G in networks does not obey SS-1. It fails because slope, a dimensionless River-Basin number, goes to 0 as network order increases, but, it cannot be eliminated from the asymptotic limit. Therefore, a generalization of SS-1, based in SS-2 is considered. It introduces two anomalous scaling exponents as free parameters, which enables us to show the existence of Horton laws for channel depth, velocity, slope and Manning's friction. The Manning's friction exponent, y, is predicted and

tested against field exponents from three studies. One of these basins shows some deviation from the theoretical prediction. A physical reason for this deviation is given. We briefly sketch how the two anomalous scaling exponents could be estimated from the transport of suspended sediment load and the bed load. Statistical variability in the Horton laws for the H-G variables is also discussed. Both are important open problems for future research.

1 Introduction

Several key papers have been published on self-similar river networks in the last twenty years. As a sample see Tokunaga (1978); Peckham (1995b); Peckham and Gupta (1999); Dodds and Rothman (1999); Veitzer and Gupta (2000); Troutman (2005); Veitzer et al. (2003); Mcconnell and Gupta (2008). Tokunaga model among them is very important because it is mathematically simple and it predicts many topological and geometrical features that are observed in real channel networks (Tokunaga, 1978; Peckham, 1995a). The predictions are made in terms of "Horton laws" that are explained in Sect. 2 on background. Our paper develops an analytical theory to predict Horton relationships for five hydraulic-geometric (H-G) variables (stream discharge Q_ω, width W_ω, depth D_ω, velocity U_ω, slope S_ω, and Manning's friction n'_ω) in self-similar Tokunaga river networks. It is the first study that generalizes the theory from topology and geometry to H-G in channel networks. The H-G exponents for W_ω and n'_ω are predicted and tested against observed exponents from three field studies. An analytical theory of H-G is a long-standing, fundamental open problem in Hydro-geomorphology that is addressed here. Next we explain the significance of this work.

Prediction of floods in river basins with sparse or no streamflow data is a hydrologic engineering problem of great practical significance. An acronym for this problem that is widely used is Prediction in Ungauged Basins (PUB) (Dawdy, 2007; Sivapalan et al., 2003). PUB is common to both developing as well as industrialized countries. A nonlinear geophysical theory is being developed for almost 30 years to solve the PUB problem for floods. Two review articles have been published on this theory Gupta and Waymire (1998); Gupta et al. (2007). Given space-time rainfall intensity field for any rainfall-runoff event, the theory attempts to predict stream flow hydrographs at all the 'junctions' (where two or three channels meet) in a channel network. The theory requires a model to transform rainfall to runoff in space and time in a basin (Furey et al., 2013), and space-time river flow dynamics in a network (Mantilla, 2007). Modeling of flow dynamics requires a theory of H-G of in a channel network, because practically no data on H-G exists. Our paper begins to fill a long-standing need in this context.

By extending the Horton laws to H-G variables, our paper shows how geomorphology, hydrology and channel hydraulics are linked in river networks. Consequently, it opens a new door to understanding how the geometry, statistics and dynamics in river networks are mutually coupled on many spatial scales, which has far-reaching implications for understanding and modeling river flows, as explained above, and transport of sediments and pollutants in river networks. A few discrete research efforts have been made on understanding the nature of the flood scaling from physical processes on annual time scale (Poveda et al., 2007; Lima and Lall, 2010). But connecting this body of work to flood scaling for events remains an important open problemGupta et al. (2010).

Our paper is organized as follows. Sect. 2 gives a brief review of the literature. It also includes two "Methods" sections that give a self-contained review of analytical methods used in the theory. Sect. 2.1 contains a review of the Horton laws for network topology and geometry as asymptotic relations in self-similar Tokunaga networks that are taken from Mcconnell and Gupta (2008). Sect. 2.2 gives a review of similarity or similitude that is based in dimensional analysis and explains the Buckingham-Pi theorem. It is followed by a review of "Asymptotic self-similarity of the first kind", or SS-1 for short. In many cases including the present case, SS-1 requires a generalization involving "Asymptotic Self-Similarity of the second kind", or SS-2 for short. Physical examples are given to explain these methodological concepts.

In Sect. 3, an application of Buckingham-Pi theorem gives a total of six independent *dimensionless River-Basin numbers*. The six numbers are specified using physical arguments rather than formal dimensional analysis. In Sect. 4, we formulate a mass conservation equation for a river network indexed by Strahler order. We apply the results from Sect. 2.1 to obtain a solution of this equation in terms of Horton bifurcation, area and discharge ratios in the limit of large network order Ω. It applies to small order streams, $\omega = 1, 2, 3 \ldots$.

In Sect. 5, we consider three H-G variables, W_ω, D_ω and U_ω, and show that they are power law functions of discharge. By definition, $Q_\omega = U_\omega W_\omega D_\omega$. Horton laws are obtained as asymptotic relations for these three H-G variables. We show that self-similar solutions involving SS-1 hold asymptotically for the width and the Reynold's number, and values for their H-G exponents are predicted.

In Sect. 6, it is assumed that the SS-1 framework from Sect. 5 is valid for S_ω, D_ω and U_ω. Horton laws for these three H-G variables are derived asymptotically, and their exponents are predicted. Our predictions of the exponents are the same as those previously predicted by (Rodríguez-Iturbe et al., 1992) for the Optimal Channel Network (OCN) model. In direct contrast to our work, the OCN model does not consider Horton Laws for these H-G variables, and uses optimality assumptions. In this sense, foundations of our theory based in self-similarity and dimensional analysis are very different from that of the OCN model. The deviations between the observed and predicted exponents are substantial suggesting that H-G in network does not obey SS-1.

In Sect. 7, we explain that the reason for the failure of SS-1 is that slope, one of the dimensionless numbers, goes to 0 as network order increases. But, slope cannot be eliminated from the asymptotic limit. Therefore, a generalization of SS-1 requiring SS-2 is needed. This section is divided into four subsections. The first one introduces two anomalous scaling exponents, α and β, in the theory. It enables us to show the existence of Horton laws for channel depth, velocity, and slope, and derive expressions for their exponents as functions of α and β. These two scaling exponents are free parameters, which cannot be predicted from dimensional considerations. To make progress with developing and testing of theory, we consider Manning's friction coefficient as the fifth H-G variable in the second subsection, which can be estimated from values of slope and velocity. We predict a Horton law for the Manning's friction coefficient and derive an expression for its exponent. The third subsection tests the prediction of the exponent against three field studies. One of these basins shows some deviation from the theoretical prediction. A physical reason for this deviation is investigated in the fourth subsection.

Two fundamental physical processes that shape the H-G of channels are transport of suspended sediment load and the bed load that are not considered here. In Sect. 8, we sketch in a preliminary manner how these two physical processes could be used to determine the two anomalous scaling exponents. Inclusion of statistical variability in the Horton laws for the H-G variables is also discussed. Both are important open problems for future research. The paper is concluded in Sect. 9.

2 Background and Methods

Leopold and Maddock (1953) first introduced the hydraulic-geometry (H-G) of rivers 'at-a-station' and in the 'downstream direction'. At a station H-G relations refer to temporal variability of width, depth, velocity, slope, channel roughness, and sediment transport as functions of discharge, and 'downstream' H-G relations refer to their spatial variability as functions of discharge. An extensive literature has developed on these topics; see Singh (2003) for a recent review of the literature. This body of literature, though important, is not directly relevant to the objectives of our paper. Indeed, Singh (2003) concluded his review paper with the statement, "The work on hydraulic geometry of channels serves as an excellent starting point to move on to the development of a theory of drainage basin geometry and channel network evolution. This will permit integration of channel hydraulics and drainage basin hydrology and geomorphology."

In a classic paper, Leopold and Miller (1956) extended the H-G relations to drainage networks. A brief review of pertinent concepts is given that is required to understand this work. Horton (1945) first discovered 'Horton laws' in quantitative geomorphology with the aid of maps. The original motivation was to define stream size based on a hierarchy of tributaries. The most common method for defining a spatial scale in a hierarchical branched network is the method of Horton–Strahler ordering, or Strahler ordering for short, because Strahler (1952, 1957) modified the ordering system that Horton had introduced. Strahler ordering assigns, $\omega = 1$ to all the unbranched streams. They contain the highest level of spatial resolution for a network and thereby define a spatial scale. Continuing downstream through the network, where two streams of identical order ω meet, they form a stream of order $\omega + 1$. Where two streams of different orders meet, the downstream channel is assigned the higher of the two orders. This continues throughout the network, labeling each stream, and ending with the stream of order Ω. By definition, any network contains only one stream of order Ω called the network order. Strahler ordering defines a one-to-one map under pruning, i.e., if the streams of order 1 are pruned and the entire tree is renumbered, the order 2 streams identically become the new order 1 streams, the order 3 streams become order 2, and so on throughout the network. The order of the entire network decreases by one. It is a necessary condition for defining self-similarity for a hierarchical branched network that is reviewed in Sect. 2.1.

Strahler ordering led to the discovery of the "Horton laws of drainage composition". They are often referred to simply as the Horton laws. The most famous of the Horton's laws is the law of stream numbers for N_ω, denoting the number of streams of order ω in a network of order Ω. It is traditionally written as

$$\frac{N_\omega}{N_{\omega+1}} = R_\mathrm{B}, \quad 1 \leq \omega \leq \Omega. \tag{1}$$

The number R_B is called the bifurcation ratio. Observations from real river networks show a limited range of R_B values between three and five. These are not formal laws because they have not been proved from first principles, however, they are widely observed in real river networks. Similar relationships are observed for lengths, slopes, and areas.

Shreve (1967) developed the 'random model' of channel networks and thereby offered the first theoretical understanding of Eq. 1. He introduced the basic topological concepts of channel links (defined as the segment of channel between two adjacent junctions where two or three channels meet) and magnitude, (defined as the total number of non-branching or source channel links). Shreve (1967) showed that in the limit as magnitude goes to infinity, R_B converges to 4. This demonstration showed for the first time that the empirical Eq. 1 can be derived as an asymptotic result from a suitable mathematical model of channel networks. We review this foundational issue in greater detail in the next subsection using the mathematical model of self-similar Tokunaga River networks.

Leopold and Miller (1956) extended the Horton laws to H-G variables. Their reasoning was that channel discharge varies as a function of drainage area as a power law, $Q = kA^c$. At the time, the Horton law for drainage area was known (Jarvis and Woldenberg, 1984). They tested the Horton law for discharge, and asserted that the Horton laws hold for the entire suite of H-G variables as functions of discharge, e.g., width, depth, velocity, slope, channel roughness, and sediment transport. Until this paper was published, the Horton laws had been observed for only the topologic and geometric variables (Jarvis and Woldenberg, 1984). By extending the Horton laws to H-G variables, the Leopold and Miller (1956) paper showed how river basin geomorphology, hydrology and channel hydraulics are linked. Consequently, it opened a new door to understanding how the geometry, statistics and dynamics in river networks are mutually coupled on many spatial scales, which has far-reaching implications for understanding and modeling river flows and sediment transport in river networks. This major objective has not been realized because the theoretical underpinning of the Horton laws and the H-G exponents in channel networks has remained elusive. It remains a fundamental, long-standing open problem in Hydro-geomorphology that is addressed here.

