

# Time Dependent Long's Equation

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## Abstract.

Long's equation describes steady-state two-dimensional stratified flow over terrain. Its numerical solutions under various approximations were investigated by many authors. Special attention was paid to the properties of the gravity waves that are predicted to be generated as a result. In this paper we derive a time-dependent generalization of this equation and investigate analytically its solutions under some simplifications. These results might be useful in the experimental analysis of gravity waves over topography and their impact on atmospheric modeling.

## 1 Introduction

Long's equation (Long 1953,1954,1955,1959) model the flow of inviscid stratified fluid in two dimensions over terrain. When the base state of the flow (that is the unperturbed flow field far upstream) is without shear the solutions of this equation are in the form of steady lee waves. Solutions of this equation in various settings and approximations were studied by many authors (Drazin 1961, Drazin and Moore and 1967, Durran 1992, Lily and Klemp 1979, Peltier and Clark 1983, Smith 1980, 1989, Yih 1967). The most common approximation in these studies was to set Brunt-Väisälä frequency to a constant or a step function over the computational domain. Moreover the values of the parameters  $\beta$  and  $\mu$  which appear in this equation were set to zero. In this (singular) limit of the equation the nonlinear terms and one of the leading second order derivatives in the equation drop out and the equation reduces to that of a linear harmonic oscillator over two dimensional domain. Careful studies (Lily and Klemp 1979) showed that these approximations are justified unless wave breaking is present in the solution (Peltier and Clark 1983, Miglietta and Rotunno 2014).

Long's equation provides also the theoretical framework for the analysis of experimental data (Fritts and Alexander 2003, Shutts et al 1988, Vernin et al 2007, Jumper et al 2004) under the assumption of shearless base flow. (An assumption which, in general, is not supported by the data). An extensive list of references appears in (Fritts and Alexander 2003, Baines 1995, Nappo 2012, Yhi 1980).

An analytic approach to the study of this equation and its solutions was initiated recently by the current author (Humi 2004). We showed that for a base flow without shear and under rather mild restrictions the nonlinear terms in the equation can be simplified. We also identified the "slow variable" that controls the nonlinear oscillations in this equation and using phase averaging approximation derived a formula for the attenuation of the stream function perturbation with

height. This result is generically related to the presence of the nonlinear terms in Long's equation. We explored also different formulations of this equation [Humi 2007,2009] and the effect of shear on the solutions of this equation. (Humi 2006,2010)

One of the major obstacles to the application of Long's equation in realistic applications is due to the fact that it is restricted to the description of steady states of the flow. It is therefore our objective in this paper to derive a time-dependent generalization of this equation and study the properties of its solutions. The resulting system contains two equations for the time evolution of the density and the stream function. While the equation for the stream function is rather complicated it can be simplified in two instances. The first corresponds to the classical (steady state) Long's equation while the second is time dependent and new (as far as we know). In this paper we explore the properties of the flow in this second case which might find some applications in the analysis of experimental data about gravity waves (Vernin 2007, Jumper 2004, Nappo 2012) and its application to atmospheric modeling (Jadwiga et al 2010, Geller et al 2013).

The plan of the paper is as follows: In Sec. 2 we derive the time dependent of Long's equation. In Sec. 3 we consider the time evolution and proper boundary conditions on shearless flow over topography. We end up with summary and conclusion in Sec 4.

## 2 Derivation of the time dependent Long's Equation

In the paper we consider the flow in two dimensions  $(x, z)$  of an inviscid, stratified and **weakly compressible** fluid that is modeled by the following equations:

$$u_x + w_z = 0 \quad (1)$$

$$\rho_t + u\rho_x + w\rho_z = 0 \quad (2)$$

$$\rho(u_t + uu_x + ww_z) = -p_x \quad (3)$$

$$\rho(w_t + uw_x + ww_z) = -p_z - \rho g \quad (4)$$

where subscripts indicate differentiation with respect to the indicated variable,  $\mathbf{u} = (u, w)$  is the fluid velocity,  $\rho$  is its density,  $p$  is the pressure and  $g$  is the acceleration of gravity.

