

Data assimilation Experiments using the Diffusive Back and Forth Nudging for the NEMO ocean model

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Abstract. The Diffusive Back and Forth Nudging (DBFN) is an easy-to-implement iterative data assimilation method based on the well-known Nudging method. It consists in a sequence of forward and backward model integrations, within a given time window, both of them using a feedback term to the observations. Therefore in the DBFN, the Nudging asymptotic behavior is translated into an infinite number of iterations within a bounded time domain. In this method, the backward integration is carried out thanks to what is called backward model, which is basically the forward model with reversed time step sign. To maintain numeral stability the diffusion terms also have their sign reversed, giving a diffusive character to the algorithm. In this article the DBFN performance to control a primitive equation ocean model is investigated. In this kind of model non-resolved scales are modeled by diffusion operators which dissipate energy that cascade from large to small scales. Thus, in this article the DBFN approximations and their consequences on the data assimilation system setup are analyzed. Our main result is that the DBFN may provide results which are comparable to those produced by a 4Dvar implementation with a much simpler implementation and a shorter CPU time for convergence. The conducted sensitivity tests show that the 4Dvar profits of long assimilation windows to propagate surface information downwards, and that for the DBFN, it is worth using short assimilation windows to reduce the impact of diffusion-induced errors. Moreover, the DBFN is less sensitive to the first guess than the 4Dvar.

Keywords. Data Assimilation, Nudging, Back and Forth Nudging, NEMO

19 1 Introduction

20 The well-known Nudging method is based on the second Newton axiom and consists in adding a
21 forcing term in the right hand side of a given system in order to gently push the model toward a
22 prescribed value. The first appearance of nudging in the geophysical literature was in 1974 (Anthes,
23 1974). In this work the authors proposed the use of nudging to mitigate initialization problems in at-
24 mospheric models. However, a similar algorithm had already been developed by Luenberger (1966).
25 This algorithm has been called "Luenberger observer" or "asymptotic estimator", since under lin-
26 earity and observability hypothesis the estimator error converges to zero for time tending to infinity.
27 It is quite interesting to note that there is no mention of the Luenberger observer in the geophysical
28 literature except in the recent work of Auroux and Blum (2005). More recently, a comprehensive
29 study on the nudging method and its variants was produced by Blum et al. (2008) and Lakshmivara-
30 han and Lewis (2012).

31 The first appearance of a successful application of nudging to ocean Data Assimilation (DA) was
32 in 1992 in a work that assimilated sea surface height derived from satellite measurements into a
33 quasi-geostrophic layered model (Verron, 1992). Since then, the method has been successfully ap-
34 plied to several oceanographic numerical problems such as the estimation of boundary conditions
35 (Marchesiello et al., 2001; Chen et al., 2013), downscaling (Li et al., 2012), and other DA problems
36 (Verron, 1992; Haines et al., 1993; Blayo et al., 1994; Lewis et al., 1998; Killworth et al., 2001;
37 Thompson et al., 2006). Concerning applications to DA problems, the weights given to the model
38 and the observations are generally not based on any optimality condition, but are rather scalars or
39 Gaussian-like functions constructed based on physical assumptions or empirical considerations. The
40 appeals of this method are the simplicity of implementation in complex numerical models, the low
41 computational power required and the time smoothness of the solution.

42 The increasing availability of computing power has allowed to use more advanced data assimi-
43 lation methods. In general, these methods use information on the model statistics and observations
44 errors to weight the model-observations combination. Two of these methods that are widely used by
45 prediction centers are the ensemble Kalman filter- EnKF (Evensen, 1994) and its variations (Pham,
46 2001; Hunt et al., 2007), and the four dimensional variational method 4Dvar (Le Dimet and Tala-
47 grand, 1986; Courtier et al., 1994). For the first, the numerical costs are due to the propagation of the
48 ensemble, usually formed by tenths of members, to calculate the forecast. For the second, the costs
49 are due to the need of minimizing a cost function in a very large state space (10^8 variables). This
50 requires several iterations of the minimization algorithm, which involves several integrations of the
51 direct and adjoint models.

52 However, even with the growing interest in these complex techniques built on solid theoretical
53 arguments, nudging has not been left aside. Recent works have used nudging along with more
54 advanced methods such as Optimal interpolation (Clifford et al., 1997; Wang et al., 2013), EnKF
55 (Ballabrera-Poy et al., 2009; Bergemann and Reich, 2010; Lei et al., 2012; Luo and Hoteit, 2012),

4DVar (Zou et al., 1992; Stauffer and Bao, 1993; Vidard et al., 2003; Abarbanel et al., 2010) or particle filters (Luo and Hoteit, 2013; Lingala et al., 2013) to extract the best of each method. In the particular case of the hybridization with the EnKF proposed by Lei et al. (2012), the resulting algorithm takes the advantage of the dynamical propagation of the covariance matrix from the EnKF and uses nudging to mitigate problems related to the intermittence of the sequential approach, which among other things entails the possible discarding of some observations.

Recently, Auroux and Blum (2005) revisited the nudging method and proposed a new observer called Back and Forth Nudging (BFN). The BFN consists in a sequence of forward and backward model integrations, both of them using a feedback term to the observations, as in the direct nudging. The BFN integrates the direct model backwards in time avoiding the construction of the adjoint and/or tangent linear models needed by 4DVar. Therefore, it uses only the fully non-linear model to propagate information forward and backward in time. The nudging gain, which has an opposite sign with respect to the forward case, has a double role: push the model toward observations and stabilize the backward integration, which is especially important when the model is not reversible.

The BFN convergence was proved by Auroux and Blum (2005) for linear systems of ordinary differential equations and full observations, by Ramdani et al. (2010) for reversible linear partial differential equations (Wave and Schrödinger equations), by Donovan et al. (2010) and Leghtas et al. (2011) for the reconstruction of quantum states and was studied by Auroux and Nodet (2012) for non-linear transport equations. The BFN performance in numerical applications using a variety of models, including non-reversible models such as a Shallow Water (SW) model (Auroux, 2009) and a Multi-Layer Quasi-Geostrophic (LQG) model (Auroux and Blum, 2008), are very encouraging. Moreover, by using a simple scalar gain, it produced results comparable to those obtained with 4DVar but with lower computational requirements (Auroux, 2009; Auroux et al., 2012).

In this article we present for the first time a BFN application to control a primitive equation ocean model. The numerical model used is NEMO (Madec, 2008), currently used by the French operational center, Mercator Océan (<http://www.mercator-ocean.fr/fre>), to produce and deliver ocean forecasts. The well-known idealized double gyre configuration at eddy-permitting resolution is used. This configuration has the advantage of being simple from the geometry and forcings point of view at the same time it reproduces most of features found in a middle latitude ocean basin.

The BFN application to control a primitive equation ocean model represents a new challenge due to the increased model complexity. Among the differences between NEMO and the simplified oceanic models used by Auroux and Blum (2008) and Auroux (2009) stand out the more complex relationship between the variables in the former since no filtering technique is used in the derivation of the physical model (except the Boussinesq approximation which is also considered by the SW and LQG models), and the inclusion of an equation for the conservation of the thermodynamical properties. The latter requires the use of a nonlinear state equation to couple dynamical and thermodynamical variables.

Furthermore, the vertical ocean structure represented by NEMO is more complex than the vertical ocean structure represented by the SW and LQG used by Auroux and Blum (2008) and Auroux (2009). This is because the SW model has no vertical levels and the LQG was implemented with only 3 layers, while in this article NEMO is configured with 11 vertical layers. In addition, NEMO considers vertical diffusion processes, mostly ignored by the LQG model. Vertical diffusion plays an important role in maintaining the ocean stratification and meridional overturning circulation, which is directly related to the transport of heat in the ocean. Moreover from the practical point of view, the diffusion/viscosity required to keep the NEMO simulations stable is by far greater than for the SW or LQG at the same resolution.

These issues call into question the validity of the approximations made by the BFN under realistic conditions. Thus, our primary objective is to study the possibility of applying the BFN in realistic models and evaluate its performance compared to the 4Dvar. This appears as being the next logical step before using the BFN to assimilate real data.

This article is organized as follows. In Sect 2 the BFN and the 4Dvar are described. Section 3 describes the model physics and the model set-up. Section 4 discusses some practical aspects of the backwards integration. Section 5 presents the BFN and the 4Dvar set-up and the designed data assimilation experiments. Finally, the data assimilation results are presented in the Sect 6, on which we discuss the impact of the length of the data assimilation window on the method performances as well as the sensitivity of each method to the observation network and the initial condition.

2 Data Assimilation Methods

In this section the Back and Forth Nudging (BFN) is introduced and the 4Dvar used to assess the BFN performance is briefly described.

2.1 The Back and Forth Nudging

The conventional nudging algorithm consists in adding a forcing term (feedback term) to the model equations, proportional to the difference between the data and the model at a given time. More generally, given a model described by a set of ordinary equations (or discretized partial differential equations), nudging consists in adding to them the forcing term $\mathbf{K}(\mathbf{x}_{obs} - \mathcal{H}(\mathbf{x}))$:

$$\frac{d\mathbf{x}}{dt} = \mathcal{F}(\mathbf{x}) + \mathbf{K}(\mathbf{x}_{obs} - \mathcal{H}(\mathbf{x})) \quad (1)$$

where \mathbf{x} represents the state vector, \mathcal{F} is the model operator, \mathcal{H} is the observation operator allowing one to compare the observations $\mathbf{x}_{obs}(t)$ to the corresponding system state $\mathcal{H}(\mathbf{x})$, and \mathbf{K} is the nudging gain matrix. In this algorithm the model appears as a weak constraint. The feedback term changes the dynamical equations and forces the state variables to fit the observations as well as possible.

In the linear case, i.e. when \mathcal{F} and \mathcal{H} may be written as matrices \mathbf{F} and \mathbf{H} , and in the absence

of noise in the system, nudging is nothing else than the Luenberger observer (Luenberger, 1966). In this case, and assuming that the observability of the pair (F, H) holds, there is a class of possible values of K that guarantees the estimator convergence when $t \rightarrow \infty$ (Gelb et al., 1974). This should be one possible explanation why nudging usually works quite well and the converged state is not strongly affected by the choice of K . However, when constructing K (which units is s^{-1}), the aim is to obtain an estimator response faster than the time scale of the studied processes.

The BFN is an iterative algorithm which sequentially solves the forward model equations with a feedback term to the observations (Eq. 1) and the backward model equations with an opposite sign for the feedback term. The initial condition of the backward integration is the final state obtained after integration of the forward nudging equation. At the end of each iteration a new estimation of the system's initial state is obtained. The iterations are carried out until convergence is reached.