The Strahler ordering and the Horton laws concepts had a big impact on several areas, for example, (i) model growth of plants and other hierarchical biological structures such as animal respiratory and circulatory systems, (ii) in the order of register allocation for compilation of high level programming languages, and (iii) in the analysis of social networks (Jarvis and Woldenberg, 1984; Pries and Secomb, 2011; Viennot and Vauchaussade de Chaumont, 1985; Park, 1985; Horsfield, 1980; Borchert and Slade, 1981; Berry and Bradley, 1976). The widespread appearance of Horton laws suggests that perhaps a "fundamental principle" underlies them. Indeed, recent research has shown that Horton laws are asymptotic relations that have been proved in theoretical self-similar river network models. Self-similarity is a form of scale invariance. Specifically, we have selected the self-similar Tokunga network model to develop the theory in this paper. This model is briefly reviewed in the next Sect. 2.1.

An attempt to predict the H-G exponents in river networks without the Horton laws is the theory of optimal channel networks (OCN) (Rodríguez-Iturbe et al., 1992). OCNs have been analytically shown to produce three universality classes in terms of scaling exponents, but none of these predictions agree with data (Maritan et al., 1996). Two comprehensive field programs were carried out in New Zealand (NZ) to test the OCN predictions (Ibbitt et al., 1998; McKerchar et al., 1998). However, the observed values of the H-G exponents substantially deviated from the OCN predictions that is explained in greater detail in Sect. 6 and Sect. 7.3. Other attempts building on optimality ideas have used data from these two New Zealand basins (Molnar and Ramirez, 2002). But, a foundational understanding of the geophysical origins of Horton laws for the H-G variables and their exponents has remained elusive.

West et al. (1997) recently tackled a somewhat similar problem in the allometric theory of biological networks. Our treatment of the H-G problem has some similarities but major differences with their approach. For example, West et al. (1997) appeal to an "optimality assumption" by maximizing or minimizing a function. By contrast the present theory uses no optimality assumption, but uses "self-similarity" as its basic building block.

The complexity resulting from space-time variability in climate, hydrology and lithology can be contrasted with the empirical observations like the Horton laws that suggest regularities related to similarity across scales, or self-similarity. We briefly review key results for the self-similar Tokunaga model in the next Subsection regarding Horton laws of stream numbers and magnitudes.

2.1 Method: A brief review of self-similar Tokunaga River networks

Eiji Tokunag introduced the Tokunaga model (Tokunaga, 1978). It is based on Strahler ordering and involves the concept of self-similar topology in its construction. Unlike the random model of Shreve (1967), this model is deterministic and does not include any statistical variability that is observed in real networks. Therefore, the link lengths are assumed to be equal to l throughout the paper. (Peckham, 1995a) gave empirical evidence that supports the idea that the average topologic features of medium to large river networks can be well described by the Tokunaga model. For example, in real networks, generally $3 < R_B < 5$ that the Tokunaga model can exhibit but the random model predicts $R_B = 4$ as mentioned above. Moreover, the mathematics in the Tokunaga model is simplified that is necessary to make progress in the H-G problem addressed here.

The key building block of the model is a generator for side tributaries, $T_{\omega,\omega-k}$, which may be interpreted as the mean number of streams of order $\omega - k$ joining streams of order ω since real networks exhibit statistical variability in their branching structures (topology). Self-similarity in the network topology is reflected in the assumption that, $T_{\omega,\omega-k} = T_k$, $k = 2, 3, \ldots$. Tokunaga further restricted his model by requiring that, $T_k/T_{k-1} = c$, $k = 2, 3, \ldots$, and $T_1 = a$, where c and a are constant parameters associated with network topology. It leads to the generator expression,

$$T_k = ac^{k-1}, \quad k = 1, 2, \ldots, \tag{2}$$

representing mean self-similar Tokunaga trees (Dodds and Rothman, 1999). These parameters are observable quantities in natural basins.

Predictions are based on a fundamental recursion equation governing N_k, the number of streams of order k,

$$N_k = 2N_{k+1} + \sum_{j=1}^{\Omega-k} T_j N_{k+j}. \tag{3}$$

Mcconnell and Gupta (2008) proved that the ratio N_k/N_{k+1} converges to R_B in the limit as $\Omega - k \to \infty$. The limit applies to small order streams, $k = 1, 2, , \ldots$, as the network order $\Omega \to \infty$. Mcconnell and Gupta (2008) gave a physical interpretation of the limit as follows, “Note that if the overall order Ω could be increased, we would expect to see more streams of higher and lower orders. However, when all the streams of smaller orders k and $k + 1$ in a basin of very large order Ω are counted, we expect that we have captured a significant and representative portion of the side tributaries, and the observed bifurcation ratio more closely matches the predicted ratio. This physical argument clarifies the use of the limit, $\Omega - k \to \infty$´´. The solution to Eq. (3) is given by,

$$R_B = \frac{(2 + a + c) + \sqrt{(2 + a + c)^2 - 8c}}{2}. \tag{4}$$

Mcconnell and Gupta (2008) also proved a Horton law for stream magnitudes, $M_\omega, \omega = 1, 2, \ldots$ from the above result. Magnitude is defined in Sect 2, and is the topological equivalent of drainage area. The Horton law for magnitudes is expressed as,

$$\lim_{\Omega-\omega\to\infty} \frac{M_{\omega+1}}{M_\omega} = R_M = \lim_{\Omega-\omega\to\infty} \frac{A_{\omega+1}}{A_\omega} = R_A. \tag{5}$$

Where the Horton magnitude ratio, $R_M = R_B = R_A$, and R_A is the Horton area ratio. We remark that the random model obeys self-similarity in a mean sense, and it has, $a = 1, c = 2$. Above equation predicts $R_B = 4$ that agrees with the random model (Shreve, 1967).

Finally, we review two results, which were previously reported by others, and which have been shown to hold for Tokunaga networks. The topological fractal dimension D_T for Tokunaga networks is given by, $D_\mathrm{T} = \log R_\mathrm{B} / \log R_\mathrm{C}$, where R_B is Horton bifurcation ratio and R_C is the link ratio (Peckham, 1995a). Since $R_\mathrm{B} = R_\mathrm{A}$, and assuming constant link lengths, $R_\mathrm{C} = R_\mathrm{L}$, it follows that, $R_\mathrm{L} = R_\mathrm{A}^{1/D_\mathrm{T}}$, where R_L is the length ratio. For OCNs, $D_\mathrm{T} = 2$ (Maritan et al., 1996). For natural river networks data sets show that typically, $1.7 < D_\mathrm{T} < 1.8$. The class of Tokunaga networks predicts values of D_T less than or equal to 2. The Hack exponent for Tokunaga networks is, $\beta = 1/D_\mathrm{T} \geq 1/2$, and the area exponent $\alpha \leq 1/2$ as observed empirically (Peckham, 1995b). Moreover, for Tokunaga networks, $\alpha + \beta = 1$ (Peckham and Gupta, 1999). A new theory of random self-similar networks (RSN) includes statistical variability, and the Tokunaga is shown as a special case for a subclass of RSN that obey mean self-similarity (Veitzer and Gupta, 2000). The RSN theory provides the topologic and geometric foundations on which a H-G theory incorporating statistical fluctuations can be developed in the future. RSN theory is not used here.

2.2 Method: Asymptotic Self-Similarity of Type-1 and Type-2

The material covered in this subsection is a huge topic with a long history. We have selected Barenblatt (1996, 2003) because they are well written and offer a comprehensive reference on this important topic in science and engineering. We have limited our review to some key concepts that are used in our paper. The reader may consult Barenblatt (1996, 2003) as required for further explanations.

Dimensional analysis is based on the simple idea that the laws of nature are independent of the arbitrarily chosen basic unit of measurements. As a consequence these laws are invariant under a change of scale. Mathematically this is expressed as a generalized homogeneity that manifests as power laws. The famous Buckingham–Pi theorem, or simply Π-theorem, is a conceptualization of this powerful idea. It enables one to reduce the number of arguments in the functions expressing the physical laws, thereby making it simpler to study nature's regularities either experimentally or theoretically. The Π-theorem says (Barenblatt, 2003, p. 25): "A physical relationship between a dimensional quantity and several dimensional governing parameters can be rewritten as a relationship between a dimensionless parameter and several dimensionless products of the governing parameters. The number of dimensionless products is equal to the total number of governing parameters minus the number of governing parameters with independent dimensions". Let, a be the dependent dimensional quantity, and $a_1, \ldots, a_k, b_1, \ldots, b_m$ the $n = k + m$ governing parameters, k of them with independent dimensions. With this notation $a = f(a_1, \ldots, a_k, b_1, \ldots, b_m)$ can be transformed to $\Pi = \Phi(\Pi_1, \ldots, \Pi_m)$, with $\Pi = a a_1^{-p} \cdots a_k^{-r}$, and for $i = 1, \ldots, m$ the dimensionless products for the governing parameters with independent dimensions are $\Pi_i = b_i a_1^{-p_i} \cdots a_k^{-r_i}$. Because of the definitions, the exponents $p_i, \ldots, r_i$ can always be obtained by solving elementary linear equations. A classical example to illustrate the Π-theorem is the formula for the period θ of the small oscillations of a pendulum of mass m and length ℓ and the gravitational acceleration g. The three governing parameters (m, ℓ, g) have three independent dimensions (M, L, LT^{-2}). Therefore the number of dimensionless products of the governing parameters is $n-k=m=3-3=0$, which implies that the dimensionless product involving the period, Π, is a constant. It can be written as, $\theta g^{1/2}\ell^{-1/2} = \text{constant}$ (Barenblatt, 2003, p. 132). The constant cannot be obtained from dimensional analysis. It must be determined either from a theory or from observations, and is 2π for the pendulum example.

Majority of the successful examples of the applications of dimensional analysis can be found in many textbooks, which share another important property that is not always emphasized but is necessary in our context. For such problems there is a clear way of separating the important variables from the ones that do not play a significant role because they are either too small or too large. For instance the textbook example (Gibbings, 2011, p. 119) of the derivation of Kepler´s third law from dimensional analysis needs three non-dimensional numbers, and expresses the ratio of the period of rotation, T, in terms of the other two, namely, the ratio of the mass of the planet m to the mass of the Sun M and the ratio of axes of the ellipsis, a/b, that can be expressed in terms of the eccentricity, $e^2 = 1-(b/a)^2$. Let G denote the constant of gravitation. Therefore

$$\frac{GMT^2}{a^3} = f(m/M, e). \tag{6}$$

The crucial observation is that two of the dimensionless numbers are small and therefore don't play a significant role in the limit when they go to zero. The consequence is that the limit of the function that expresses the non-dimensional number involving the period of the rotation in terms of the other two goes to a constant. It is well known from the theory that the limit is $4\pi^2$. This is self-similarity characterized by the existence of a non-trivial (different from zero or infinity) limit of the function when some of the other non-dimensional products become very small or large. This is called "Asymptotic self-similarity of the first kind", or SS-1 for short, (Barenblatt, 2003, p. 84). SS-1 is applied to our problem in Sect. 5 and 6.