One possible interpretation of (1), is that the fluid is incompressible while (2) is an advection equation for a scalar (viz.  $\rho$ ) by the flow. However since we consider in the following derivatives of the density we refer to this formulation as representing a "weakly compressible fluid".

We can non-dimensionalize these equations by introducing

$$\begin{aligned} \bar{x} &= \frac{x}{L}, \quad \bar{z} = \frac{N_0}{U_0} z, \quad \bar{u} = \frac{u}{U_0}, \quad \bar{w} = \frac{LN_0}{U_0^2} w \\ \bar{\rho} &= \frac{\rho}{\rho_0}, \quad \bar{p} = \frac{N_0}{gU_0\rho_0} p \end{aligned} \quad (5)$$

where  $L$ ,  $U_0$ , and  $\rho_0$  represent respectively characteristic length, velocity and density.  $N_0$  is the characteristic Brunt-Väisälä frequency

$$N_0^2 = -\frac{g}{\bar{\rho}} \frac{d\bar{\rho}}{dz}. \quad (6)$$

where  $\bar{\rho}$  is the ambient density profile of the atmosphere. In the following we let  $N_0$  to be a constant.

In these new variables eqs (1)-(4) take the following form (for brevity we drop the bars)

$$u_x + w_z = 0 \quad (7)$$

$$\rho_t + u\rho_x + w\rho_z = 0 \quad (8)$$

$$\beta\rho(u_t + uu_x + ww_z) = -p_x \quad (9)$$

$$\beta\rho(w_t + uw_x + ww_z) = -\mu^{-2}(p_z + \rho) \quad (10)$$

where

$$\beta = \frac{N_0 U_0}{g} \quad (11)$$

$$\mu = \frac{U_0}{N_0 L}. \quad (12)$$

$\beta$  is the Boussinesq parameter (Shutts et al 1988, Baines 1995) which controls stratification effects (assuming  $U_0 \neq 0$ ) and  $\mu$  is the long wave parameter which controls dispersive effects (or the deviation from the hydrostatic approximation). In the limit  $\mu = 0$  the hydrostatic approximation is fully satisfied, (Baines 1995, Nappo 2012). **It should be observed that these two parameters  $\beta$  and  $\mu$  encapsulate the atmospheric conditions which impact the creation of gravity waves over terrain although one of them ( $\mu$ ) can be suppressed by additional scaling.**

In view of eq. (7) we can introduce a stream function  $\psi$  so that

$$u = \psi_z, \quad w = -\psi_x. \quad (13)$$

Using this stream function we can rewrite eq. (8) as

$$\rho_t + J\{\rho, \psi\} = 0 \quad (14)$$

where for any two (smooth) functions  $f, g$

$$J\{f, g\} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial x} \quad (15)$$

Using  $\psi$  the momentum equations (9),(10) become

$$\beta\rho(\psi_{zt} + \psi_z\psi_{zx} - \psi_x\psi_{zz}) = -p_x \quad (16)$$

$$\beta\mu^2\rho(-\psi_{xt} - \psi_z\psi_{xx} + \psi_x\psi_{xz}) = -p_z - \rho \quad (17)$$

We can suppress  $\mu$  from the system (14), (16) and (17) if we introduce the following normalized independent variables

$$\bar{t} = \frac{t}{\mu}, \quad \bar{x} = \frac{x}{\mu}, \quad \bar{z} = z, \quad \mu \neq 0. \quad (18)$$

Eqs. (14), (16) remain unchanged and (17) becomes

$$\beta\rho(-\psi_{xt} - \psi_z\psi_{xx} + \psi_x\psi_{xz}) = -p_z - \rho \quad (19)$$

where we dropped the bars on  $t, x, z$ . However we observe that in these coordinates  $\psi_z = u$  and  $\psi_x = -\mu w$ .