The BFN novelty with respect to conventional nudging methods is the model integration backward in time. This allows to recover initial conditions as well as to use more than once the same observations set. Consequently, the BFN may be seen as a sub-optimal iterative smoother.

Under the hypothesis of a linear model a variational interpretation is possible. In this case, if we choose $K = k H^T R^{-1}$, where R is the observation error covariance matrix, and k is a scalar, the solution of the forward nudging is a compromise between the minimization of the system's energy and the minimization of the distance between the data and the model (Auroux and Blum, 2008).

However, the backward integration is problematic when the model is diffusive or simply not reversible. In the case of ocean models, there are two main aspects requiring the inclusion of diffusion: i) the control of numerical noise, and ii) the modeling of sub grid-scale processes, i.e. to parameterize the energy transfer from explicitly resolved to non-resolved scales. Indeed, diffusion naturally represents a source of uncertainty in ocean forecasts, even for the purely forward model, and has been investigated from the point of view of the optimal control theory in Leredde et al. (1999).

To address the problem of the backward model instability in this article the Diffusive Back and Forth Nudging-DBFN (Auroux et al., 2011) is used. In this algorithm the sign of the diffusion term remains physically consistent and only the reversible part of the model equations are really solved backward. Practical consequences of this assumption are analysed in Sect 4. A similar solution was proposed by Pu et al. (1997) and Kalnay et al. (2000) to stabilize their Quasi-Inverse Linear model.

To describe the DBFN algorithm, let us assume that the time continuous model satisfies dynamical equations of the form:

$$\frac{\partial x}{\partial t} = \mathcal{F}(x) + \nu \Delta x, \quad \text{for} \quad 0 < t < T, \quad (2)$$

with an initial condition $x(0) = x_0$, where \mathcal{F} denotes the nonlinear model operator without diffusive terms, ν is a diffusion coefficient and Δ represents a diffusion operator. If nudging is applied to the forward system (2) it gives:

$$\frac{\partial x_k}{\partial t} = \mathcal{F}(x_k) + \nu \Delta x_k + K(x_{obs} - \mathcal{H}(x_k)) \quad (3)$$

$$163 \quad \mathbf{x}_k(0) = \tilde{\mathbf{x}}_{k-1}(0), \quad 0 < t < T,$$

164 where $k \in \mathbb{N}_{\geq 1}$ stands for iterations and $\tilde{\mathbf{x}}_0(0)$ is a given initial guess. Nudging applied to the
 165 backward system with the reversed diffusion sign gives:

$$166 \quad \frac{\partial \tilde{\mathbf{x}}_k}{\partial t} = \mathcal{F}(\tilde{\mathbf{x}}_k) - \nu \Delta \tilde{\mathbf{x}}_k - \mathbf{K}'(\mathbf{x}_{obs} - \mathcal{H}(\tilde{\mathbf{x}}_k)) \quad (4)$$

$$167 \quad \tilde{\mathbf{x}}_k(T) = \mathbf{x}_k(T), \quad T > t > 0.$$

168 The system composed by equations (3) and (4) is the basis of the DBFN algorithm. They are iterated
 169 until convergence.

170 Therefore, one important aspect of the DBFN algorithm is the convergence criterion. Ideally,
 171 at convergence the nudging term should be null or small comparable to the other equation terms.
 172 Otherwise, when the nudging is switched off, which is the case in the forecast phase, the system
 173 may return to a state close to the background state or to a state which is not consistent to the one at
 174 convergence. The convergence is calculated as:

$$175 \quad \frac{\|\mathbf{x}_k(t=0) - \mathbf{x}_{k-1}(t=0)\|}{\|\mathbf{x}_{k-1}(t=0)\|} \leq \epsilon, \quad (5)$$

176 where $\|\bullet\|$ is the L_2 norm, and the choice for $\epsilon = 0.005$ is based on sensitivity tests (not presented
 177 in this article).

178 Data Assimilation is the ensemble of techniques combining the mathematical information pro-
 179 vided by the equations of the model and the physical information given by the observations in order
 180 to retrieve the state of a flow. In order to show that the DBFN algorithm achieves this double ob-
 181 jective, let us give a formal explanation of the way DBFN proceeds. If $\mathbf{K}' = \mathbf{K}$ and the forward
 182 and backward limit trajectory are equal, i.e $\tilde{\mathbf{x}}_\infty = \mathbf{x}_\infty$, then taking the sum between Eqs.(3) and (4)
 183 shows that \mathbf{x}_∞ satisfies the model equations without diffusion:

$$184 \quad \frac{\partial \mathbf{x}_\infty}{\partial t} = \mathcal{F}(\mathbf{x}_\infty) \quad (6)$$

185 while taking the difference between Eqs.(3) and (4) shows that \mathbf{x}_∞ satisfies the Poisson equation:

$$186 \quad \Delta \mathbf{x}_\infty = -\frac{\mathbf{K}}{\nu}(\mathbf{x}_{obs} - \mathcal{H}(\mathbf{x}_\infty)) \quad (7)$$

187 which represents a smoothing process on the observations for which the degree of smoothness is
 188 given by the ratio $\frac{\nu}{K}$ (Auroux et al., 2011). Equation (7) corresponds, in the case where \mathcal{H} is a matrix
 189 \mathbf{H} and $\mathbf{K} = k\mathbf{H}^T \mathbf{R}^{-1}$, to the Euler equation of the minimization of the following cost-function

$$190 \quad \mathcal{J}(\mathbf{x}) = k < \mathbf{R}^{-1}(\mathbf{x}_{obs} - \mathbf{H}\mathbf{x}), (\mathbf{x}_{obs} - \mathbf{H}\mathbf{x}) > + \nu \int_{\Omega} \|\nabla \mathbf{x}\|^2 \quad (8)$$

191 where the first term represents the quadratic difference to the observations and the second one is a
 192 first order Tikhonov regularisation term over the domain of resolution Ω . The vector \mathbf{x}_∞ , solution
 193 of (7), is the point where the minimum of this cost-function is reached. It is shown in Sect 6.1 that
 194 at convergence the forward and backward trajectories are very close, which justifies this qualitative
 195 justification of the algorithm.

196 The description of the used \mathbf{K} matrix is given in the Sect (5.1).

197 2.2 Four Dimensional Variational Method - 4DVar

198 Variational methods minimize a cost function that measures the distance between the estimated
 199 state and the available observations. Let us assume that observations are available at every instant
 200 $(t_i)_{1 \leq i \leq N}$. Given a first guess \mathbf{x}^b of the initial state, the 4DVar algorithm will find an optimal initial
 201 condition that minimizes the distance between the model trajectory and the observations in a given
 202 assimilation window. This optimal state is found by minimizing the following cost function:

$$203 \quad J(\mathbf{x}_0) = \frac{1}{2}(\mathbf{x}_0 - \mathbf{x}^b)^T \mathbf{B}^{-1}(\mathbf{x}_0 - \mathbf{x}^b) \\
 204 \quad + \frac{1}{2} \sum_{i=0}^N (\mathcal{H}_i[\mathcal{M}_{0,i}(\mathbf{x}_0)] - \mathbf{y}_i)^T \mathbf{R}_i^{-1}(\mathcal{H}_i[\mathcal{M}_{0,i}(\mathbf{x}_0)] - \mathbf{y}_i) \quad (9)$$

205 where \mathbf{B} is the background error covariance matrix and $\mathcal{M}_{0,i}$ represents the model integration from
 206 time t_0 to time t_i . $\mathbf{R}_i, \mathcal{H}_i$ and \mathbf{y}_i are the observations error covariance matrix, the observation
 207 operator and the available observations at time t_i , respectively.

208 The optimal initial state is found by solving:

$$209 \quad \nabla J(\mathbf{x}^a(t_0)) = 0 \quad (10)$$

210 The calculation of this gradient is done using the adjoint method proposed by Lions (1971) and
 211 brought to the meteorological context by Le Dimet and Talagrand (1986).

212 If \mathcal{H} or \mathcal{M} are nonlinear, the solution of the problem is not unique, i.e. the functional (9) may
 213 have multiple local minima, and the minimization procedure may not stop at the global minimum. To
 214 overcome this problem, Courtier et al. (1994) proposed to solve a sequence of quadratic problems,
 215 expecting this sequence would converge to the solution of the problem given by (9) and (10). This
 216 algorithm is called the incremental 4Dvar. In this case, the cost function will not be minimized
 217 with respect to the initial state but with respect to an increment $\delta \mathbf{x}_0$ defined by $\mathbf{x}_0 = \mathbf{x}^b + \delta \mathbf{x}_0$. The
 218 operators \mathcal{H} or \mathcal{M} are linearized in a neighborhood of \mathbf{x}^b as:

$$219 \quad \mathcal{M}_{0,i}(\mathbf{x}^b + \delta \mathbf{x}_0) \approx \mathcal{M}_{0,i}(\mathbf{x}^b) + \mathbf{M}_{0,i} \delta \mathbf{x}_0 \quad \forall i \quad (11)$$

$$220 \quad \mathcal{H}_i(\mathbf{x}^b + \delta \mathbf{x}_0) \approx \mathcal{H}_i(\mathbf{x}^b) + \mathbf{H}_i \delta \mathbf{x}_0 \quad \forall i \quad (12)$$

221 and the new cost function is given by:

$$222 \quad J(\delta \mathbf{x}_0) = \frac{1}{2} \delta \mathbf{x}_0^T \mathbf{B}^{-1} \delta \mathbf{x}_0 + \frac{1}{2} \sum_{i=0}^N (\mathbf{H}_i \mathbf{M}_{0,i} \delta \mathbf{x}_0 - \mathbf{d}_i)^T \mathbf{R}_i^{-1} (\mathbf{H}_i \mathbf{M}_{0,i} \delta \mathbf{x}_0 - \mathbf{d}_i) \quad (13)$$

223 where $\mathbf{d}_i = \mathbf{y}_i - \mathcal{H}_i(\mathcal{M}_{0,i}(\mathbf{x}^b))$ is called the innovation vector. It is possible that after some iterations
 224 of the minimizer the increments become too large and a new linearization of \mathcal{H} and \mathcal{M} should be
 225 done. This gives rise to what is called the inner loop and outer loop iterations. The algorithm
 226 implemented in NEMO, called NEMOVAR (Mogensen et al., 2009), uses this technics. It can be
 227 summarized as follows:

–**Initialisation** : $\mathbf{x}_0^0 = \mathbf{x}^b$
 –**While** $k \leq k_{max}$ or $\|\delta \mathbf{x}_0^{a,k}\| > \epsilon$ (**Outer Loop**)

Do

• $\mathbf{d}_i^k = \mathbf{y}_i - \mathcal{H}_i(\mathcal{M}_{0,i}(\mathbf{x}_0^k))$

• Search the $\delta \mathbf{x}_0^{a,k}$ that minimises (**Inner Loop**):

$$J(\delta \mathbf{x}_0^k) = \frac{1}{2}(\delta \mathbf{x}_0^k)^T \mathbf{B}^{-1}(\delta \mathbf{x}_0^k) + \frac{1}{2} \sum_{i=0}^N (\mathbf{H}_i \mathbf{M}_{0,i} \delta \mathbf{x}_0^k - \mathbf{d}_i^k)^T \mathbf{R}_i^{-1} (\mathbf{H}_i \mathbf{M}_{0,i} \delta \mathbf{x}_0^k - \mathbf{d}_i^k) \quad (14)$$

• $\mathbf{x}_0^{k+1} = \mathbf{x}_0^k - \delta \mathbf{x}_0^{a,k}$

The description of the matrices \mathbf{B} and \mathbf{R} is given in the Sect (5.2).