In many other cases a dimensionless number despite being too small (or large if you consider its reciprocal), cannot be ignored in the limit. Mathematically this corresponds to the case that the limit of a function does not exist, or is zero or infinity. The simplicity of SS-1, that consists in discarding small dimensionless variables and obtaining the scaling exponents from dimensional analysis is lost in this case. In such cases small variables continue to play a role in the problem, and require a generalization of the dimensional analysis. The concept of "Asymptotic Self-Similarity of the second kind", or SS-2 for short, discussed in Barenblatt (1996, chap. 5). It is known as the 'Renormalization group theory' in statistical physics. For our purposes, the fluid-mechanical approach is more natural than the statistical physical approach, because it is based on a generalization of the dimensional analysis framework. We follow the fluid-mechanical approach in this paper. Barenblatt (1996, p. 172) has explained that these two approaches are equivalent.

A simple example of SS-2 is the determination of the length of a fractal curve, that can be contrasted with a smooth curve (Barenblatt, 2003, p. 132). Let L_η be the length of a broken line of segment length ξ that approximates the continuous curve between two points that are separated by a distance η. L_η depends on the two dimensional parameters η and ξ. Dimensional analysis gives $L_\eta = \eta f(\eta/\xi)$. For a smooth curve, say a semicircle, as $\xi \to 0$ the argument $\eta/\xi \to \infty$ and the function f goes to a limit, namely $\pi/2$. Whereas for a fractal curve, the limit of f when $\eta/\xi \to \infty$ is infinity. In fact from fractal geometry we know that $f(\eta/\xi) \simeq (\eta/\xi)^{D-1}$. The anomalous exponent $D > 1$ is the fractal dimension, that cannot be estimated from dimensional analysis. Barenblatt (1996) gives a recipe for the applications of similarity analysis and SS-2 to obtain the exponents along with many physical examples that include turbulent shear flows, fractals, biological allometry, and groundwater hydrology. We apply SS-2 in Sect. 7.

3 Dimensionless River-Basin Numbers

The fundamental physical parameters governing the H-G in drainage networks are defined at the bottom of a complete Strahler stream of order $\omega \geq 1$ as follows. Q_ω is river discharge rate $[\mathrm{L}^3/\mathrm{T}]$, A_ω is the cumulative drainage area, and $D_\omega, U_\omega, W_\omega$ are channel depth, velocity, and width, respectively. H_ω is the elevation drop that is defined as the elevation difference between the beginning and the end junctions of a complete Strahler stream. L_ω denotes the corresponding stream length. Slope is defined as, $S_\omega = H_\omega/L_\omega$. Kinematic viscosity is ν, water density is ρ, and the gravitational acceleration is g. R is the mean runoff rate per unit area from the hillslopes along a channel network, and has dimension $[\mathrm{L}\,\mathrm{T}^{-1}]$. The spatial uniformity of R implies that river basin is being assumed to be homogeneous with respect to mean runoff generation. This assumption can be relaxed, but we want to keep this presentation simple.

From the set of twelve variables listed above, only nine are independent, because three variables are dependent: $Q_\omega = U_\omega W_\omega D_\omega$, $S_\omega = H_\omega/L_\omega$, and $L_\omega = A_\omega^{1/D_\mathrm{T}}$. These nine independent variables include three basic dimensions, Length (L), time (T) and mass (M). The Buckingham-Pi theorem explained in Sect. 2.2 gives that the number of independent dimensionless numbers is 6, but it does not give what they are. They can be specified either using formal dimensional analysis or physical arguments. We adopt the later approach because it is physically insightful. Some of these dimensionless numbers were considered in Peckham (1995b).

The first dimensionless number is given by,

$$\Pi_1(\omega) = \frac{Q_\omega}{\mathrm{RA}_\omega}. \tag{7}$$

Discharge rate Q_ω is taken to be a linear function of drainage area given by, $Q_\omega = RA_\omega$, that is observed in many humid climates for low and mean flows. (Leopold et al., 1964) used mean flow in their H-G investigations. Low flow has been used in recent field H-G observations that are analyzed in Sect. 7.

The second dimensionless number is,

$$\Pi_2(\omega) = \frac{R\sqrt{A_\omega}}{D_\omega U_\omega}. \tag{8}$$

It is suggested by mass conservation involving the ratio of runoff per unit width of drainage basin in the numerator, and discharge per unit channel width in the denominator.

The relation between gravitational and inertia forces in river networks suggests the third dimensionless number $\Pi_3(\omega)$. Specifically, we define the "Basin Froude Number" as,

$$\Pi_3(\omega) = \frac{U_\omega}{\sqrt{gH_\omega}} = \frac{U_\omega}{\sqrt{gS_\omega L_\omega}}, \tag{9}$$

where the channel slope,

$$\Pi_4(\omega) = S_\omega = H_\omega / L_\omega \tag{10}$$

is the fourth dimensionless number. The drop H_ω defines the length scale governing gravitational force. It should be differentiated from a channel Froude number in open channel hydraulics where flow depth defines the length scale.

The fifth dimensionless number is given by the Reynolds number. Leopold et al. (1964, p. 158) have discussed its significance in the context of laminar and turbulent flows. In natural streams, the flow is largely turbulent.

$$\Pi_5(\omega) = \frac{U_\omega D_\omega}{\nu}. \tag{11}$$

The sixth dimensionless number incorporates the factors controlling flow velocity. Total frictional force along the channel boundary is, $\tau_\omega(2D_\omega + W_\omega)L_\omega \approx \tau_\omega W_\omega L_\omega$, where τ_ω is the shear stress per unit area. It is proportional to the square of the mean velocity for turbulent flows if the boundary does not change with variations in flow (Leopold et al., 1964, p. 157). Gravitational force due to the mass of water along the channel length L_ω is given by $\rho g W_\omega D_\omega L_\omega S_\omega$. Dimensionless ratio of these two forces gives,

$$\Pi_6(\omega) = \frac{U_\omega^2}{gD_\omega S_\omega}. \tag{12}$$

The term $\sqrt{gD_\omega S_\omega}$ is known as the shear velocity. Π_6 is proportional to the Darcy-Weisbach resistance coefficient. Leopold et al. (1964, Fig. 6.5) illustrated that for the Bryandywine Creek, PA, $1/\sqrt{\Pi_6}$ is linearly related to the logarithm of relative roughness defined by the ratio of flow depth to the height of roughness elements.

4 Mass conservation in self similar Tokunaga networks

The discharge Q_ω is computed using a mass conservation equation for a network indexed by the Strahler order $\omega \geq 1$. We show that a mass conservation equation for a channel network indexed by link magnitudes (Gupta and Waymire, 1998) is a special case of it. We further assume that our channel network is self-similar Tokunaga with link lengths l , and no statistical fluctuations in the topology of channel network are considered in developing the H-G theory as mentioned in Sect. 2.1.

Let $\overline{S}_\omega(t)$ denote the storage in a Strahler stream of order $\omega \geq 1$ defined by,

$$\overline{S}_\omega(t) = W_\omega(t) D_\omega(t) L_\omega. \tag{13}$$

The dependence of storage on time t comes from temporal variations of streamflows in the network, which results in width and depth to vary with time.

Total number of junctions denoted by C_ω is the same as the total number of links in a complete Strahler stream of order ω. Let t_i, $i = 1,2,3,\ldots,C_\omega$ be a sequence of travel times for water to reach the bottom of a complete Strahler stream from successive junctions enumerated from the bottom. This means that t_1 represents the travel time from the first junction from the bottom, t_2 from the second junction and so on. For example, since all the links are assumed to have the same length l, and if water flows with a uniform velocity u, then $t_i = il/u$.

Let $R_i(t)$, $i = 1,2,3,\ldots,C_\omega$ denote the volumetric runoff rate from the ith hill along a complete Strahler stream of order ω. Let Q_{k_i}, $i = 1,2,\ldots,C_\omega$ denote the discharge from the side tributary at the ith junction from the bottom. Here the subscripts $k_1, k_2, \ldots$ denote the Strahler orders of the side tributaries coming into the junctions counted from the bottom of a stream. Let $Q^1_{\omega-1}(t)$ and $Q^2_{\omega-1}(t)$ denote the discharges in the two tributaries at the top of the stream. Each of them is of order $\omega - 1$ by definition of the Strahler ordering given in Sect. 2.

Considering a Strahler stream as a finite control volume, the mass conservation equation can be written as,

$$\frac{\mathrm{d}\overline{S}_\omega(t)}{\mathrm{d}t} + Q_\omega(t) = Q^1_{\omega-1}(t - t_{\omega-1}) + Q^2_{\omega-1}(t - t_{\omega-1}) + \sum_{i=1}^{C_\omega} Q_{k_i}(t - t_i) + 2\sum_{i=1}^{C_\omega} R_i(t). \tag{14}$$

For $\omega = 1$, Eq. (14) reduces to the link magnitude-based mass conservation equation in Gupta and Waymire (1998) that is easy to check.

As a first step, we have chosen to focus solely on the spatial analysis in the context of H-G. In particular, we seek a spatial solution of Eq. (14) by ignoring the time dependence of $Q_\omega(t)$, and denoting it as $Q_\omega(t) = Q_\omega$. This is tantamount to assuming that $\mathrm{d}\overline{S}_\omega(t)/\mathrm{d}t = 0$, $R_i(t) = 0, \forall i, t > 0$, and the travel times $t_i = 0, \forall i$. Physically, these sets of assumptions can be interpreted to mean that R, is applied uniformly throughout the network at time $t = 0$. Moreover, water is assumed to travel in a very short time throughout the network so that travel times are ignored.

In a recent paper on a space-time theory of low flows for river networks, travel times were ignored throughout the basin compared to the subsurface response time for hillslopes, and R was computed from hillslope processes under idealized conditions. The theoretical results so obtained compared well with observations (Furey and Gupta, 2000). Similarly, in the present context, the idealized assumptions made above are necessary to make progress on this complex problem.