Thus after all these transformations the system of equations governing the flow is (14), (16) and (19).

To eliminate  $p$  from (16) and (18) we differentiate these equations with respect to  $z, x$  respectively and subtract. This leads to

$$\begin{aligned} & \beta\rho_z(\psi_{zt} + \psi_z\psi_{zx} - \psi_x\psi_{zz}) + \\ & \beta\rho(\psi_{zzt} + \psi_z\psi_{zzx} - \psi_x\psi_{zzz}) - \\ & \beta\rho_x(-\psi_{xt} - \psi_z\psi_{xx} + \psi_x\psi_{xz}) - \\ & \beta\rho(-\psi_{xxt} - \psi_z\psi_{xxx} + \psi_x\psi_{xxz}) = \rho_x \end{aligned} \quad (20)$$

The sum of the second and fourth terms in this equation can be rewritten as

$$\beta\rho[\nabla^2\psi]_t + J\{\nabla^2\psi, \psi\}. \quad (21)$$

(However observe that when  $\mu \neq 1$ ,  $\nabla^2\psi$  does not represent the flow vorticity due to the transformation (18) and therefore the sum of the two terms in (21) is not zero in general.)

To reduce the first and third terms in (20) we use (14). We obtain

$$\begin{aligned} & \beta[\rho_z(\psi_{zt} + \psi_z\psi_{zx} - \psi_x\psi_{zz})] - \\ & \beta[\rho_x(-\psi_{xt} - \psi_z\psi_{xx} + \psi_x\psi_{xz})] = \\ & \beta[\rho_z(\psi_{zt} + \rho_z\psi_z\psi_{zx} - (\rho_t + \rho_x\psi_z)\psi_{zz} + \\ & \rho_x\psi_{xt} + (\psi_x\rho_z - \rho_t)\psi_{xx} - \rho_x\psi_x\psi_{xz})] = \\ & \beta\left\{\rho_z\psi_{zt} + \rho_x\psi_{xt} - \rho_t\nabla^2\psi + \frac{1}{2}J\{(\psi_x)^2 + (\psi_z)^2, \rho\}\right\} \end{aligned} \quad (22)$$

Combining the results of (21) and (22) eq. (20) becomes

$$\begin{aligned} & \rho[(\nabla^2\psi)_t + J\{\nabla^2\psi, \psi\}] + \rho_z\psi_{zt} + \rho_x\psi_{xt} + \\ & \left[-\rho_t\nabla^2\psi + \frac{1}{2}J\{(\psi_x)^2 + (\psi_z)^2, \rho\}\right] = \frac{J\{\rho, z\}}{\beta} \end{aligned} \quad (23)$$

Thus we have reduced the original four equations (1)-(4) to two equations (14) and (23). This system of equations can be considered as the generalization of Long's equation to time dependent flows.

While eq. (23) is rather complicated in general it can be simplified further in some special cases. The first is when one considers the steady state of the flow. (This simplifies also eq. (14)). This restriction leads to Long's equation [Long 1953, 1954, 1955, 1959, Baines 1995, Yhi 1980]. Furthermore if the density derivatives associated with the momentum terms are neglected (23) reduces to the 2D Bousinesq equation. (Tabaei, Akylas and Lamb, 2005). Another case happens when  $\nabla^2\psi = 0$  i.e  $\psi$  is harmonic. (Note however, that this does not imply that the physical vorticity  $\nabla \times \mathbf{u}$  is zero due to the transformation (18) unless  $\mu = 1$ ). Eq. (23) becomes

$$\rho_z\psi_{zt} + \rho_x\psi_{xt} + \frac{1}{2}J\{(\psi_x)^2 + (\psi_z)^2, \rho\} = \frac{J\{\rho, z\}}{\beta}. \quad (24)$$

Observe that the derivatives of  $\rho$  with respect to time are not present in this equation and this is consistent with (1).