3 Ocean Model and Experimental set-up

The ocean model used in this study is the ocean component of NEMO (Nucleus for European Modelling of the Ocean; Madec, 1996). This model is able to represent a wide range of ocean motions, from the basin scale up to the regional scale. Currently, it has been used in operational mode by the French Mercator Océan group (<http://www.mercator-ocean.fr>) and the European Center for Medium Range Weather Forecast (ECMWF).

The model solves six prognostic equations, namely the momentum balance, the hydrostatic equilibrium, the incompressibility equation, the heat and salt conservation equations and a nonlinear equation of state which couples the two tracers to the fluid fields. In this study, a linear free surface formulation is used along with the approach developed by Roullet and Madec (2000) to filter out the external gravity waves.

Equations are discretized using spherical coordinates in a Arakawa C grid. The model advances in time using a leap-frog scheme for all terms except for the vertical diffusive terms, which are treated implicitly. At every time step the model uses a Robert-Asselin (RA) temporal filter to damp the computational mode. The leap-frog scheme followed by the RA filter leads to a first order temporal scheme (Williams, 2009). Spatial discretization uses a centered second order formulation for both the advective and the diffusive terms.

The double gyre configuration, extensively used to study jet instabilities (Chassignet and Gent, 1991; Primeau, 1998; Chang et al., 2001), meso and submeso-scale dynamics (Levy et al., 2010) and data assimilation methods (Molcard et al., 2004; Krysta et al., 2011; Cosme et al., 2010), is used for the present study. The double gyre configuration simulates the ocean middle latitude dynamics and has the advantage of being simple, when compared to real applications, but still considering full dynamics and thermodynamics.

In our experiments we use a homogeneous horizontal grid with a 25km resolution and a verti-

cal resolution ranging from 100m near the upper surface to 500m near the bottom. The bottom topography is flat and the lateral boundaries are closed and frictionless. The only forcing term considered is a constant wind stress of the form $\tau = \left(\tau_0 \cos\left(\frac{2\pi(y-y_0)}{L}\right), 0 \right)$, where y is the latitude geographic coordinate with $y_0 = 24^\circ$ and $y_0 \leq y \leq 44^\circ$, $L = 20^\circ$ and $\tau_0 = -0.1 N/m^2$. Horizontal diffusion/viscosity are modeled by a bilaplacian operator meanwhile a laplacian operator is used in the vertical. They all use constant coefficients in time and space: $\nu_h^{u,v} = -8 \times 10^{10} m^4/s$ and $\nu_v^{u,v} = 1.2 \times 10^{-4} m^2/s$ for the momentum equations and $\nu_h^{t,s} = -4 \times 10^{11} m^4/s$ and $\nu_v^{t,s} = 1.2 \times 10^{-5} m^2/s$ for temperature and salinity. The initial condition is similar to that used by Chassignet and Gent (1991) and consists of a homogeneous salinity field of 35psu and a temperature field created to provide a stratification which has a first baroclinic deformation radius of 44.7km. Velocity and sea surface height (SSH) fields are initially set to zero.

This double gyre configuration is currently used as the NEMO data assimilation demonstrator and as the experimentation and training platform for data assimilation activities (Bouttier et al., 2012). For the present work, the model was integrated for 70 years, in order to reach the statistical steady state. Afterwards, ten years of free model run were performed, that were used to calculate the regression models which are used to calculate the nudging matrix \mathbf{K} (see Sect 5.1), and then two additional years were finally completed to be used as the truth, from which the observations were extracted.

4 The backward integration without Nudging: Practical aspects

The backward model uses exactly the same numerical scheme as the forward model. Since most of the model is solved using centered finite differences, the inverse version of the discretized model is similar to the discrete version of the inverse continuous model. The only distinction between the forward and the backward model is the change in the sign of the diffusive terms when stepping backwards, this making the backward integration stable. If this is not taken into account the model blows up after a few days.

Reversing the diffusion sign in the backward model is a numerical artifact and being so its effects should be carefully analysed. In this section, the backward integration accuracy is studied, as well as its sensitivity with respect to the choice of the diffusion coefficient. The errors are analysed calculating the L2 error norm at the end of one forward-backward integration relative to a typical one day model variation:

$$R_{err} = \frac{\|\mathbf{x}(0) - \tilde{\mathbf{x}}(0)\|}{< \|\mathbf{x}(t + \Delta t) - \mathbf{x}(t)\| >} \quad (15)$$

where $\Delta t = 1\text{day}$ and the brackets represent the empirical mean.

Figure 1 shows the global error, R_{err} , for different window sizes. The errors grow linearly with the window size for all variables. Temperature is the most affected variable, followed by sea level and velocities. Temperature errors exceed 18 times a typical one-day variation for the 30 days exper-

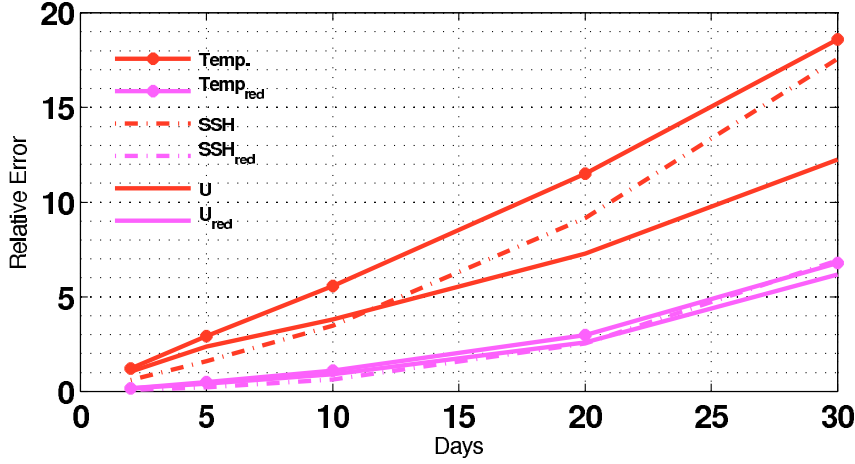


Fig. 1. Errors on the initial condition after one forward-backward model integration perfectly initialized and without nudging. Red curves were obtained using the same diffusion coefficients as in the reference experiment ($\nu_h^{u,v} = -8 \times 10^{10} m^4/s$ and $\nu_h^{t,s} = -4 \times 10^{11} m^4/s$) and magenta curves were obtained using reduced diffusion ($\nu_h^{u,v} = -8 \times 10^9 m^4/s$ and $\nu_h^{t,s} = -8 \times 10^{10} m^4/s$). The abscissa represents the length of the time window.

289 iment and 1.2 times for the 2 days. The use of reduced diffusion/viscosity coefficients reduces the
 290 errors to 6.8 and 0.16 times the one-day variation for 30 and 2 days experiments, respectively. Ve-
 291 locities errors were reduced by 50% for 30 days and 85% for 2 days, while ssh errors were reduced
 292 by 60% and 88% for 30 and 2 days, respectively.

293 As shown on Fig. 2 velocity and temperature errors are depth-dependent. Whereas for velocity
 294 they are larger at the surface and decrease with depth, for temperature they are larger in the ther-
 295mocline. In the cases for which the forward-backward integrations use the same diffusion/viscosity
 296 coefficients as in the reference simulation, the temperature errors at thermocline depths exceed 3
 297 times the typical one day variation for the 5 days experiments and reaches 15 times for 20 days ex-
 298periments. Considering the velocities, errors are proportional to 4 one-day variations for the 5 days
 299 experiment and to 8 one-day variations for the 20 days experiments. For time windows of 10, 20 and
 300 30 days, velocities at the thermocline depths start to be influenced by temperature errors.

301 Reduction of the diffusion/viscosity coefficients greatly reduced the errors especially in the ther-
 302mocline for the temperature and at the surface for the velocity. It can be noted that when the diffusion
 303 coefficient is decreased the errors converge to a limit. This limit changes with respect to the window
 304 length and should be related to the diffusion required to stabilize the numerical method, which is of
 305 second order in our case, and hence oscillatory. Therefore, there is a compromise between the errors
 306 induced by the extra diffusion and errors due to spurious oscillations.