In view of above assumptions, Eq. (14) simplifies to,

$$Q_\omega = Q^1_{\omega-1} + Q^2_{\omega-1} + \sum_{i=1}^{C_\omega} Q_{k_i}. \tag{15}$$

The key problem is to compute a solution for Q_ω. In view of the definition of self-similarity given in Sect. 2.1, Eq. (15) reduces to,

$$Q_\omega = 2Q_{\omega-1} + \sum_{k=1}^{\omega-1} T_k Q_{\omega-k}, \tag{16}$$

where $T_k = T_{\omega,\omega-k}$, $k = 1,2,\ldots,\omega-1$ denotes the number of side tributaries of order $\omega-k$ joining a stream of order ω. Equation (16) has been solved rigorously under the assumption that T_k's obey Tokunaga self-similarity. The solution is given by Eq. (4). Because the recursion equation (Eq. 16) for Q_ω is the same as the ones for A_w, we assert from the arguments given in Sect. 2.1 that

$$R_Q = \lim_{\Omega-\omega\to\infty} \frac{Q_{\omega+1}}{Q_\omega} = R_\mathrm{B} = \lim_{\Omega-\omega\to\infty} \frac{N_\omega}{N_{\omega+1}} = R_\mathrm{A} = \lim_{\Omega-\omega\to\infty} \frac{A_{\omega+1}}{A_\omega} \tag{17}$$

and

$$R_Q = R_\mathrm{B} = R_\mathrm{A} = \frac{(2+a+c)+\sqrt{(2+a+c)^2-8c}}{2}.$$

Equation (17) implies that $Q_\omega = \mathrm{RA}_\omega$ that is used in defining the first dimensionless number in Sect. 3. It follows from the definitions of Horton ratios, and from the equality, $R_Q = R_\mathrm{B}$ that,

$$Q_{\omega+1}N_{\omega+1} = Q_\omega N_\omega, \quad \text{as } \Omega-\omega\to\infty. \tag{18}$$

This is a foundational result governing mass conservation in self-similar Tokunaga networks. Even though, Eq. (18) is valid in the limit of large network order, the result holds for small values of ω as explained in Sect. 2.1. It should be noted that if one substitutes A_ω for Q_ω in Eq. (18), then it loses its physical interpretation. The reason is that Q_ω is a dynamic variable but A_ω is a fixed geometrical variable. Moreover, a power law relation between these two variables plays a fundamental role in the H-G investigations as explained above in Sect. 2 and later in Sect. 7.3.

West et al. (1997) used mass conservation equation for perfect branching biological networks in which no side tributaries are present and each parent branch bifurcates into two branches. In that case, it is simple to write down Eq. (18) as a special case of mass conservation without involving any limit. West et al. (1997) used it to obtain some remarkable results governing allometry in biological networks.

We apply Eq. (18) to extend the geometric and topological Horton laws in self-similar Tokunaga networks to include the H-G variables. Figure 1 shows a Horton law for channel widths in a drainage network that was mentioned along with other H-G variables in Sect. 2. As mentioned there, the key equation providing this link is a power-law relation between discharge and drainage area, and a Horton law for drainage areas (Leopold and Miller, 1956, p. 19-20). Both these features are derived above for Tokunaga networks. It is discussed in greater detail in the context of H-G data from two field studies in Sect. 7.3.

5 Derivation of Horton Law and the exponent of width using SS-1

5.1 Horton laws for channel width, depth and velocity

It follows from the definition of Π_1 (Eq. 7) and the fact that $R_Q = R_{\mathrm{A}}$ (Eq. 17),

$$\lim_{\Omega-\omega\to\infty} \frac{\Pi_1(\omega+1)}{\Pi_1(\omega)} = R_{\Pi_1} = 1, \quad \omega = 1,2,\ldots \ll \Omega. \tag{19}$$

This important result comes from the self-similarity of Tokunaga networks and the assumption of spatial homogeneity of runoff R. It probably is a valid assumption for widely varying climatic regions and a broad range of spatial scales. For example, the three river basins, one from the United States (US) and two from New Zealand (NZ) that we use to test the predictions of our theory in Sect. 7.3 have different climates, and different sized drainage areas. We also test if $R_Q = R_{\mathrm{A}}$ holds for one of the NZ basins in Sect. 7.3.

All the five H-G variables, U, W, D, S and the Manning's friction coefficient n' considered in this paper vary as discharge Q varies. Therefore, we assume that all the H-G variables are homogeneous functions of Q. This is the simplest mathematical assumption because it means that the functions do not depend on any other parameter except Q. It is widely used in the H-G literature reviewed in Sect. 2. We can write it as $U = f_1(Q)$, $W = f_2(Q)$ etc. To determine the form of these functions, one needs a functional equation depending on the physical context. Peckham (1995b, p. 53) has reviewed four functional equations with solutions known as Cauchy equations that repeatedly come up in similarity type investigations. Of these four, the most pertinent in our context is the functional equation based in self-similarity. Consider $U = f_1(Q)$. Self-similarity can be represented by the functional equation $f_1(Q_1+Q_2) = f_1(Q_1)f_2(Q_2)$ (Gupta and Waymire, 1998, p. 102–103), whose solution is a power law.

$$U = f_1(Q) \propto Q^m.$$

The above argument applies to all the functions. Therefore, the H-G variables can be written as power law functions of discharge,

$$U_\omega \propto Q_\omega^m, \quad W_\omega \propto Q_\omega^b, \quad D_\omega \propto Q_\omega^f, \quad S_\omega \propto Q_\omega^z \quad \text{and} \quad n'_\omega \propto Q_\omega^y. \tag{20}$$

The power-law representation of the H-G variables as functions of discharge has been widely used in the literature in analyzing data. Our notations for the H-G exponents are the same as in Leopold et al. (1964, p. 244). Eq. (17) showed that the ratio, $Q_{\omega+1}/Q_\omega$, converges to R_Q, and thereby obeys a Horton law. To extend the Horton laws to the H-G variables, let us consider velocity

$$\lim_{\Omega-\omega\to\infty} \frac{U_{\omega+1}}{U_\omega} = \frac{Q_{\omega+1}^m}{Q_\omega^m} = R_Q^m = R_U, \tag{21}$$

which follows from the fact that the ratios are positive and monotonic in ω as shown in (Eq. 20) (Rudin, 1976, p. 44).

Similarly, $R_W = R_Q^b$ and $R_D = R_Q^f$. By definition, $Q_\omega = U_\omega W_\omega D_\omega$. Therefore, the Horton ratios for velocity, width and depth can be written as,

$$R_U = R_Q^m, \quad R_W = R_Q^b, \quad R_D = R_Q^f, \quad m+b+f = 1. \tag{22}$$

The derivation of Eq. (22) required that, (i) Horton laws for channel widths, depths and velocities hold in Tokunaga self-similar networks, (ii) runoff generation R is spatially homogeneous, and (iii) channel width, depth and velocity depend monotonically on channel order.

5.2 Prediction of the width exponent and Reynolds number exponent

We will now use the above results to show that the Horton laws for the topologic and geometric variables explained in Sect. 2.1 extend to channel widths. Our arguments are based in dimensional analysis as explained in Sect. 2.2. Consider the dimensionless number $\Pi_2(\omega)$ defined by Eq. (8), and the ratio given by,

$$R_{\Pi_2}(\omega) = \frac{\sqrt{A_{\omega+1}}}{\sqrt{A_\omega}} \times \frac{D_\omega U_\omega}{U_{\omega+1} D_{\omega+1}}. \tag{23}$$

Substituting, $D_\omega U_\omega = Q_\omega / W_\omega$, in the above expression gives,

$$R_{\Pi_2}(\omega) = \frac{\sqrt{A_{\omega+1}}}{\sqrt{A_\omega}} \times \frac{Q_\omega W_{\omega+1}}{Q_{\omega+1} W_\omega}. \tag{24}$$

We have already shown that the right hand side converges to a constant in Sect. 5.1. It follows that the left side of Eq. (24) also converges to a constant. Stated mathematically,

$$\lim_{\Omega-\omega\to\infty} \frac{\sqrt{A_{\omega+1}}}{\sqrt{A_\omega}} \frac{Q_\omega W_{\omega+1}}{Q_{\omega+1} W_\omega} = \frac{R_{\mathrm{A}}^{1/2} R_W}{R_Q} = \lim_{\Omega-\omega\to\infty} R_{\Pi_2}(\omega) = R_{\Pi_2}. \tag{25}$$

The asymptotic constancy of the ratio $R_{\Pi_2}(\omega)$ of the dimensionless number Π_2 across different Strahler orders holds in Tokunaga networks. Since, $R_Q = R_{\mathrm{A}}$, Eqs. (25) and (22) can be combined to obtain,

$$R_W = R_{\Pi_2} R_Q^{1/2} = R_Q^b. \tag{26}$$

Therefore, $R_{\Pi_2} = 1$, and the channel width H-G exponent is,

$$R_W = R_Q^{1/2}, \quad b = 1/2. \tag{27}$$

It directly follows from Eqs. (11), (22) and (27) that a Horton law for Reynolds number can be written as,

$$R_{\Pi_5} = R_U R_D = R_Q^{m+f} = R_Q^{1/2}. \tag{28}$$

We test these predictions against data from three field studies in Sect. 7.3.

6 Predictions of Horton Laws and the H-G exponents assuming SS-I, and comparison with OCN model exponents

In the following developments, we assume that SS-1 applies to slope, S_ω, and that the Horton ratio for slope converges to R_S following a similar reasoning as given in Eq. (21). We predict the Horton laws for width, depth, velocity and slope and test our predictions of their exponents against the optimal channel network (OCN) model of Rodríguez-Iturbe et al. (1992). We show that the four predicted scaling exponents from our theory agree with the OCN model. However, our theory differs from it in a fundamental manner because we predict Horton laws for these variables, but the OCN model does not address this issue. Moreover, our theory uses self-similarity, but OCN uses 'optimality'. SS-1 in the H-G context is not the correct assumption as explained in the next section. It is being made here only to compare the predictions of the H-G exponents from our theory with the OCN model.