However if  $\nabla^2\psi = 0$  we can define  $v_1 = \psi_z$  and  $v_2 = -\psi_x$ . (These definitions use the stretched coordinates of (18)) and then

$$(v_1)_z - (v_2)_x = 0.$$

This implies that there exists a function  $\eta$  so that

$$\eta_x = v_1, \quad \eta_z = v_2.$$

That is

$$\eta_x = \psi_z, \quad \eta_z = -\psi_x$$

Physically these relations imply that  $\eta_x = u$  and  $\eta_z = \mu w$ .

Replacing  $\psi$  by  $\eta$  in (24) yields

$$J\{\eta_t + \frac{1}{2}[(\eta_x)^2 + (\eta_z)^2] + \frac{z}{\beta}, \rho\} = 0 \quad (25)$$

Hence

$$\eta_t + \frac{1}{2}[(\eta_x)^2 + (\eta_z)^2] + \frac{z}{\beta} = R(\rho) \quad (26)$$

where  $R(\rho)$  is a parameter function that can be determined from the asymptotic conditions on the flow. This equation is formally similar to Bernoulli equation with  $\eta$  playing the role of the velocity potential. When  $\mu = 1$ , and the term  $\frac{z}{\beta}$  is interpreted as potential energy  $\eta$  represents potential flow.

To summarize: The equations of the flow in this case are

$$\rho_t + \eta_x\rho_x + \eta_z\rho_z = 0 \quad (27)$$

(which replaces (14)), and (26).

## 2.1 Other Reductions of (23)

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(35)

The reduction of eq. (23) was carried out above under the assumption  $\nabla^2\psi = 0$ . However it can be generalized to case  $\nabla^2\psi = a$  where  $a$  is a constant. To this end we define

$$v_1 = \psi_z, \quad v_2 = -\psi_x + ax$$

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Therefore

$$(v_1)_z - (v_2)_x = 0,$$

which implies that there exists a function  $\eta$  so that

$$\eta_x = v_1, \quad \eta_z = v_2.$$

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Hence

$$\eta_x = \psi_z, \quad \eta_z = -\psi_x + ax \quad (28)$$

Using these relations to substitute  $\eta$  for  $\psi$  in (23) leads to

$$\begin{aligned} \rho_z \eta_{xt} - \rho_x (\eta_z - ax)_t + \left[ -a\rho_t + \frac{1}{2} J\{(\eta_z - ax)^2 + (\eta_x)^2, \rho\} \right] \\ = \frac{J\{\rho, z\}}{\beta}. \end{aligned} \quad (29)$$

Therefore

$$J\{\eta_t, \rho\} - a\rho_t + \frac{1}{2} J\{(\eta_z - ax)^2 + (\eta_x)^2, \rho\} = \frac{J\{\rho, z\}}{\beta}. \quad (30)$$

Hence

$$-a\rho_t + J\{\eta_t + \frac{1}{2}[(\eta_z - ax)^2 + (\eta_x)^2] + \frac{z}{\beta}, \rho\} = 0. \quad (31)$$

Using (14) we have

$$-aJ\{\psi, \rho\} + J\{\eta_t + \frac{1}{2}[(\eta_z - ax)^2 + (\eta_x)^2] + \frac{z}{\beta}, \rho\} = 0. \quad (32)$$

It follows then that

$$-a\psi + \eta_t + \frac{1}{2}[(\eta_z - ax)^2 + (\eta_x)^2] + \frac{z}{\beta} = R(\rho). \quad (33)$$

We can eliminate  $\psi$  from this equation by differentiating with respect to  $z$  and use (28)

$$-a\eta_x + \left[ \eta_t + \frac{1}{2}[(\eta_z - ax)^2 + (\eta_x)^2] \right]_z = -\frac{1}{\beta} + R(\rho)_z \quad (34)$$