307 Numerical errors were assessed by changing the model time step from 900s to 90s. The resulting
 308 errors (not shown) do not change, suggesting that the errors induced by the diffusion are domi-

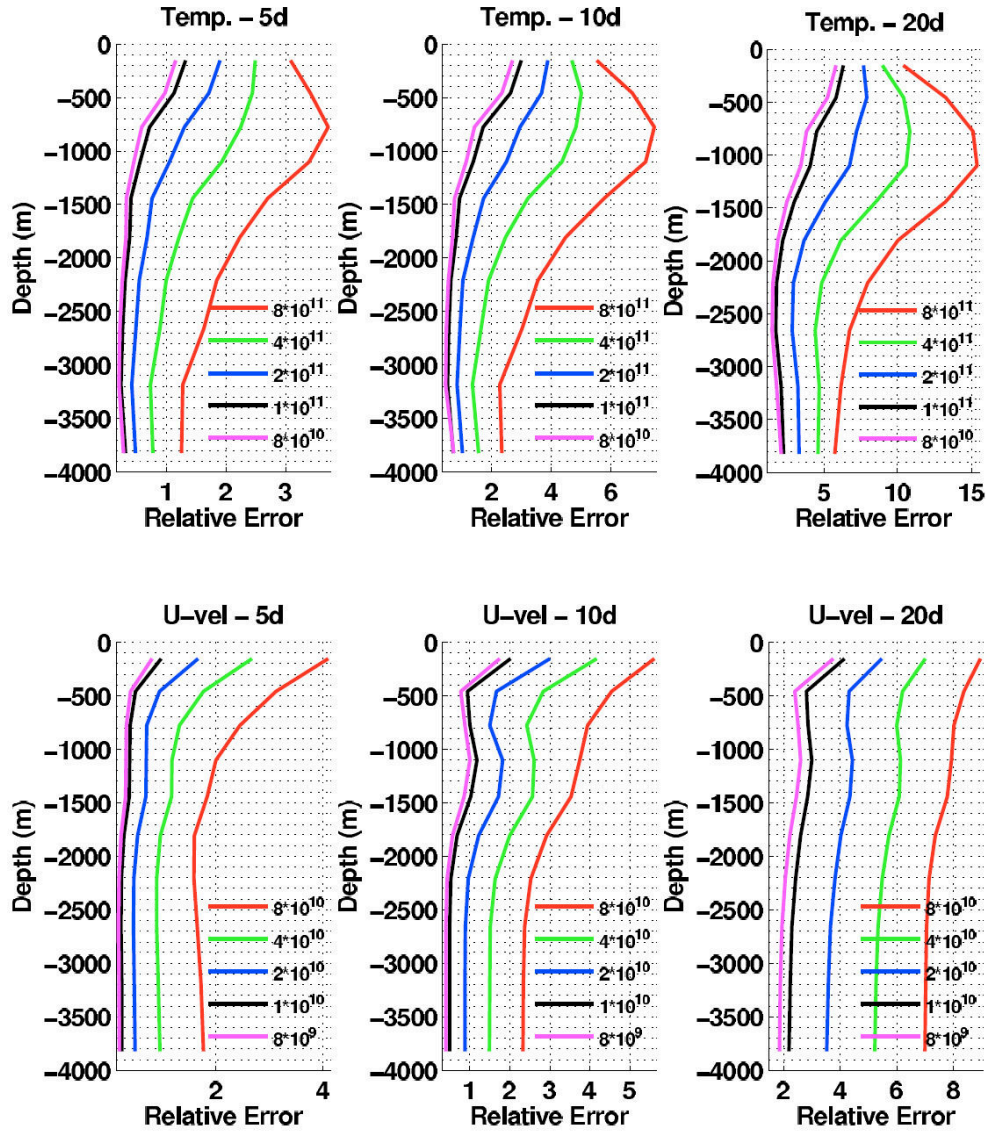


Fig. 2. Vertical profiles of relative errors on the initial condition after one forward-backward model integration without nudging. Each color refers to an experiment performed using the diffusion coefficient indicated in the figures legend. Red curves were obtained using the same diffusion coefficients as in the reference experiment. Top panel: temperature errors; bottom panel: zonal velocity errors. The length of the time window is indicated in the title of each figure.

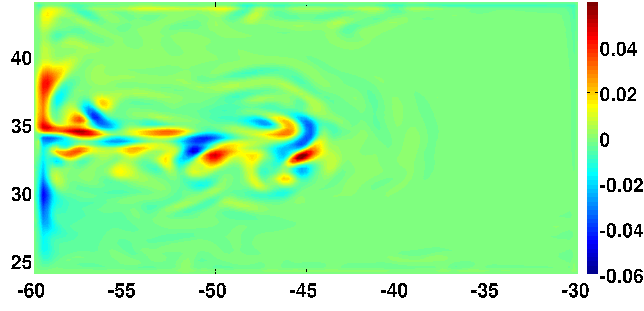


Fig. 3. Sea level errors after one forward-backward model integration. The time window is of 10 days.

309 nant. On the one hand, this is important because the complete rewriting of the model's code can be
 310 difficult, similarly to the adjoint model programming used by the 4Dvar, but on the other hand if
 311 the assimilation cannot control the diffusion errors it may represent a fundamental problem of the
 312 method when it is applied to non-reversible geophysical systems such as the ocean.

313 Figure 3 shows the spatial structures of the sea level error for the 10 days experiment. The errors
 314 are highly variable in space, being larger along the main jet axis. This is probably due to the fact that
 315 the backward integration smooths the gradients and so the largest errors are found near the fronts.
 316 Therefore, the errors structures may be of high variability in space and time since they are state
 317 dependent.

318 Figure 4 shows the surface kinetic energy spectrum calculated from the experiment employing
 319 the reference diffusion coefficient and a reduced diffusion coefficient. The backward integration
 320 introduces an extra diffusion, coarsening the effective model resolution, which is defined as the por-
 321 tion of the spectra for which there is a change in the spectrum slope. In the reference simulation the
 322 effective model resolution is estimated to be 190km, which is coherent with the $\approx 7 \times \Delta x$ estimation
 323 of Skamarock (2004).

324 The longer the time window the greater the portion of the spectra affected. For the experiment
 325 employing the reference diffusion coefficient, the divergence between the true spectra and the spec-
 326 tra obtained from the backward integration is observed at 126, 314 and 627km for 5, 10 and 20 days
 327 experiments, while for the experiments considering a reduced diffusion coefficient there is almost
 328 no differences for the 5 days experiment, and the divergence is observed at 126 and 314km for the
 329 10 and 20 days experiments. If on the one hand using the reduced diffusion helps to keep the en-
 330 ergy distribution coherent with the true distribution, on the other hand it creates noise in the range
 331 of 126km to 25km. This confirms that there is a trade-off between the errors due to the excessive
 332 smoothing and the errors due to high frequency numerical modes.

333 In this section we have seen that there are large backward-errors induced by over-diffusion.
 334 Therefore, short time windows with reduced diffusion coefficients would be preferable to be used
 335 in DA experiments. Two regions have to be cautiously analyzed: the surface and the thermocline.

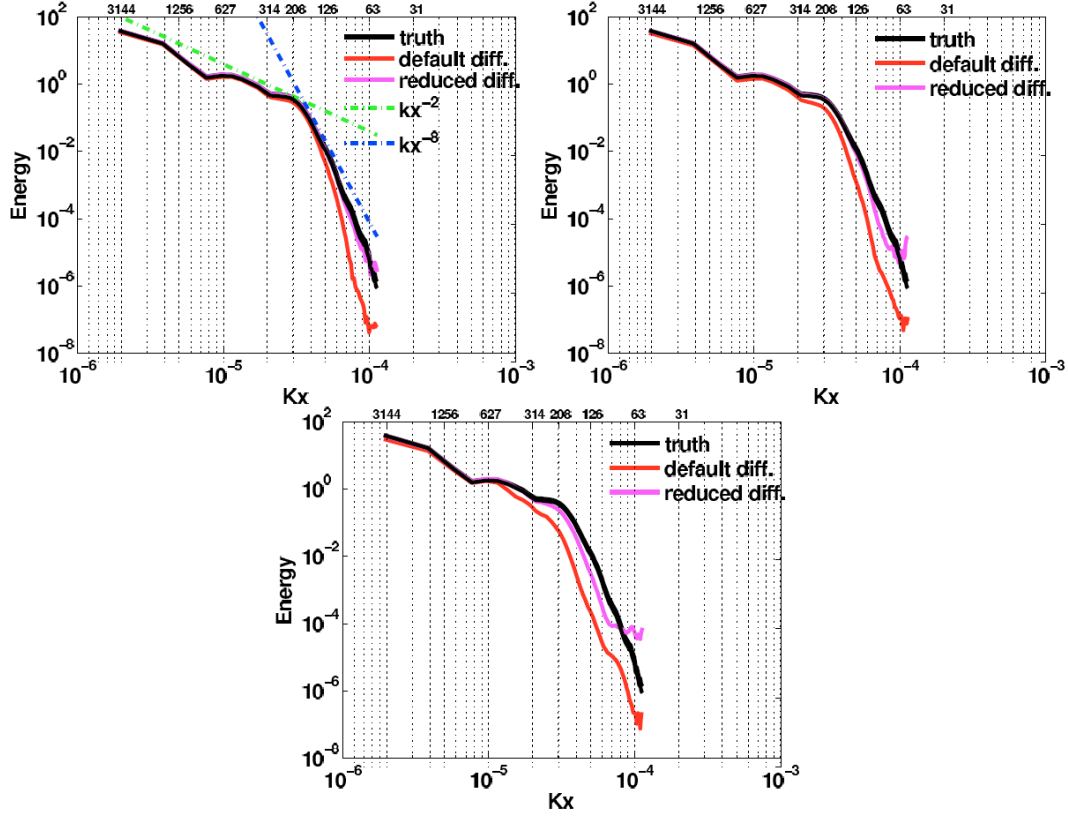


Fig. 4. Kinetic energy mean power spectra calculated using the first layer velocity fields. Black curves represent the “true” initial condition power spectra; Red curves represent the power spectra calculated after one forward-backward iteration without the nudging term and employing the reference diffusion coefficient; Magenta curves represent the power spectra calculated after one forward-backward iteration without the nudging term and employing a reduced diffusion coefficient. Top left: 5 days assimilation window. Top right: 10 days assimilation window. Bottom: 20 days assimilation window. In the bottom abscissa the ticklabels stand for longitudinal wave-number (rad/m) while in the top abscissa the ticklabels stand for the corresponding wavelengths in km units.

Surface layers are prone to feature errors due to their role on the wind energy dissipation while at the thermocline strong density gradients contribute to high diffusion rates.

5 Data Assimilation experiments

5.1 Prescription of the DBFN gain

In this study the increments corresponding to the term $\mathbf{K}(\mathbf{x}^{obs} - \mathcal{H}(\mathbf{x}))$ are calculated in two operations: first the increments of the observed variables are calculated using a prescribed weight and subsequently the increments of the other state variables are calculated using linear regression. More precisely, defining $\mathbf{y} = \mathcal{H}(\mathbf{x})$ as the observed part of the state vector, the first step may be written as:

$$\delta \mathbf{y} = \Theta(\mathbf{x}^{obs} - \mathbf{y}^b) \quad (16)$$

where the superscript b denotes the background field or the model field available from the last time step. The prescribed weight is given by:

$$\Theta = \frac{\sigma_m^2}{\sigma_m^2 + \gamma \sigma_o^2} \quad (17)$$

where σ_m^2 is the mean spatial value of SSH variance calculated from the free model run, σ_o^2 is the observation error variance and γ is a parameter used to adjust the variance of the observation errors. When $\gamma = 1$ the Eq.(17) for the weight Θ has the same form of the scalar Kalman gain (Gelb et al., 1974). For values greater than one, γ is an inflation factor, i.e. it increases the variance of the observation errors resulting in more weight given to the model in the Eq.(16).