Define the Horton ratio for the Basin Froude number from Eq. (9). Following similar arguments as given in Eq. (21), and given about the length ratio in Sect. 2.1. We assert the convergence of the Basin Froude number because the Horton ratio of each term in it converges.

$$\lim_{\Omega-\omega\to\infty} \frac{\Pi_3(\omega+1)}{\Pi_3(\omega)} = R_{\Pi_3} = \frac{R_U}{\sqrt{R_\mathrm{L} R_\mathrm{S}}}. \tag{29}$$

From Eq. (20) $R_\mathrm{S} = R_\mathrm{A}^z = R_Q^z$. Invoking, $R_\mathrm{L} = R_\mathrm{A}^{1/D_\mathrm{T}} = R_Q^{1/D_\mathrm{T}}$ from Sect. 2.1 and assuming that the Tokunaga network is space filling as discussed there for the OCN model, it follows that $D_\mathrm{T} = 2$. Substituting $R_U = R_Q^m$ from Eq. (21) into Eq. (29) gives,

$$R_U = R_Q^m = R_{\Pi_3} R_Q^{1/4} R_Q^{z/2}. \tag{30}$$

Equation (30) predicts that $R_{\Pi_3} = 1$, and

$$m = \frac{1}{2}(z + 1/2). \tag{31}$$

Similarly, consider the Horton ratio for the dimensionless number proportional to the Darcy–Weisbach resistance coefficient given by Eq. (12), and take limit. We have demonstrated the convergence of each term in it. Therefore,

$$\lim_{\Omega-\omega\to\infty} \frac{\Pi_6(\omega+1)}{\Pi_6(\omega)} = R_{\Pi_6} = \frac{R_U^2}{R_D R_\mathrm{S}}. \tag{32}$$

We get an expression for the depth exponent by rewriting Eq. (32) as

$$R_D = R_Q^f = \frac{R_U^2}{R_{\Pi_6} R_\mathrm{S}} = \frac{R_Q^{2m}}{R_{\Pi_6} R_Q^z}. \tag{33}$$

It predicts, $R_{\Pi_6} = 1$, and

$$f = 2m - z. \tag{34}$$

Solving Eqs. (31) and (34) gives, $f = 1/2$, $m = 0$, $z = -1/2$, which also satisfy the constraint that $m + f = 1/2$. Our predictions may be summarized as: (1) Horton laws hold for the H-G variables in self-similar Tokunaga networks, (2) the H-G exponents are, $b = 1/2$, $m = 0$, $f = 1/2$, $z = -1/2$. Our second prediction agrees with the OCN model Rodríguez-Iturbe et al. (1992). We have already mentioned that the OCN model does not consider Horton laws for the H-G variables. That the exponents match is not a surprise because our theory is built on dimensional analysis. In conclusion, we state that our theory is fundamentally different from the OCN model.

To test the OCN predictions, two comprehensive field measurement programs were conducted in NZ. Figure 2 illustrates the measurement sites in channel network of the Taieri River Basin. These two basins are described in Sect. 7, where the field-measured values of the H-G exponents are given. Except for b in one of the two basins, other measured H-G exponents don't agree with theoretical predictions. The deviations are substantial suggesting that H-G in network does not obey SS-1. We address this foundational issue in the next section.

7 Application of SS-2 to predict Horton Laws and Exponents for the H-G variables

Slope appears in dimensionless numbers given by Eqs. (9), (10) and (12). The stream drop in Eq. (10) is bounded but stream length increases with order. Therefore, slope $S_\omega \to 0$ as $\Omega - \omega \to \infty$. Moreover, slope cannot be eliminated from the asymptotic limit. Therefore, a generalization of the dimensional analysis as explained in Sect. 2.2 is required to develop the H-G theory. It is the focus of this section.

7.1 Horton laws and theoretical expressions for the H-G exponents

Following Barenblatt, we define two "renormalized dimensionless numbers" in which slope appears. Equations (9) and (12) modify to,

$$\Pi_3^*(\omega) = \frac{U_\omega}{\sqrt{gL_\omega S_\omega^\alpha}}, \tag{35}$$

$$\Pi_6^*(\omega) = \frac{U_\omega^2}{gD_\omega S_\omega^\beta}. \tag{36}$$

Here α and β are "anomalous scaling exponents" that cannot be predicted from dimensional analysis. In principle they can be predicted from physical arguments involving sediment transport. This is a task for future research as explained in Sect. 8.

Following similar arguments as given in Sect. 5 to justify Eq. (21), we assert the convergence of the normalized dimensionless numbers,

$$\lim_{\Omega-\omega\to\infty} \frac{\Pi_3^*(\omega+1)}{\Pi_3^*(\omega)} = R_{\Pi_3^*} = \frac{R_U}{\sqrt{R_{\mathrm{L}} R_{\mathrm{S}}^\alpha}}. \tag{37}$$

and,

$$\lim_{\Omega-\omega\to\infty} \frac{\Pi_6^*(\omega+1)}{\Pi_6^*(\omega)} = R_{\Pi_6^*} = \frac{R_U^2}{R_D R_\mathrm{S}^\beta}. \tag{38}$$

Recall from Sect. 2.1 that $R_\mathrm{L} = R_\mathrm{A}^{1/D_\mathrm{T}} = R_Q^{1/D_\mathrm{T}}$ and from Sect. 5 that, $R_\mathrm{S} = R_\mathrm{A}^z = R_Q^z$, and $R_U = R_Q^m$. Therefore, Eq. (37) gives,

$$R_U = R_Q^m = R_{\Pi_3}^* R_Q^{1/2D_\mathrm{T}} R_Q^{z\alpha/2}. \tag{39}$$

Equation (39) predicts that $R_{\Pi_3}^* = 1$, and

$$m = \frac{1}{2}(z\alpha + 1/D_\mathrm{T}). \tag{40}$$

Since $m + f = 1/2$, an expression for the depth exponent follows directly from Eq. (40),

$$f = \frac{1}{2}(1 - z\alpha - 1/D_\mathrm{T}). \tag{41}$$

We get a second expression for the depth exponent by rewriting Eq. (38) as

$$R_D = R_Q^f = \frac{R_U^2}{R_{\Pi_6^*} R_\mathrm{S}^\beta} = \frac{R_Q^{2m}}{R_{\Pi_6}^* R_Q^{z\beta}}. \tag{42}$$

It predicts, $R_{\Pi_6^*} = 1$, and, in view of Eq. (40),

$$f = 2m - z\beta = z(\alpha - \beta) + 1/D_\mathrm{T}. \tag{43}$$

Equating the expressions for f from Eqs. (43) and (41), we obtain an expression for the slope scaling exponent as,

$$z(3\alpha - 2\beta) = 1 - 3/D_\mathrm{T}. \tag{44}$$

Equations (40) and (44) together generalize the H-G theory for a channel network based on an application of the renormalization group theory, and SS-2.

To summarize, given the topological fractal dimension D_T, and prediction of the width exponent, $b = 1/2$ by Eq. (28), we have two Eqs. (40), (44) that give theoretical expressions for H-G exponents m, z in terms of two unknown parameters, α, β. Our theoretical expressions for the H-G exponents can be written as,

$$\begin{aligned} b &= 1/2 \\ z &= (1 - 3/D_\mathrm{T})/(3\alpha - 2\beta) \\ m &= (z\alpha + 1/D_\mathrm{T})/2 \\ f &= 1/2 - m. \end{aligned} \tag{45}$$

7.2 Horton law for the Manning's friction and a theoretical expression for its exponent

The scaling exponents α and β are free parameters, which are not predicted by our theory. It implies that m, $f = 1/2 - m$ and z are not predicted and tested against data in this paper. This poses a big challenge for our work. To make progress on this crucial aspect of the theory we consider Manning's friction coefficient that can be estimated from the observed values of depth and slope. We derive a theoretical expression for the Manning's friction exponent here, and test it against data as explained in the next subsection.

Rewrite $\Pi_6^*(\omega)$ given by Eq. (36) as,

$$\Pi_6^*(\omega) = 1 = \frac{U_\omega^2}{gD_\omega S_\omega^\beta} = \frac{U_\omega^2}{gD_\omega S_\omega S_\omega^{-1+\beta}}, \tag{46}$$

so it may be expressed in the form of the well-known Chezy's equation,

$$U_\omega = (gD_\omega S_\omega)^{1/2} S_\omega^{(-1+\beta)/2}. \tag{47}$$

Therefore an expression for the Chezy's friction parameter is given as,

$$C_\omega^* = (g)^{1/2} S_\omega^{(-1+\beta)/2}, \tag{48}$$

provided, $\beta < 1$. This constraint is mentioned because the slope exponent must be negative to be consistent with data. Manning's friction coefficient n'_ω is related to Chezy's by means of (Leopold et al., 1964, p. 158),

$$U_\omega = C_\omega^* \sqrt{D_\omega S_\omega} = \frac{1.49}{n'_\omega} D_\omega^{2/3} S_\omega^{1/2} = \frac{1.49}{n'_\omega} D_\omega^{1/6} \sqrt{D_\omega S_\omega}.$$

Therefore,

$$n'_\omega = 1.49 D_\omega^{1/6} / C_\omega^*. \tag{49}$$

It can be expressed as the ratio using Eqs. (48) and (49) as,

$$\frac{C_{\omega+1}^*}{C_\omega^*} = \left[\frac{D_{\omega+1}}{D_\omega}\right]^{1/6} \left[\frac{n'_\omega}{n'_{\omega+1}}\right]. \tag{50}$$

Taking the limit as, $\Omega - \omega \to \infty$, gives

$$R_{n'} = R_D^{1/6} R_\mathrm{S}^{-(-1+\beta)/2}. \tag{51}$$

Equation (20) defined $R_{n'} = R_Q^y$. Using $R_\mathrm{S} = R_Q^z$ as before gives an expression for the H-G scaling exponent related to the Manning's equation,

$$R_{n'} = R_Q^y = R_Q^{f/6} R_Q^{-z(-1+\beta)/2}, \tag{52}$$

which gives a theoretical prediction for the Manning friction exponent as,

$$y = f/6 - z(-1+\beta)/2 \tag{53}$$

provided, $\beta < 1$. There are no adjustable parameters in this expression once β is estimated as explained in the next subsection. We use the same sets of field data for channel networks to support that estimated $\beta < 1$, and $\beta < 3\alpha/2$ from Eq. (44).

Mantilla et al. (2006) have described the H-G form of Chezy's friction coefficient that they deduced from empirical observations. They showed that the expression for Chezy's friction coefficient played a key role in testing the presence of statistical self-similarity involving Hortonian relationship in peak flows in the Walnut Gulch basin, Arizona.

7.3 Test of Manning Scaling exponent for Three Field Studies

As mentioned in Sect. 7.1, we estimate the anomalous scaling exponents α, β using the empirical values of f and z for three field studies. We first check that Eqs. (46) and (44) hold as required by the theory. We use the published data for H-G exponents from these three basins, and test the prediction of the scaling exponents b and y corresponding to the width and the Manning friction as a test of our theory.

The first basin is the classic Brandywine creek, PA in the US as given in Leopold et al. (1964, Table 7.5, p. 244), where the H-G exponents are also given. It has humid subtropical climate with cool to cold winters, hot, humid summers, and generous precipitation throughout the year, approx $1100\,\mathrm{mm\,yr^{-1}}$. Köppen climate classification lists it as type Cfa. It has a drainage Area of $777\,\mathrm{km}^2$ at the mouth. Average discharge is $12\,\mathrm{m^3\,s^{-1}}$.