## 3 Time Evolution of Stratified Flow

In this section we shall consider the time evolution of a stratified shearless base flow viz. a flow which satisfies as  $t \rightarrow -\infty$

$$\lim_{x \rightarrow -\infty} \rho^0(t, x, z) = \frac{H-z}{H}, \quad \lim_{x \rightarrow -\infty} u = 1, \quad \lim_{x \rightarrow -\infty} v = 0$$

i.e. the far upstream flow is independent of time and satisfies asymptotically  $u = 1$ ,  $v = 0$  and  $\rho^0$  is stratified with height ( $H$  is a height at which  $\rho^0 \approx 0$ ). The conditions on  $u, v$  imply that asymptotically  $\eta^0 = x$ . **We note that this is the standard setup that has been used to analyze experimental observations of gravity waves (Jumper et al 2004, Vermin et al 2007, Shutts et al, 1988). The solutions of (26) and (27) which we discuss below represent therefore gravity waves which are generated by low lying topography.**

In these limits Eq. (27) is satisfied. Substituting these limiting values in (26) we obtain that

$$R(\rho) = \frac{z}{\beta} + \frac{1}{2} = \frac{H(1-\rho)}{\beta} + \frac{1}{2} \quad (36)$$

However it is obvious that different profiles of the base flow will yield different  $R(\rho)$ .

We now consider perturbations from the (shearless) base flow described by (35) due to shape of the topography viz.

$$\eta = \eta^0 + \epsilon\phi, \quad \rho = \rho^0 + \epsilon\zeta. \quad (37)$$

From eqs. (26), (27) we obtain to first order in  $\epsilon$  the following equations for  $\phi$  and  $\zeta$

$$\frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x} + \frac{H\zeta}{\beta} = 0 \quad (38)$$

$$\frac{\partial\zeta}{\partial t} + \frac{\partial\zeta}{\partial x} - \frac{1}{H} \frac{\partial\phi}{\partial z} = 0 \quad (39)$$

To find the general form of the solution of these equations we use (38) to express  $\zeta$  in terms of  $\phi$  and substitute in (39). This yields the following equation for  $\phi$

$$\frac{\partial^2\phi}{\partial t^2} + 2\frac{\partial^2\phi}{\partial t\partial x} + \frac{\partial^2\phi}{\partial x^2} + \frac{1}{\beta} \frac{\partial\phi}{\partial z} = 0 \quad (40)$$

It is possible to find "elementary solutions" to this equation by separation of variables if we let

$$\phi = p(t, x)F(z),$$

where  $c$  is arbitrary positive constant so that  $\phi$  to represents a perturbation moving forward in time. This leads to

$$\frac{\partial^2 p}{\partial t^2} + 2\frac{\partial^2 p}{\partial t\partial x} + \frac{\partial^2 p}{\partial x^2} = -\frac{1}{\beta} \frac{F(z)'}{F(z)} = -\omega^2 \quad (41)$$

where  $\omega^2$  is the separation of variables constant. Primes denote differentiation with respect to the appropriate variable.

Solving (41) we obtain the following elementary solution for  $\phi$

$$\phi_\omega = C_\omega \exp[\beta\omega^2 z] [G(x-t) \cos \omega t + K(x-t) \sin \omega t]$$

255 where  $G(x-t)$ ,  $K(x-t)$  are arbitrary smooth functions and  $C_\omega$  is a constant.

The corresponding solution for  $\zeta$  can be obtained by substituting this result in (38)

$$\zeta = C_\omega \frac{\beta\omega}{H} \exp[\beta\omega^2 z] [G(x-t) \cos\omega t - K(x-t) \sin\omega t]. \quad (43)$$

260 Hence the general solution for  $\phi$  can be written as

$$\phi = \int_0^\infty \exp[\beta\omega^2 z] [G_\omega(x-t) \cos\omega t + K_\omega(x-t) \sin\omega t] d\omega \quad (44)$$

with similar expression for  $\zeta$ .