The use of the inflation factor is theoretically justified in the linear Kalman filtering context. In this case, it is well-known that the Kalman Filter provides the best linear unbiased estimator. Therefore, there is no need to use more than once the observations. Consequently, when one is iterating the Kalman Filter the inflation parameter should be used to avoid overfitting and the introduction of correlated errors in the system (Kalnay and Yang, 2010). In this study $\gamma = 18$, which means that theoretically the best solution would be reached in 9 iterations. However, since in this study the Kalman Filter equations are not fully used and the system is not linear, the γ parameter is used as a guide on how strong the model is nudged toward the observations. Indeed, the iterations are not limited to 9. The used values for the other parameters are $\sigma_m = 0.017m$ and $\sigma_o = 0.03m$ consistently with the perturbations added to the observations (see Sect 5.4).

Then, the increments of the non-observed variables, $\delta \mathbf{x}$, are calculated by using a regression equation of the form:

$$\delta \mathbf{x} = \hat{\mathbf{B}}^{PLS} \delta \mathbf{y} \quad (18)$$

where $\hat{\mathbf{B}}^{PLS}$ is the Partial Least Squares (PLS) regression coefficients which are described below. It is worth noting that in Sect 6 we also apply this update scheme to an ordinary direct nudging

368 experiment. In this case γ is equal to one.

369 The PLS can be seen as an improvement to the Ordinary Least Square (OLS) regression. The most
 370 important difference between OLS and PLS is that the later assumes that the maximum information
 371 about the non-observed variables is in those directions of the observed space which simultaneously
 372 have the highest variance and the highest correlation with the non-observed variables.

373 In the PLS description (Tenenhaus, 1998), $\mathbf{Y} \in \mathbb{R}^{n \times M}$ is considered as the observed or predictor
 374 variables and $\mathbf{X} \in \mathbb{R}^{n \times N}$ as the non-observed or response variables. In our notation n is the sample
 375 size and M and N are respectively the size of the state space of \mathbf{Y} and \mathbf{X} . Besides, \mathbf{Y} and \mathbf{X} are
 376 centered and have the same units. The PLS regression features two steps: a dimension reduction step
 377 in which the predictors from matrix \mathbf{Y} are summarized in a small number of linear combinations
 378 called "PLS components". Then, that components are used as predictors in the ordinary least-square
 379 regression.

380 The PLS as well as the principal component regression can be seen as methods to construct a
 381 matrix of p mutually orthogonal components \mathbf{t} as linear combinations of \mathbf{Y} :

$$382 \quad \mathbf{T} = \mathbf{Y}\mathbf{W}, \quad (19)$$

383 where $\mathbf{T} \in \mathbb{R}^{n \times p}$ is the matrix of new components $\mathbf{t}_i = (t_{1i}, \dots, t_{ni})^T$, for $i = 1, \dots, p$, and $\mathbf{W} \in \mathbb{R}^{M \times p}$
 384 is a weight matrix satisfying a particular optimality criterion.

385 The columns $\mathbf{w}_1, \dots, \mathbf{w}_p$ of \mathbf{W} are calculated according to the following optimization problem:

$$386 \quad \mathbf{w}_i = \operatorname{argmax}_{\mathbf{w}} \{ \operatorname{cov}(\mathbf{Y}\mathbf{w}, \mathbf{X})^2 \} \quad (20)$$

387 subject to $\mathbf{w}_i^T \mathbf{w}_i = 1$ and $\mathbf{w}_i^T \mathbf{Y}^T \mathbf{Y} \mathbf{w}_j = 0$ for $j = 1, \dots, i-1$.

388 The PLS estimator $\hat{\mathbf{B}}^{PLS}$ is given by:

$$389 \quad \hat{\mathbf{B}}^{PLS} = \mathbf{W}(\mathbf{W}^T \mathbf{Y}^T \mathbf{Y} \mathbf{W})^{-1} \mathbf{W}^T \mathbf{Y}^T \mathbf{X} \quad (21)$$

390 An immediate consequence of Eq. (21) is that when $\mathbf{W} = \mathbf{I}$ the Ordinary Least Squares solution is
 391 obtained.

392 The number of components p is chosen from cross-validation. This method involves testing a
 393 model with objects that were not used to build the model. The data set is divided in two contiguous
 394 blocks; one of them is used for training and the other to validate the model. Then the number of
 395 components giving the best results in terms of mean residual error and estimator variance is sought.

396 The weight Θ and the regression model $\hat{\mathbf{B}}^{PLS}$ are kept constant over the assimilation cycles
 397 and the correction steps (16) and (17) are applied at the end of the loop of time. Thus, our updat-
 398 ing scheme can be seen as a rough approximation of the two steps update for EnKF presented by
 399 Anderson (2003).

where $\mathbf{\Lambda}$ is a diagonal matrix of error standard deviation, for which the climatological standard deviation are the entries, and \mathbf{C} is an univariate correlation matrix modeled using the generalized diffusion equation (Weaver and Courtier, 2001; Weaver et al., 2005). In this method the user should chose typical decorrelation lengths. In this study the horizontal decorrelation length is set to $400km$ and the vertical decorrelation length is set to $1500m$. In addition, the 4Dvar is configured to perform one outer-loop and a maximum of thirty inner-loop for each assimilation cycle.

5.3 Assimilation cycle

One assimilation cycle is defined as the process of identifying an initial condition through the iterative process followed by a forecast spanning the assimilation window, which provides a new background to the next assimilation cycle.

The objective of cycling is to provide a background state for the next assimilation window that is closer to the true state than the very first background field. This usually reduces the number of iterations needed by the algorithms to reach convergence.

The length of the Data Assimilation window (DAw) used in the reference experiments (Sect 6.1) is 10 days. For the sensitivity experiments presented in the Sect 6.2 the lengths of the the assimilation window are 5 days and 30 days.

5.4 Observation network

In this article, every four days an observation network simulating Jason-1 satellite density sample is available. The data is perturbed with white Gaussian noise with standard deviation equals to $3cm$. With this observation network a new set of 5000 observations is available every four days.

The data assimilation problem we proposed to solve is to recover the full model state at the beginning of the assimilation window. The model state space is composed of five variables: sea surface height (η), meridional and zonal velocities (u and v), temperature and salinity ($temp$ and $salt$). Since we have a horizontal mesh of size 81×121 and 11 vertical layers the total size of the state space is 441045. Therefore, the problem is undetermined, since the observations represent only a 1.1% of the total state space. This means that the background term, and accordingly the \mathbf{B} matrix for the 4Dvar and the regression model $\hat{\mathbf{B}}^{PLS}$ for the DBFN, have quite a strong importance on the method performances since they project the increments of the observed variables onto the numerous non-observed variables.

To study at which extent the results are depend on the amount of assimilated observations and on the first guess, in Sect 6.2.2 two additional experiments assimilating complete daily fields of SSH are conducted: one using the same first guess of the experiments of Sect 6.1, and another using a perturbed initial condition. In despite of the fact that the problem continues to be underestimated, in this case the SSH analysis is no more dependent on the SSH spatial covariance, and the unstable

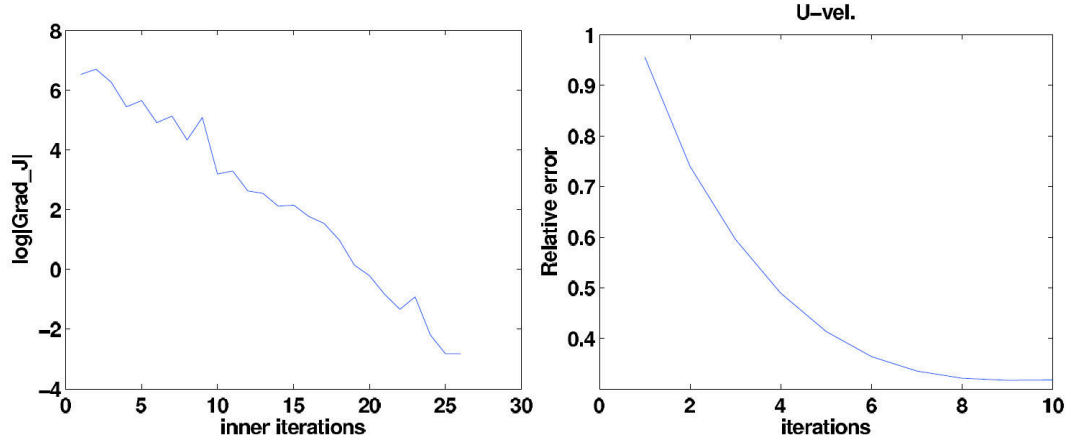


Fig. 5. Figure shows the gradient of the cost function after each inner iteration (left) and the reduction of the relative error for zonal velocity for the DBFN experiment (right).

461 modes associated with the SSH dynamics are certainly observed. The analysis produced for the
 462 other state vector variables remains dependent on the matrices \mathbf{B} for the 4Dvar case and \hat{B}^{PLS} for
 463 the DBFN case.

464 6 Data Assimilation Results

465 6.1 Reference experiment

466 In this section the results produced by the DBFN, the 4Dvar method, the Ordinary Nudging (ONDG)
 467 and the control experiment are presented. All assimilation methods include the five prognostic vari-
 468 ables in the state vector. This is possible thanks to the PLS regression method in the case of the
 469 DBFN and ONDG and thanks to the multivariate balance operator G in the case of the 4Dvar ex-
 470 periments. The diffusion and viscosity coefficients used in the DBFN experiments are those which
 471 produced the smaller errors in the experiments without Nudging, as reported in Sect 4.

472 First the minimization performance of the 4Dvar implementation is analysed. Figure 5 shows the
 473 reduction of the cost function gradient for the 4Dvar and the reduction of the relative error of the
 474 zonal velocity for the DBFN, both of them for the first assimilation cycle. 4Dvar takes 26 iterations
 475 to approximately achieve the optimality condition $\nabla J = 0$. This represents 3 times the number of it-
 476 erations required by the DBFN to converge, i.e., after which the errors cease to decrease. Moreover,
 477 the 4Dvar numerical cost is more than 3 times the DBFN cost since one execution of the adjoint
 478 model costs four times the cost of the direct model in terms of CPU time.

479 We note that the minimum error for the DBFN is reached after 9 iterations. This is quite consis-
 480 tent with our choice $\gamma = 18$, since theoretically it allows the use of the same set of observations for
 481 18 times.

At this point we find appropriate to present the fact that the trajectories of the forward and backward nudging are very close to each other at convergence, which justifies the qualitative explanation of the DBFN algorithm given by Eqs. (6) and (7). This fact can be seen in the Fig 6 that shows the forward and backward surface zonal velocity mean trajectories at convergence as well as the surface zonal velocity trajectories for a point located on the unstable jet, at 34° North and 52.6° West.