The observed values of the H-G exponents are $b = 0.42$, $f = 0.45$, $m = 0.05$, $z = -1.07$ and $y = -0.28$. We fix $D_\mathrm{T} = 7/4$ since river networks show a value between 1.7 and 1.8 (La Barbera and Rosso, 1989; Maritan et al., 1996). The empirically computed values of the scaling exponents, which correspond to these H-G exponents, are: $\alpha = 0.441$, and $\beta = 0.327$. They satisfy the theoretical constraints, $\beta < 1$ and $\beta < 3\alpha/2$. Moreover, Eq. (53) correctly predicts the empirical value of the Manning's exponent, $y = -0.285$. However, the empirical values related to the width, depth and velocity do not satisfy $b + f + m = 1$, instead they add to 0.92. Assuming that depth and velocity exponents are correct, because they correctly predict the Manning's exponent, the value of $b = 1 - f - m = 1/2$ agrees with our theoretical prediction.

The second and the third basins are from NZ. The largest part of NZ has a pleasant sea climate with mild winters and warm summers. Köppen climate classification lists it as type Cf. The second field study was conducted in the Taieri River Basin (Ibbitt et al., 1998) that was introduced earlier. It has an estimated mean annual precipitation of $1400\,\mathrm{mm}$. Basin Area is $158\,\mathrm{km}^2$. Mean discharge, as measured over the discontinuous 14 year period 1983–1996, is $4.90\,\mathrm{m^3\,s^{-1}}$, representing an average runoff rate of $980\,\mathrm{mm\,yr^{-1}}$ from the basin.

The field values of the H-G exponents are, $b = 0.517$, $z = -0.315$, $m = 0.238$ and $f = 0.247$. The empirical width exponent is close to the predicted value, $b = 1/2$. Ibbitt et al. (1998) do not give an empirical value of Manning's friction exponent, but it can be computed from other exponents given above, and the empirical Manning equation,

$$U_\omega = 1.49 D_\omega^{2/3} S_\omega^{1/2}/n'_\omega,$$

as,

$$y = (2/3)0.247 - (1/2)0.315 - 0.238 = -0.231. \tag{54}$$

To make a theoretical prediction of y, we take, $D_\mathrm{T} = 7/4$. Then, using the empirically computed values of the scaling exponents f and z in Eq. (43) we get

$$\alpha - \beta = (f - 1/D_\mathrm{T})/z = 1.030, \tag{55}$$

and from Eq. (44)

$$3\alpha - 2\beta = (1 - 3/D_\mathrm{T})/z = 2.268. \tag{56}$$

Solving Eqs. (55) and (56) gives the values of the scaling exponents as, $\alpha = 0.208$, and $\beta = -0.822$, which satisfy the constraints on α, β described above. The predicted value of the Manning-scaling exponent from Eq. (53) is,

$$y = f/6 - z(-1+\beta)/2 = -0.246, \tag{57}$$

which is very close to the observed Manning exponent given in Eq. (54), which supports our theory.

The third field study was conducted in the $121\,\mathrm{km}^2$ Ashley River Basin (McKerchar et al., 1998). Annual precipitation increases in a northwesterly direction across the basin from 1200 to about $2000\,\mathrm{mm\,yr^{-1}}$. Mean discharge, as measured at the stream gauge over the 20-year period 1977–1996 is $3.99\,\mathrm{m^3\,s^{-1}}$, representing an average runoff rate of $1040\,\mathrm{mm\,yr^{-1}}$ from the basin.

The observed H-G scaling exponents are, $b = 0.44$, $z = -0.317$, $m = 0.318$ and $f = 0.242$. The empirical width exponent, $b = 0.44$ shows some deviation from the predicted value, $b = 1/2$. The Horton law for the width, taken from Mantilla (2014) is shown in Fig. 6. The observed value of $R_W = 1.61$, which shows some deviation from the predicted Horton ratio for the width exponent $R_Q^{1/2} = 1.74$. Since $R_Q^{0.44} = 1.63$, the width exponent that McKerchar et al. (1998) presented is consistent with the observed value of the Horton width ratio of 1.61 that Mantilla (2014) obtained. Other H-G variables not shown here support that the Horton laws hold as predicted in our work, and $R_W R_D R_U = 2.96$ that is close to the value of $R_Q = 3.05$ (see Eq. 22).

McKerchar et al. (1998) do not give an empirical value of the Manning's friction exponent, but it can be computed from above exponents and the empirical Manning equation. The value is,

$$y = (2/3)0.242 - 0.318 - (1/2)0.317 = -0.315. \tag{58}$$

We take, $D_\mathrm{T} = 7/4$. Using the observed values of f and z, the empirically computed values of the scaling exponents are (using Eq. 55):

$$\alpha - \beta = \frac{0.242 - 4/7}{-0.317} = \frac{0.329}{0.317} = 1.038. \tag{59}$$

Similarly, using Eq. (56) and the empirical exponents we obtain

$$3\alpha - 2\beta = \frac{1-12/7}{-0.317} = \frac{0.714}{0.317} = 2.253. \tag{60}$$

Solving Eqs. (59) and (60) gives, $\alpha = 0.175$, $\beta = -0.864$, which satisfy the constraints on α and β described above.

The predicted value of the Manning scaling exponent using Eq. (53) is,

$$y = \frac{f}{6} - \frac{z(\beta-1)}{2} = \frac{0.242}{6} + \frac{0.317(-1-0.864)}{2} = -0.255. \tag{61}$$

There is some discrepancy between the observed and the predicted values. It seems to come from the observed exponents of width and velocity. The discrepancy in the width exponent affects the velocity exponent, which in turn affects the Manning friction exponent. The reader may compare the empirical H-G exponents in the Taieri basin with those in the Ashley basin to get a comparative idea of the field measured values of the H-G exponents in these two basins that have comparable scales and climates. The measured values of f and z are comparable as one expects, but not of b and m. Reasons for this potential discrepancy may lie in $Q = kA^c$ relationship if $c < 1$. This important topic is examined next.

7.4 Test of Horton laws and $Q = kA^c$ relationship for the Ashley basin

Mantilla (2014) is testing the Horton laws for the H-G variables in the two NZ basins considered here and a few others basins for which he has data. He has kindly shared some of his analysis with us for the Ashley basin that has $\Omega = 6$. Mantilla (2014) extracted the Ashley basin geomorphology from the Digital Elevation Model (DEM) data using the software CUENCAS (Mantilla and Gupta, 2005). His first set of results pertains to the Horton laws for drainage area and stream numbers as shown in Figs. 3 and 4. The Horton laws hold quite well, and the observed $R_\mathrm{A} = 4.47$ agrees well with $R_\mathrm{B} = 4.5$ as predicted for the Tokunaga network in Sect. 2.1.

Next, the Horton law for mean stream flow is considered. In making this plot, the theoretical condition $\Omega - \omega \to \infty$ is incorporated by omitting order 6 and 5 streams from the analysis. Mantilla (2014) found that the basin has a large number of the 1-st order streams that are mostly missed in the map that McKerchar et al. (1998) presented. Therefore, the Horton plot is made for streams of order $\omega = 2, 3, 4$, shown in Fig. 5. $R_Q = 3.05$ is observed. It shows that $R_Q = R_\mathrm{A}^\theta$, where $\theta = 0.74$. The reason is that all the streams in a network need not contribute to stream flows. Many physical processes play a role, like space-time variable rainfall, state of dryness or wetness of soil in a basin at the time rainfall begins, which governs infiltration into soil and evaporation from it and so on. The physical parameter, θ, represents the aggregate behavior of the physical processes governing runoff generation, and can take a values less than or equal to 1. It is written as,

$$R_Q = R_\mathrm{A}^\theta. \tag{62}$$

Galster (2007) analyzed several basins to test the relationship $Q = kA^c$. His results show that the studied watersheds could be grouped into two broad categories based on their respective c values: (1) those where $c = 1$ or nearly 1, and (2) those where c is significantly < 1 like 0.8 or 0.5. Other research efforts have been made on understanding the nature of the scaling exponent θ from physical processes (Poveda et al., 2007; Gupta et al., 2010; Furey et al., 2013). Clearly, the Ashley basin shows that $\theta < 1$. Our derivation in Eq. (17) that $R_Q = R_{\mathrm{A}}$ applies to category (1) basins in Galster (2007), but not to category (2) like the Ashley. Therefore, it needs to be generalized to incorporate such basins for which $\theta < 1$.

Table 1 presents a summary of the observed and predicted H-G scaling exponents for the three basins considered above. Predicted values for the exponent y that we presented above using Eq. (53), and for the exponent $b = 1/2$ from Eq. (45).

8 Future research problems: two examples

The above theory can be generalized along several lines. We illustrate two important problems. The first is that the anomalous scaling exponents α and β need to be predicted using physical arguments. Two fundamental physical processes that shape the H-G of channels are transport of suspended sediment load and the bed load that we have not considered so far. There is a huge literature on this subject (Leopold et al., 1964; Singh, 2003). Our ideas on how these two physical processes can be used to determine α and β are rudimentary and are only meant for illustration.

The suspended load increases in proportion to discharge. Therefore, suspended sediment concentration, defined as the ratio of the two, does not change. Leopold et al. (1964, p. 269) gave an expression for sediment concentration, $C \propto (UD)^{0.5} S^{1.5}/n^4$, constancy of C implies that $0.5m + 0.5f + 1.5z - 4y = 0$, or, $0.25 + 1.5z - 4y = 0$ since, $m + f = 1/2$. It gives the first equation in terms of α and β.

The second equation can be developed from considering stream power per unit of bed area, $\varpi = \rho g Q S/W$, which plays a basic role in the bed load transport (Molnar, 2001). Essentially all the theories of bed load transport assume that there is a threshold shear stress, stream power, or mean flow speed, and no erosion occurs below it. During floods, these variables exceed the threshold, and bed load is transported that creates erosion. We expect that a second equation can be obtained from these considerations in terms of α and β. The two equations can be solved to compute α and β.

Tradionally Horton laws have been known in terms of statistical means. Peckham and Gupta (1999) reformulated the Horton laws in terms of probability distributions that are called "generalized Horton law". They presented a framework for drainage areas and channel lengths. Specifically, they gave observational and some theoretical arguments to show that probability distributions of all drainage areas rescaled by their means, $A_\omega/\overline{A}_\omega$ collapse into a common probability distribution. Let us consider drainage areas, $A_\omega/\overline{A}_\omega$ to illustrate generalized Horton laws. There are two components to this argument.

i. A Horton law for the mean drainage areas, $\overline{A}_\omega$ of order ω, holds that can be written as,

$$\overline{A}_\omega = R_\mathrm{A}^{\omega-1}\overline{A}_1, \quad \omega = 1,2,\ldots, \tag{63}$$

where R_A, is the Horton's area ratio. It is illustrated in the Whitewater basin, Kansas, USA in Fig. 7.

ii. A generalized Horton law is defined as

$$A_{\omega+1}/\overline{A}_{\omega+1} \overset{d}{=} A_\omega/\overline{A}_\omega, \tag{64}$$

or,

$$A_{\omega+1} \overset{d}{=} \left(\overline{A}_{\omega+1}/\overline{A}_\omega\right) A_\omega, \quad \omega = 1,2,\ldots \tag{65}$$

where $\overset{d}{=}$ means that the probability distributions of the rescaled areas on both sides of Eq. (65) are the same. Since the Horton law holds for the mean areas given in Eq. (63), it follows from Eq. (65) that,

$$A_{\omega+1} \overset{d}{=} R_\mathrm{A} A_\omega, \quad \omega = 1,2,\ldots \tag{66}$$

This feature is illustrated for drainage areas in the Whitewater basin, Kansas, USA in Fig. 8.