### 3.1 Boundary Conditions

265 We consider a flow in an unbounded domain over topography with shape  $f(x)$  and maximum height  $h$  and impose the following boundary conditions on  $\rho$  and  $\psi$  in the limits  $x = -\infty$  and  $t = -\infty$

$$\psi(-\infty, -\infty, z) = z, \quad \rho(-\infty, -\infty, z) = \rho^0(z). \quad (45)$$

270 (This implies that in this limits  $\eta = x$ ).

At the topography we impose the following boundary condition on  $\rho$  at  $t = 0$

$$\rho(0, x, \epsilon f(x)) = \rho^0(\epsilon f(x)) = \frac{H - \epsilon f(x)}{H} \quad (46)$$

but

$$\rho(0, x, \epsilon f(x)) \approx \rho^0(0, x, 0) + \epsilon \zeta(0, x, z).$$

Hence at the topography

$$\zeta(0, x, \epsilon f(x)) = -\frac{f(x)}{H}. \quad (47)$$

280 To derive the corresponding boundary condition for  $\eta$  we first consider the appropriate boundary condition on the stream function  $\psi$  along the topography. To this end we assume that the topography is a line on which the stream function is constant and this constant can be chosen to be zero. For the base flow described in (45)  $\psi_0 = z$  and  $\psi = \psi_0 + \epsilon\psi_1$  where  $\psi_1$  is the perturbation due to the topography. Hence 315 along the topography

$$0 = \psi_0 + \epsilon\psi_1 = z + \epsilon\psi_1(0, x, \epsilon f(x)) = \epsilon f(x) + \epsilon\psi_1(0, x, \epsilon f(x)) \quad (48)$$

285 Therefore along the topography we let  $\psi_1(0, x, \epsilon f(x)) = -f(x)$ . We now observe that by definition  $\psi_x = \eta_z$ . But  $\psi_x = \epsilon\psi_x^1 = -\epsilon f'(x)$ , (where primes denote differentiation

with respect to  $x$ ) and  $\eta = -x + \epsilon\phi$ . Therefore we infer that the boundary condition on  $\eta$  along the topography is

$$\phi_z(0, x, \epsilon f(x)) = -f'(x) \quad (49)$$

(which is consistent with (39)).

295 As to the boundary condition on  $\eta(t, \infty, z)$  we observe that the system (26),(27) contains no dissipation terms and therefore only radiation boundary conditions can be imposed in this limit. (Physically this means that the horizontal group velocity is positive and energy is radiated outward). Similarly at  $z = \infty$  it is customary to impose (following [7]) radiation boundary conditions. However in view of (42)-(43) it is obvious that the perturbation described by these equation is propagating forward in time and this condition is satisfied. A formal verification of this constraint is possible by expressing  $F, G, K$  in these equations using Fourier transform representation.

305 For low lying topography (viz  $\epsilon \ll 1$ ) it is customary to replace the boundary conditions (46), (47) by

$$\zeta(0, x, 0) = -\frac{f(x)}{H}, \quad \phi_z(0, x, 0) = -f'(x), \quad (50)$$

**Example** If  $f(x)$  is given by a "witch of Agnesi" curve then

$$f(x) = \frac{a^2}{(a^2 + x^2)}, \quad f'(x) = -\frac{2a^2 x}{(x^2 + a^2)^2} \quad (51)$$

Let the initial perturbation in  $\rho$  be

$$\zeta(0, x, z) = e^{\beta\lambda^2 z},$$

where  $\lambda$  is a constant. From (50) we infer that the general expression for  $\zeta$  is given by (43) with  $\omega = \lambda$ . Hence at  $t = 0$  we must have

$$G(x) = -\frac{f(x)}{\beta\lambda}$$

Similarly the boundary condition on  $\phi$  yields

$$K(x) = -\frac{f'(x)}{\beta\lambda^2}$$

Figs. 1 and 2 exhibit cross sections of the perturbation at  $z = 2$  and  $x = 20$  at different times with  $C_\omega = 0.1$ ,  $a = 2$ , and  $\lambda = 1$ .