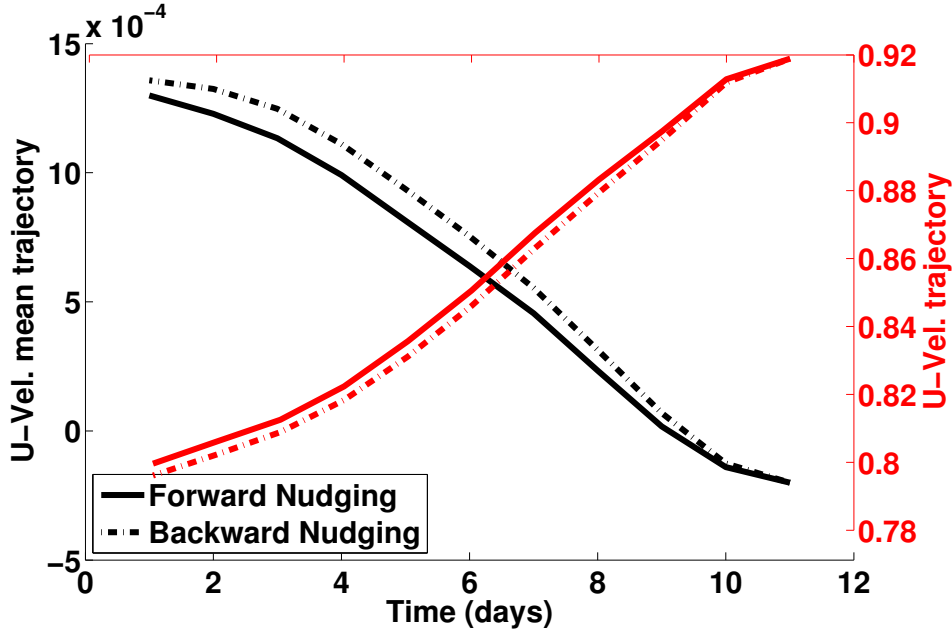


Fig. 6. Black curves represent the forward and backward surface zonal velocity mean trajectories at convergence and red curves the forward and backward surface zonal velocity trajectories at convergence for a point located at 34° North and 52.6° West, which is located on the unstable jet.

Figure 7 shows the root mean squared (rms) error for the control experiment (without assimilation), the experiment using the direct nudging with PLS regression (ONDG), the DBFN and the 4Dvar. The DBFN errors for the velocity and SSH converge to their asymptotic values after the first assimilation cycle while for ONDG and 4Dvar errors stop decreasing after 100 and 200 days, respectively. This is a benefit of the iterations performed by the DBFN when model and data are quite different. Among the experiments conducted, the DBFN produced the smallest errors for all variables, except for the zonal velocity, for which the 4Dvar has slightly smaller errors. The ONDG also showed good performance, but with errors larger than the DBFN and 4Dvar errors.

With respect to the vertical error (Fig. 8), the DBFN and the ONDG performed better for the upper ocean than 4Dvar. Clearly, the PLS also corrects the deep ocean velocity, but less accurately than 4Dvar. The first error mode is the barotropic one, i.e. it has the same sign over all depths, and accounts for 97% of the error variability for 4Dvar, 96% and 93% for DBFN and ONDG, respec-

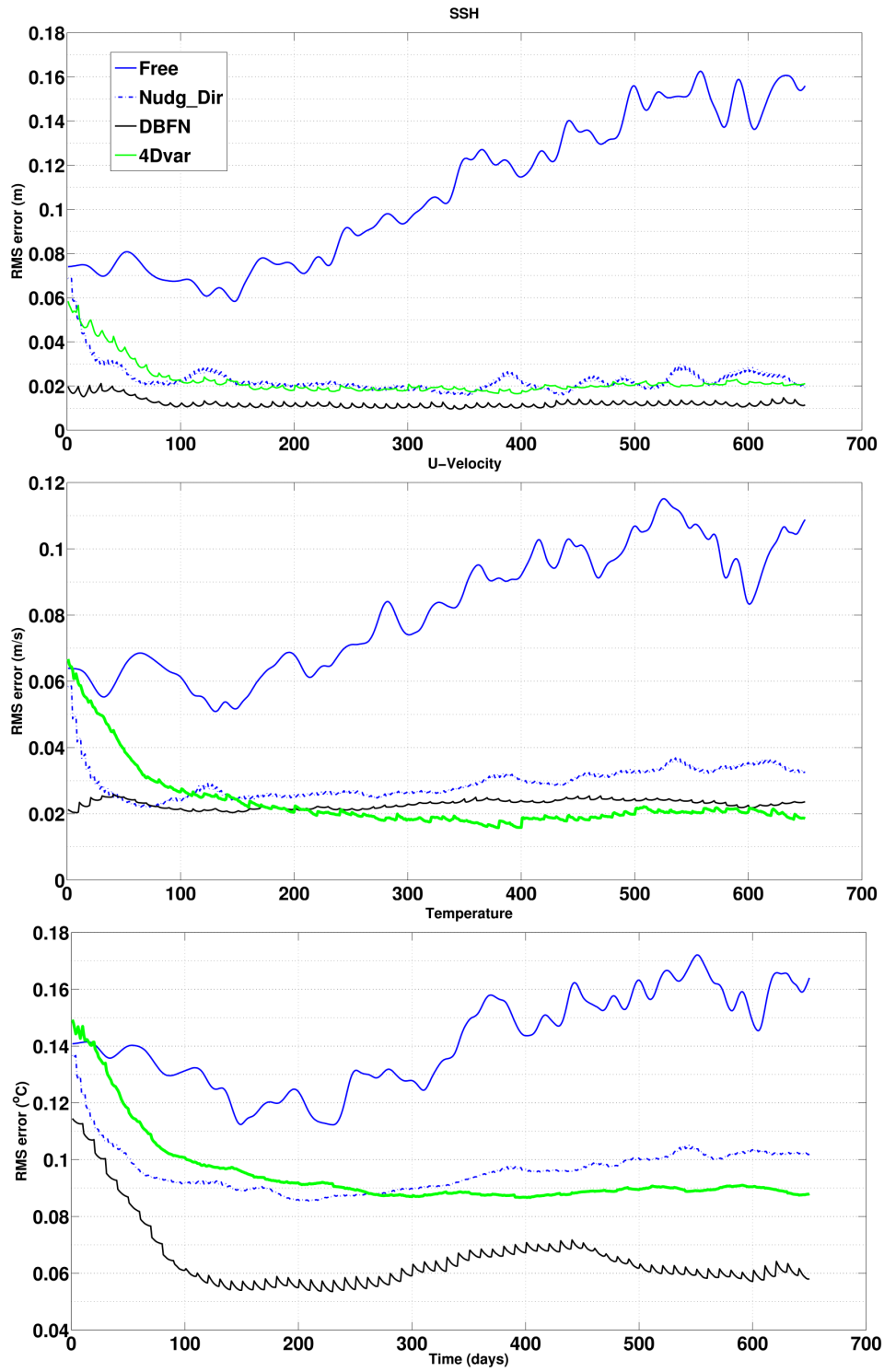


Fig. 7. The figure shows errors of the SSH (top panel), the zonal velocity (middle panel) and the temperature (bottom panel).

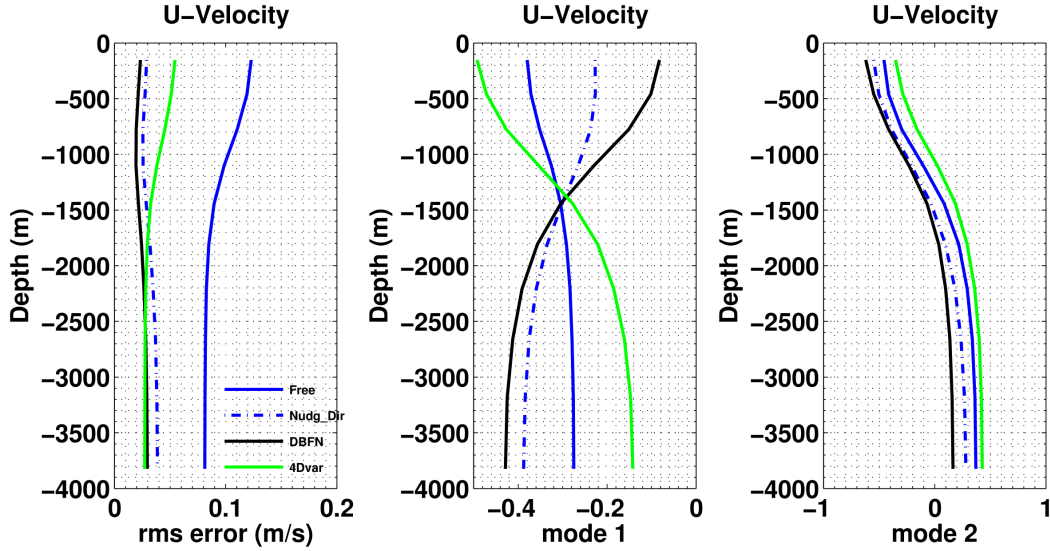


Fig. 8. Vertical profiles of rms error in zonal velocity (Left panel) and first (middle panel) and second (right panel) eof error modes calculated using forecast from day 200 to day 720.

tively. Although the first mode is the barotropic one for all methods, the 4Dvar barotropic mode of error is out of phase with respect to the PLS barotropic mode. This reflects the better performance of the 4Dvar for the deep ocean and the better performance of the DBFN and ONDG for the upper ocean.

The second mode, which accounts for almost all the remaining variability, has a sign inversion with depth and is found especially over the main axis of the jet. In this region the deep ocean velocities are overestimated due to spurious covariances between the SSH and the deep ocean velocities.

The way both methods correct the model depends on the B matrix in the 4Dvar algorithm and on the regression model \hat{B}^{PLS} in the DBFN. It means that results may be different if another approximation of B and another model regression model are used. Perhaps the main conclusion of this comparison is that the DBFN, which is easier to implement and cheaper to execute, can produce results similar to 4Dvar. Also, it is shown that iterations is an important aspect of the method. Iterations compensate for the lack of a priori information on the model errors as well as filter out noise in observations. The latter must be connected to the diffusive character of the algorithm. Moreover, the iterations allows us to put information from the observations into the model, without causing initialization problems since the nudging gain can be taken smaller than the one used for the direct nudging due to the possibility of using more than once the same set of observations.

518 6.2 Sensitivity experiments

519 6.2.1 Length of the DAw

520 Sensitivity tests with respect to the length of the DAw are presented. As we have shown in Sect 4,
521 the accuracy of the backward model is inversely proportional to the length of the DAw. Therefore,
522 in this section we present experiments using a DAw of five days and thirty days. The experiments
523 configuration is similar to those presented in the previous section.