Let us consider the dependence of channel widths on discharge. Both are treated as random variables. Therefore the results obtained in Sect. 5.2 can be interpreted as that for the means and written as, $\overline{W}(Q_\omega) = c\overline{Q}_\omega^b$. We conjecture that the generalized Horton law holds for the rescaled channel widths, and write it as,

$$W(Q_{\omega+1}) = \left(\frac{\overline{Q}_{\omega+1}}{\overline{Q}_\omega}\right)^b W(Q_\omega), \quad \omega = 1,2,\ldots \tag{67}$$

This is an equality among random variables as shown for drainage areas in Eq. (65). It means that the probability distribution of $W(Q_{\omega+1})$ can be computed from the probability distribution of $W(Q_\omega)$ provided a Horton law of mean widths and the value of b are known. Both these features are predicted in Sect. 5 for self-similar Tokunaga networks. We interpret them as those for the means. Our conjecture is made in the light of the result that the Tokunaga networks are a special case of a subclass of RSN (Veitzer and Gupta, 2000). In view of these arguments, we can write,

$$\overline{W}_\omega = R_W^{\omega-1}\overline{W_1}, \quad \omega = 1,2,\ldots, \tag{68}$$

where, $R_W = R_Q^b$ is the Horton ratio for the mean widths. We conjecture based on these arguments that Horton laws hold for all the H-G variables measured in the two New Zealand basins that were analyzed in the Sect. 7.3. We supported this conjecture for the validity of Horton laws for widths and stream flows in the Ashley basin in Sect. 7.3. Both the NZ basins have the necessary data sets to test our conjecture regarding applicability of the generalized Horton laws for all the H-G variables considered in this paper. Mantilla (2014) is conducting this research.

9 Conclusions

There has been important progress in topological and geometric theories to explain the related Horton's law for stream bifurcation, drainage areas and stream lengths as asymptotic relations. But progress on Horton laws for the H-G variables has been long overdue. We made a contribution to this important problem, and laid the theoretical foundations of a H-G theory in the SS Tokunaga networks. Our main findings are summarized below:

1. We used the Buckingham–Pi theorem and identified six dimensionless basin numbers in Sect. 3, which served as a basis to develop the theory in the subsequent sections.

2. A mass conservation equation was specified in Strahler ordered networks. A link-based equation as a special case of our equation has been known (Gupta et al., 2007). We solved it in Tokunaga SS networks using the results from Mcconnell and Gupta (2008) and derived a mass conservation equation in the limit as, $\Omega - \omega$ goes to infinity in terms of Horton bifurcation and discharge ratios in Sect. 4.

3. We gave an analytical derivation of the H-G relations as power-law functions of discharge. The derivation is based on the assumptions that the H-G variables are homogeneous and self-similar functions of discharge. The Horton laws are extended to width, depth and velocity in Tokunaga SS networks using the results from Sect. 4. Within the dimensional analysis framework, the SS-1 given in Barenblatt (1996) is used to predict the width exponent, $b = 1/2$. These results are given in Sect. 5.

4. Assuming that SS-1 holds for slope, we predicted the Horton's laws for S, U and D, and their exponents. Our predictions agree with the exponents given in the optimal channel network model (OCN) (Rodríguez-Iturbe et al., 1992), but OCN does not consider Horton laws. Our theoretical framework is based in self-similarity, and does not use any optimality assumptions. Published previous field studies cited here have shown that the OCN predictions do not agree with observations. These results are given in Sect. 6. We assert following Barenblatt (1996) that the problem lies in the assumption that SS-1 holds for slope, because it goes to zero in the limit of large basin order.

5. SS-2 is required to deal with the case when one or more dimensionless numbers go to zero in the limit (Barenblatt, 1996). Therefore, we consider that SS-2 holds for slopes, which gives

rise to two anomalous scaling exponents, α and β that come from two dimensionless numbers in Sect. 3. We derived Horton's law for S, D, and U in Sect. 7, but the H-G exponents become functions of α and β. To predict these two anomalous scaling exponents from geophysics, we suggest that consideration of sediment transport is needed, as briefly discussed in Sect. 8.

6. We tested the predictions of our theory against observations using published H-G data from three river basins in Sect. 7.3. Since we do not give a physical prediction of α and β, we back calculated them using observed exponents for D and S. In this process we lost testability. To make progress with testing our theory, we consider a fifth H-G variable, namely Manning's friction, that could be estimated from data on slope, velocity, width and depth, and predicted from our theory. The predictions are excellent for the first two but Ashley basin shows some discrepancy, as given in Sect. 7.3.

7. The empirical observation that $R_Q \neq R_{\mathrm{A}}$, illustrated here for the Ashley basin hold more generally as Galster (2007) discussed, and need further considerations. In particular, some of the assumptions leading to Eq. (16) do not hold, because it was shown rigorously that under those assumptions $R_Q = R_{\mathrm{A}} = R_{\mathrm{B}}$. In the text some possible physical explanations are suggested. However, to incorporate this hydrologic feature in Tokunaga networks, the generator expression given in Eq. (2) needs to be modified so that all the streams that don't contribute to stream flows are removed in the derivation of Eq. (16).

8. The estimation of the anomalous exponents from physical principles and the consideration of sediment transport are needed for a definite test of the theory.

9. Two NZ basins analyzed here show statistical variability in the H-G variables. We showed some results from Mantilla (2014) for the Ashley basin for the existence of Horton laws for widths and stream flows. He is testing for the presence of generalized Horton laws in all the H-G variables for a further development of this theory. Last but not least, more H-G data on river networks is needed in different climates to test theoretical predictions as they become available. This is an expensive proposal, which requires international cooperation to make progress.

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Table 1. Summary of observed and predicted H-G scaling exponents. The sources of the data are Ibbitt et al. (1998) for the Taieri River Basin in New Zealand; McKerchar et al. (1998) for the Ashley River Basin in New Zealand; Leopold et al. (1964, Table 7.5, p. 244) for the Brandywine creek, PA in the United States.

		Basin		
Variable	Exponent	Taieri	Asheley	Brandywine
		Observed		
$U \propto Q^m$	m	0.238	0.318	0.050
$W \propto Q^b$	b	0.517	0.440	0.420
$D \propto Q^f$	f	0.247	0.242	0.450
$S \propto Q^z$	z	−0.315	−0.317	−1.070
$n' = Q^y$	y	−0.231	−0.315	−0.280
Estimated using $D_{\mathrm{T}} = 7/4, f$ and z				
	α	0.208	0.175	0.441
	β	−0.822	−0.864	0.327
Predicted using Eqs. (45) and (53)				
$W \propto Q^b$	b	0.500	0.500	0.500
$n' \propto Q^y$	y	−0.246	−0.255	−0.285

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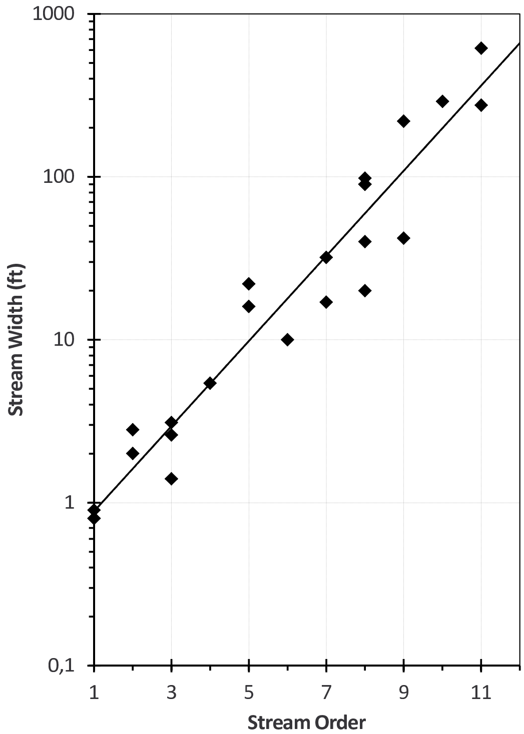

Figure 1. Reproduction of the original figure of Leopold and Miller (1956, Fig. 19, p. 23) showing the relation of stream width to stream order in arroyos. Numbers beside points correspond to different points in the network.

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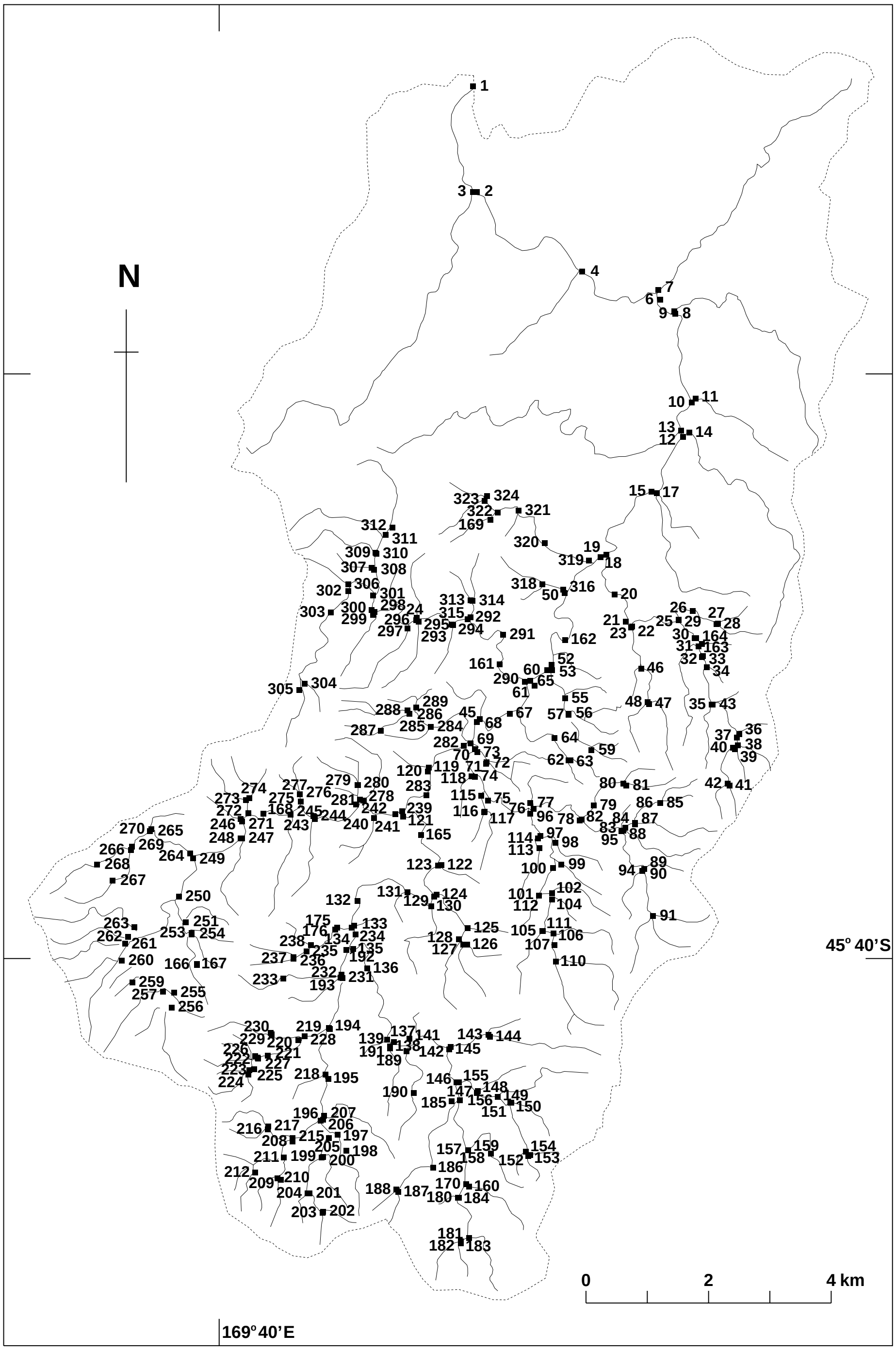