## 4 Summary and Conclusions

Steady state solutions of Long's equation model the vertical structure of plane parallel gravity waves. These solutions are useful, for example, in the parameterizations of unresolved gravity wave drag where the WKBJ approximation is invoked to describe the time dependent amplitude spectrum of a packet of gravity waves propagating in a slowly varying background.

The present paper presents an alternative analytical approach to solve (under several restrictions) this and similar time dependent problems without having to invoke the WKBJ approximation. The analytical insights derived from this approach might be used to complement and verify the numerical results obtained from the WKBJ method.

## References

- Baines P.G. : Topographic effects in Stratified flows. Cambridge Univ. Press, New York, 1995.
- Drazin P.G. : On the steady flow of a fluid of variable density past an obstacle. *Tellus* 13, 239-251, 1961.
- Drazin P.G. and Moore D.W. : Steady two dimensional flow of fluid of variable density over an obstacle. *J. Fluid. Mech.* 28, 353-370, 1967.
- Durrant D.R. : Two-Layer solutions to Long's equation for vertically propagating mountain waves. *Q.J.R. Meteorol. Soc.* 118, 415-433, 1992.
- Fritts, D. C., and Alexander M.J.: Gravity wave dynamics and effects in the middle atmosphere, *Rev. Geophys.*, 41(1), 1003, doi:10.1029/2001RG000106, 2003.
- Geller, Marvin A., and Coauthors : A comparison between gravity wave momentum fluxes in observations and climate models. *J. Climate*, 26, 6383-6405, 2013.
- Humi M : On the Solution of Long's Equation Over Terrain. II *Nuovo Cimento C* 27, 219-229, 2004.
- Humi M : On the Solutions of Long's Equation with Shear *SIAM J. Appl. Math.* 66, No. 6 1839-1852, 2006.
- Humi M : Density Representation of Long's Equation, *Nonlin. Processes Geophys.* 14, 273-283, 2007.
- Humi M : Long's Equation in Terrain Following Coordinates *Nonlin. Processes Geophys.*, 16, 533-541, 2009 .
- Humi M : The Effect of Shear on the Generation of Gravity Waves, *Nonlin. Processes Geophys* 17, 201-210, 2010.
- G.Y. Jumper, E. A. Murphy, A.J.Ratkowski and J. Vernin : Multi-sensor Campaign to correlate atmospheric optical turbulence to gravity waves, *AIAA paper AIAA-2004-1077*, 2004.
- Lily D.K. and Klemp J.B. : The effect of terrain shape on nonlinear hydrostatic mountain waves. *J. Fluid Mech.* 95, 241-261, 1979.
- Long R.R. : Some aspects of the flow of stratified fluids I. Theoretical investigation. *Tellus* 5, 42-57, 1953.
- Long R.R. : Some aspects of the flow of stratified fluids II. Theoretical investigation. *Tellus* 6, 97-115, 1954.
- Long R.R. : Some aspects of the flow of stratified fluids III. Continuous density gradients. *Tellus* 7, 341-357, 1955.
- Long R.R. : The motion of fluids with density stratification. *J. Geo Res.* 64, 2151-2163, 1959.
- Miglietta M.M. and R. Rotunno R: Numerical Simulations of Sheared Conditionally Unstable Flows over a Mountain Ridge, *J. Atmos. Sci.*, 71, 1747-1762, 2014
- Carmen J. Nappo : *Atmospheric Gravity Waves*, 2nd edition, Academic Press, Boston, 2012.
- Peltier W.R. and Clark T.L. : Nonlinear mountain waves in two and three spatial dimensions. *Q.J.R. Meteorol. Soc* 109, 527-548, 1983.
- Richter, Jadwiga H., Fabrizio Sassi, Rolando R. Garcia,: Toward a physically based gravity wave source parameterization in a general circulation model *J. Atmos.Sci.* 67, 136-156, 2010.
- G.J. Shutts, M.Kitchen and P.H. Hoare : A large amplitude gravity wave in the lower stratosphere detected by radiosonde, *Q.J. R. Meteorological Soc.* 114, 579-594, 1988.
- Smith R.B. : Linear theory of stratified hydrostatic flow past an isolated mountain. *Tellus* 32, 348-364, 1980.
- Smith R.B. : Hydrostatic airflow over mountains. *Advances in Geophysics* 31, 1-41, 1989.
- A. Tabaei,T.R. Akylas and K. G. Lamb,: Nonlinear effects in reflecting and colliding internal wave beams *J. Fluid Mechanics* 526, 217-243, 2005
- Vernin, J., Trinquet, H., Jumper, G., Murphy, E., Ratkowski, A : OHP02 gravity wave campaign in relation to optical turbulence, *Journal Environmental Fluid Mechanics*, 7, 371-382, 2007.
- Yih C-S : Equations governing steady two-dimensional large amplitude motion of a stratified fluid. *J. Fluid Mech.* 29, 539-544, 1967.
- Yih C-S : *Stratified flows*. Academic Press, New York, NY, 1980.