524 Figure 9 shows the evolution of the rms errors for the zonal velocity and temperature during the
525 DBFN iterations over the first assimilation cycle, for three DAw (including the ten day-window used
526 previously). When considering only one iteration, the best results were obtained with the 30 days-
527 window experiment. This is a consequence of the asymptotic character of the Nudging method: the
528 longer the assimilation window, the more observations accounted for, the smaller the error. This
529 changes when several iterations are considered. The observed divergence for the 30 days-window is
530 due to the errors induced by the over-diffusion that induce great increments, which by their nature,
531 are not modelled by the ensemble of model states used to construct the regression model.

532

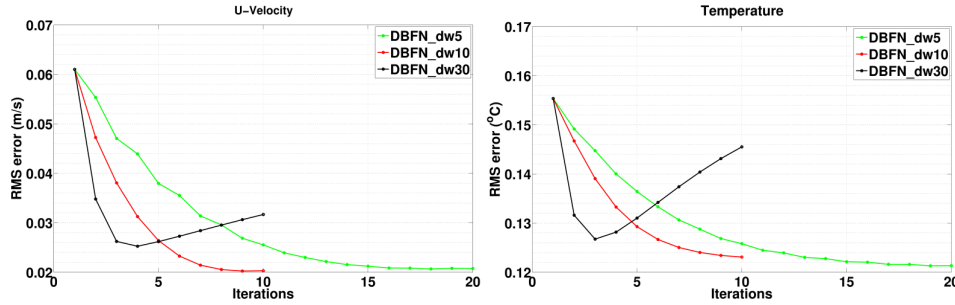


Fig. 9. Evolution of the rms errors for the zonal velocity and temperature during the DBFN iterations over the first assimilation cycle, for three DAw: 5, 10 and 30 days.

533 Figure 10 shows the rms error for the DBFN and 4Dvar experiments for three DAw: 5, 10 and
534 30 days. The methods exhibited comparable performance depending on the length of the DAw. For
535 the DBFN the 5 and 10 days DAw provided better results than the 30 days window, while for the
536 4Dvar the 30 days window provided the best estimation in terms of rms error. The DBFN and 4Dvar
537 experiments using the 30 and 5 days DAw, respectively, failed to identify the initial conditions since
538 their SSH rms errors are greater than the observation error standard deviation. The poor performance
539 of the 4Dvar for the 5 days DAw is related to spurious increments due to the fact that in one assim-
540 ilation window there is only one set of observation available. If this set is at the end of the window
541 this can complicate the minimization process and the iterations may stop before convergence.

542 Figure 11 shows the time evolution of vertical profiles of horizontally layer-wise averaged rms

543 error of zonal velocities for the DBFN and 4Dvar experiments. The 4Dvar profits of the longer DAw
544 to spread the observation to the 3-dimensional variables. This is done by the iterations of the direct
545 model and by the B matrix. For the DBFN experiments, after one year of data assimilation the
546 errors in the deep ocean start to grow. This is due to the high variance of the PLS estimator for deep
547 layers. The problem becomes more evident on the second year because at this stage the observa-
548 tions are farther from the model states used to construct the regression coefficients. Therefore, this
549 mean that this behavior is not intrinsic to the DBFN algorithm and its diffusive aspects, but due to
550 our implementation. Ideally, the regression model should evolve in time, similarly to the Kalman
551 Filter scheme. The 4Dvar has good performance at the deep ocean thanks to the use of a vertical
552 localization with a length scale of $1500m$.

553

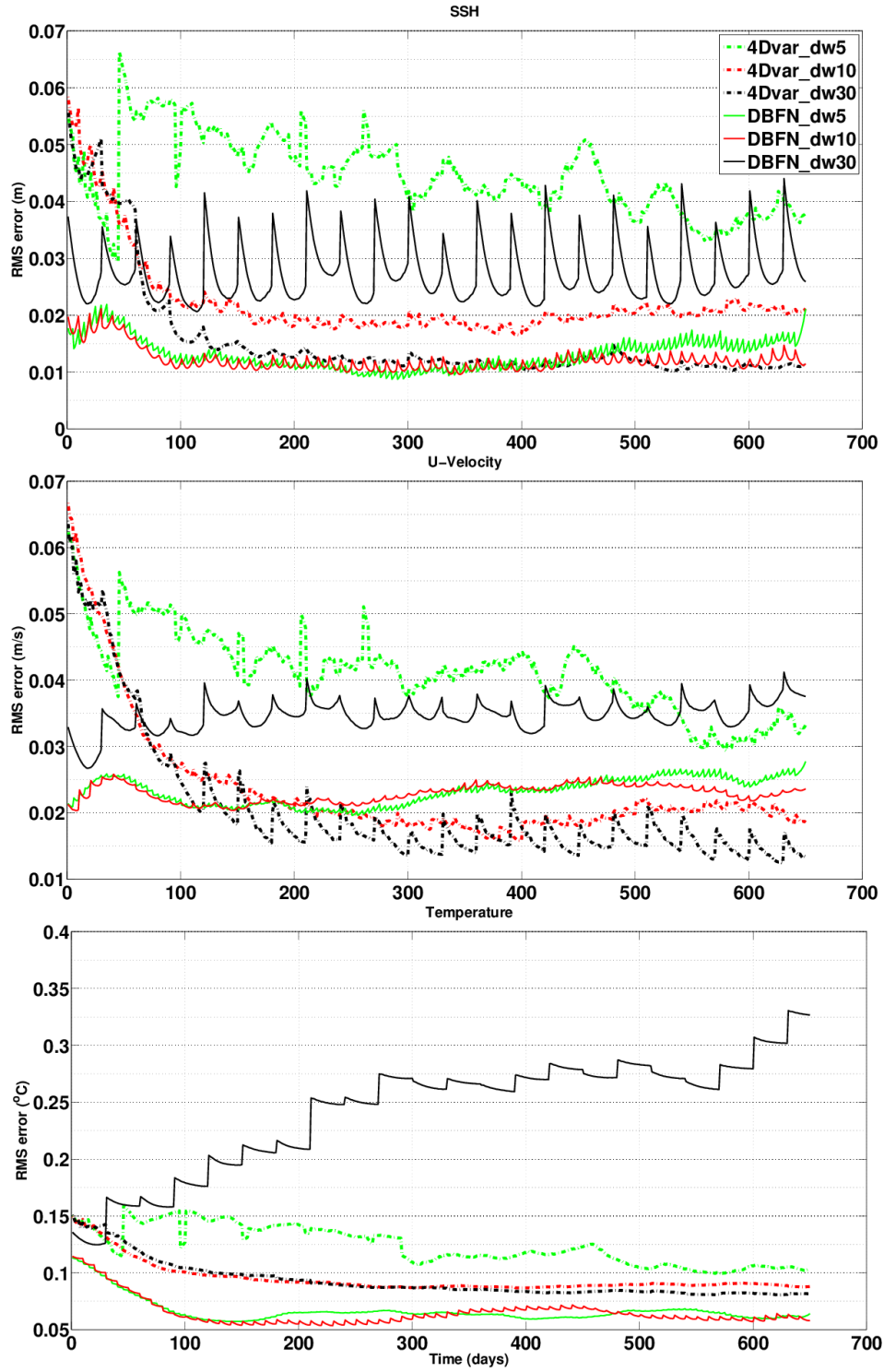


Fig. 10. RMS errors on SSH (top panel), zonal velocity (middle panel) and temperature (bottom panel) from DBFN and 4Dvar experiments with DAw of 5, 10 and 30 days.

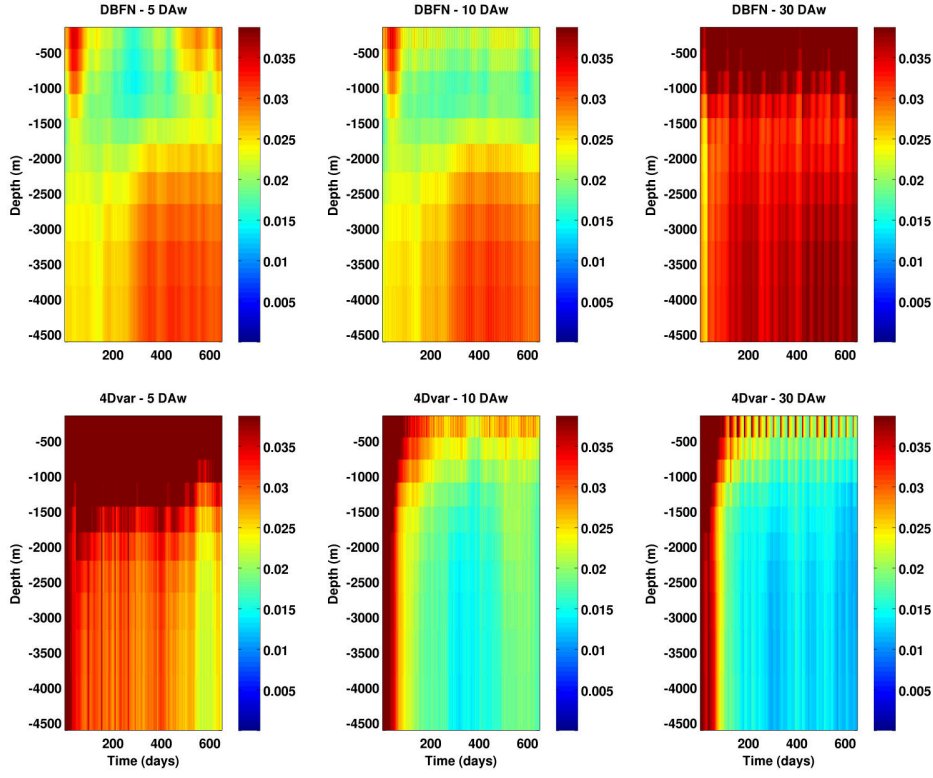


Fig. 11. Time evolution of vertical profiles of horizontally layer-wise averaged rms error of zonal velocities for the DBFN (top panels) and 4Dvar (bottom panels) experiments. Units are in (m/s) .

Next we investigate which scales are better represented by each assimilation method. This is done by comparing the surface kinetic energy spectrum and the deep ocean kinetic spectrum produced by each method. The Fig.(12) shows that the effective resolution of the model is not affected by the diffusive character of the DBFN algorithm. It is clear that there is a reduction of the energy for the scales close to the grid scale, but the energy contained in scales greater than $7 \times \Delta x$ is not affected. It means that the diffusion-induced errors presented in Sect 4 are "controlled" by the assimilation of sea surface height observations.