Figure 2. Reproduction of the original figure of Ibbitt et al. (1998, Fig. 1) showing the river network of the Taieri basin in New Zealand along with measurements sites in the network.

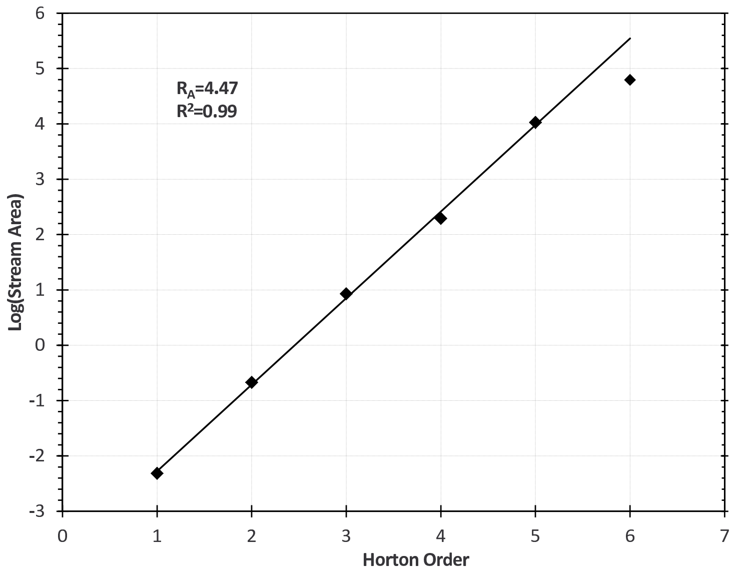

Figure 3. Horton analysis of upstream areas (including orders 2, 3, and 4) for Ashley River Basin (McKerchar et al., 1998), results kindly provided by Mantilla (2014).

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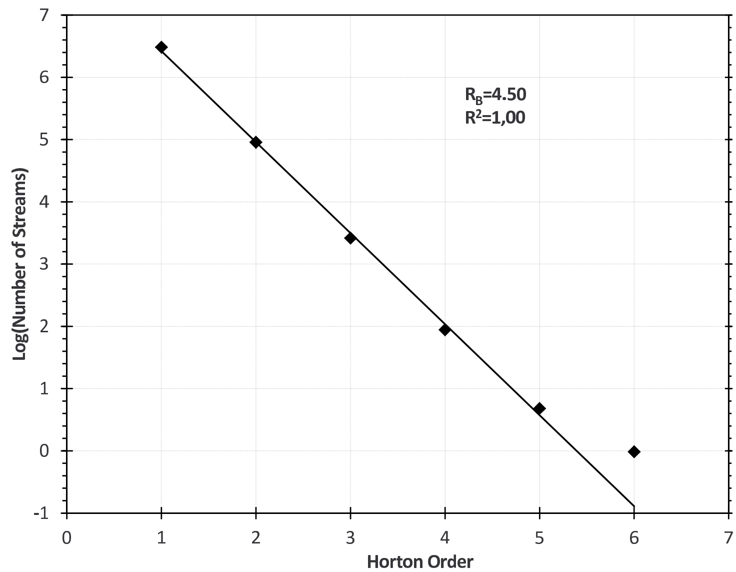

Figure 4. Horton analysis of stream numbers (including orders 2, 3, and 4) for Ashley River Basin (McKerchar et al., 1998), results kindly provided by Mantilla (2014).

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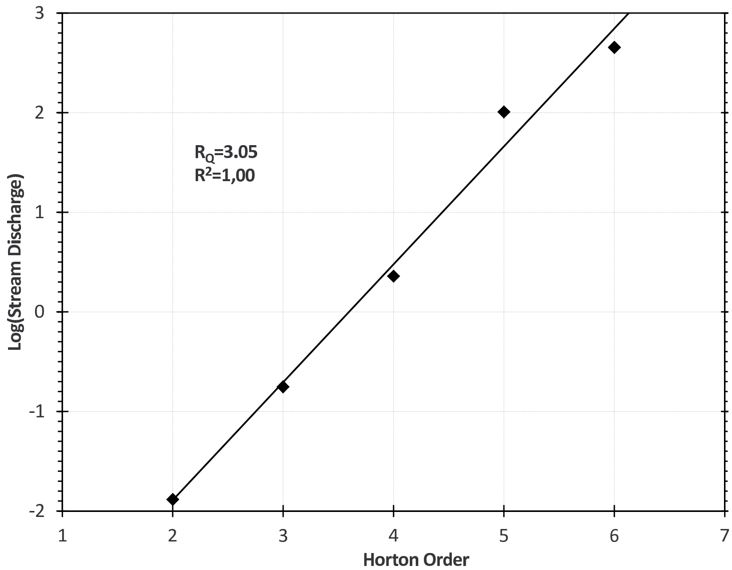

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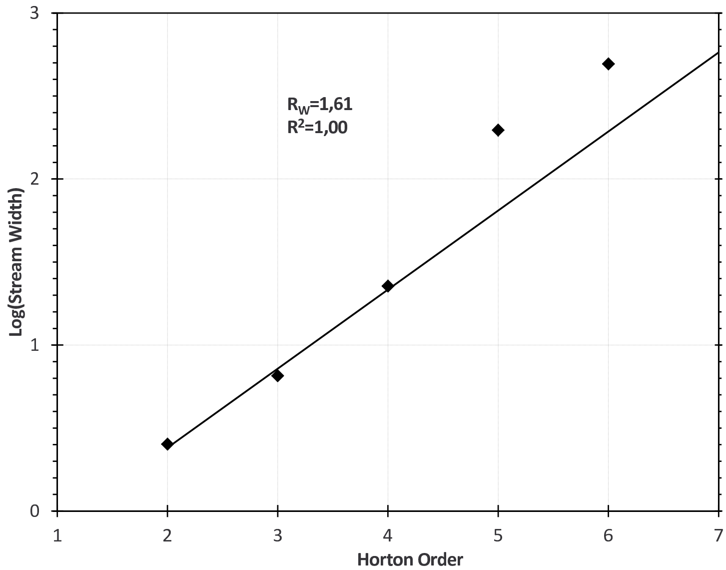

Figure 6. Horton plots for Hydraulic Geometric variables (including order 2, 3, and 4) for Ashley river basin (McKerchar et al., 1998), results kindly provided by Mantilla (2014).

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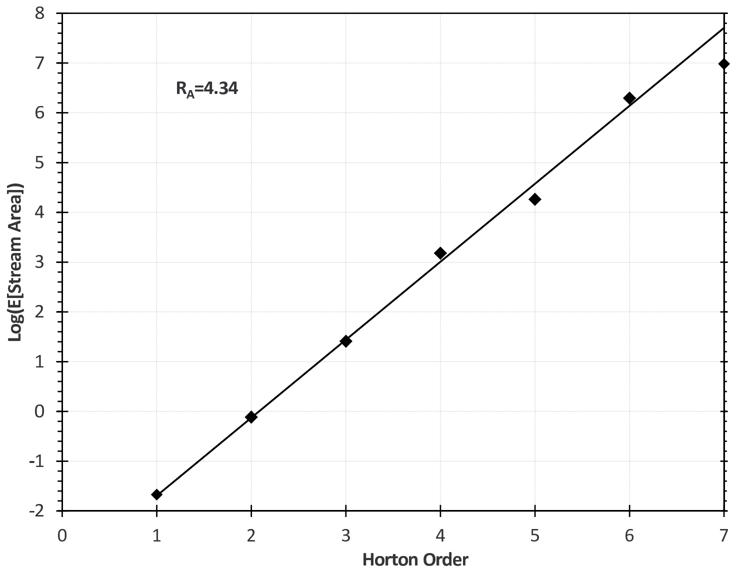

Figure 7. Reproduction of the original figure of Mantilla and Gupta (2005, Fig. 2) showing the scaling of mean drainage area with order (Horton law) of the river network of the Whitewater Basin, Kansas, US.

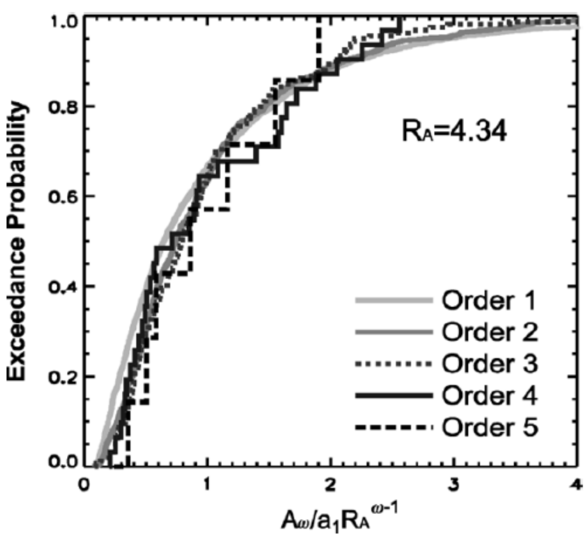

Figure 8. Reproduction of the original figure of Mantilla and Gupta (2005, Fig. 2) showing the statistical scaling of the probability distribution of drainage area with order (generalized Horton law) of the river network of the Whitewater basin, Kansas, US.

West, G. B., Brown, J. H., and Enquist, B. J.: A general model for the origin of allometric scaling laws in biology, Science, 276, 122–126, 1997.