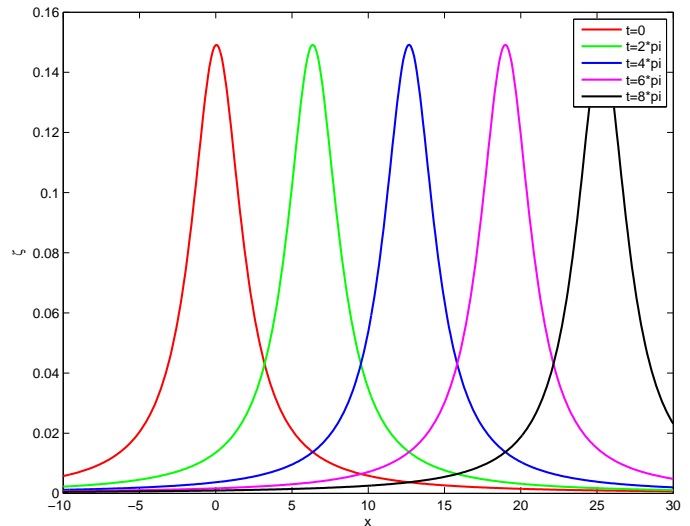


Figure 1. A cross section of the perturbation in  $\rho$  at  $z = 2$

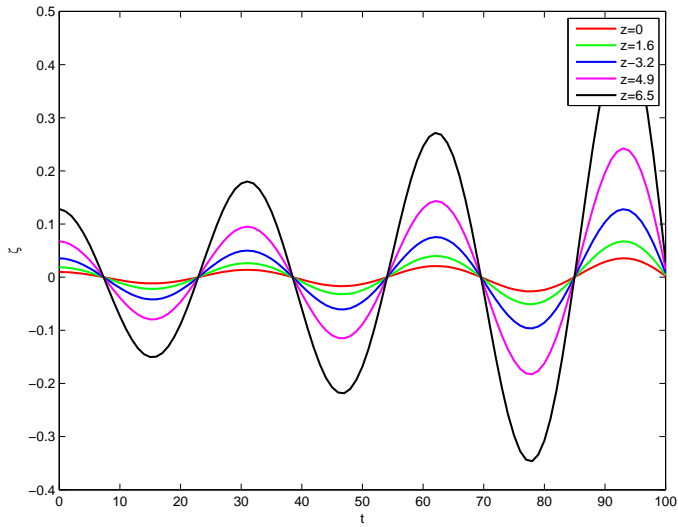


Figure 2. A cross section of the perturbation in  $\rho$  at  $x=20$