There is no great difference between the DBFN and 4Dvar surface spectrum for the assimilation windows shorter than 30 days, which once more proves the reliability of the DBFN for the assimilation of oceanic observations. The energy spectra for the deep ocean velocities produced by the DBFN contains more energy than the true spectrum independently of the used DAw. This confirms that the deep ocean velocity errors are due to the high variance of the PLS regression model.

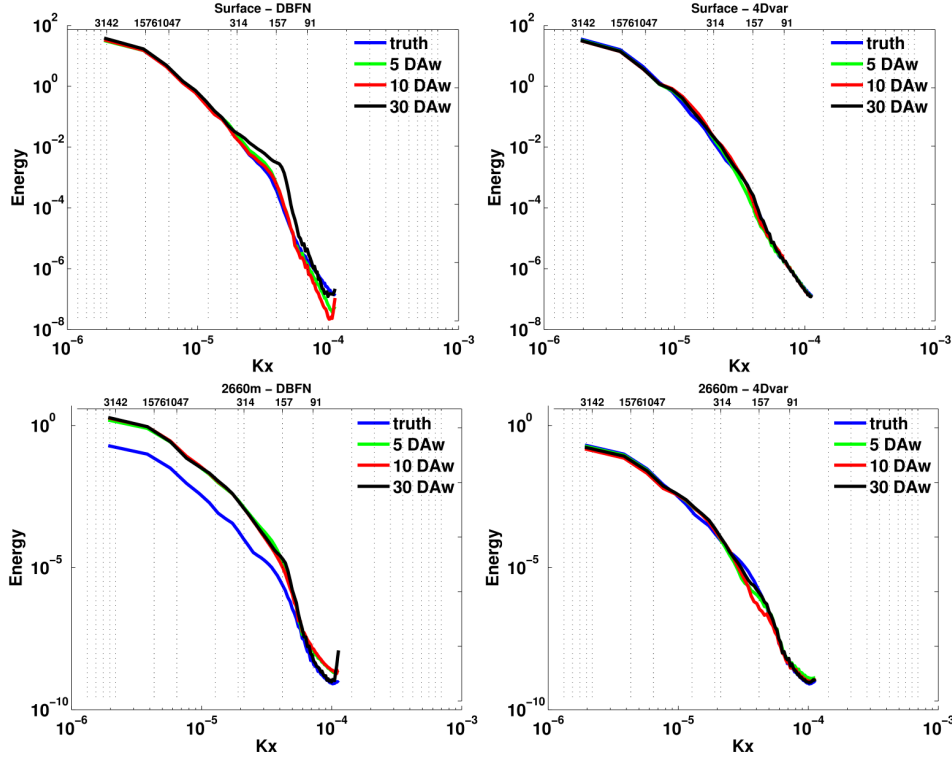


Fig. 12. Kinetic energy mean power spectra calculated using the first layer (top) and a layer at 2660m (bottom) and using the 650 days of the assimilation experiments using the DBFN (left) and the 4Dvar (right). Blue curves represent the “true” power spectra; Green curves represent the power spectra calculated for the 5 days DAw; Red curves represent the power spectra calculated for the 10 days DAw and Black curves represent the power spectra calculated for the 30 days DAw. In the bottom abscissa the tick-labels stand for longitudinal wave-number (rad/m) while in the top abscissa the tick-labels stand for the corresponding wavelengths in km units.

567 6.2.2 Observations density and first guess

568 Finally, two new experiments similar to the one presented in the Sect 6.1 and assimilating complete
 569 daily fields of SSH are presented. The first one uses the same initial condition of the previously
 570 presented experiments and its goal is to investigate the role of the amount of assimilated observa-
 571 tions on the results. In despite of the fact that the problem continues to be underestimated, in this
 572 case the SSH analysis is no more dependent on the SSH spatial covariance, and the unstable modes
 573 associated with the SSH dynamics are certainly observed. The analysis produced for the other state
 574 vector variables remains dependent on the matrices \mathbf{B} for the 4Dvar case and \hat{B}^{PLS} for the DBFN
 575 case.

576 Fig.13 shows the rms error for the SSH and zonal velocity. The results are quite similar to the
 577 results presented in Sect 6.1 with a lower rms error for all variables for both methods. Fig.14 shows
 578 the initial condition error for the zonal velocity produced by both methods for the satellite-like obser-

579 variations and the complete observations experiments. The figure reveals that while in some places the
 580 inclusion of more observations helps to reduce the error in other places it increases the error. This
 581 means that although much more information could be extracted from the new set of observations,
 582 which decreases the global rms errors, the solution remains dependent on the covariance structures
 583 contained on \mathbf{B} and $\hat{\mathbf{B}}^{PLS}$.
 584

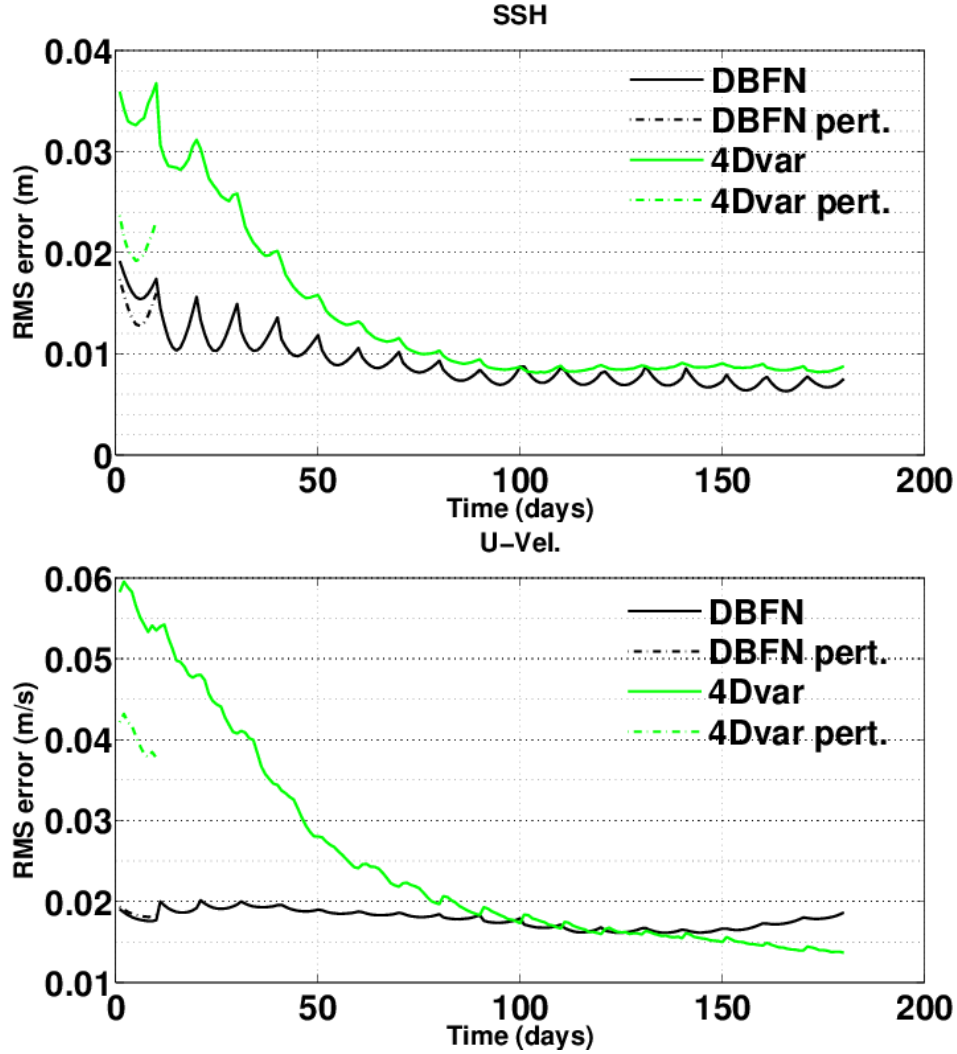


Fig. 13. RMS errors of SSH (top panel) and zonal velocity (bottom panel) from the DBFN and 4Dvar experiments with DAW of 10 days and assimilating complete daily fields of SSH. Dashed lines concern the results from the perturbed experiments.

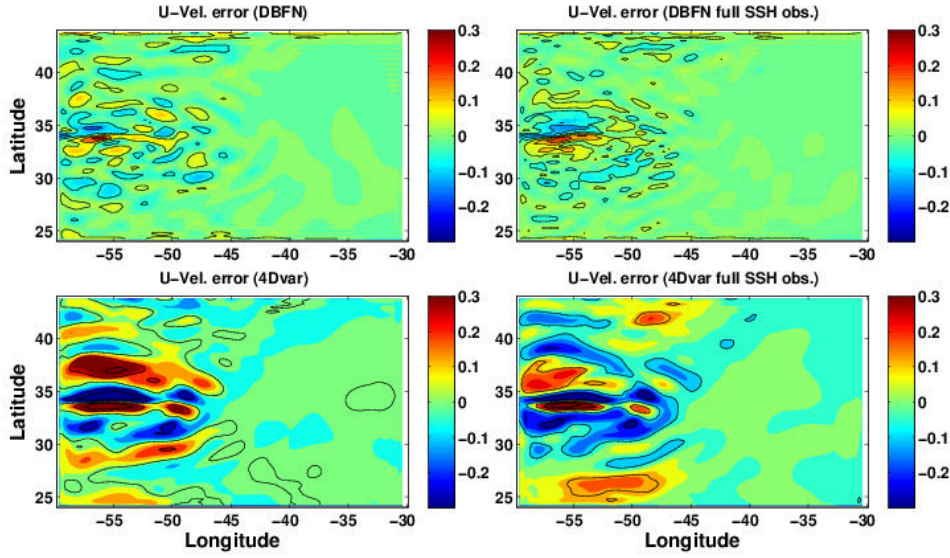


Fig. 14. Zonal velocity error (analysis - truth) for the first assimilation cycle from DBFN experiments (top panels) and 4Dvar experiments (bottom panels). Right panels show the results obtained by assimilating complete daily fields of SSH and the left panels the results from the experiments presented in the Sect 6.1.

The second experiment is initialized with an initial condition that is different from the one used previously. We call this experiment as perturbed experiment. In this case, the objective is to analyze the sensitivity of the solution to the choice of the first guess. Thus, only one assimilation cycle is performed.

Fig.15 shows the initial condition error for the SSH produced by both methods for the perturbed and non-perturbed experiments. Since the perturbed initial condition is not much different from the unperturbed one, the analysis errors have the same structure in both cases, but they differ from one method to another.

The DBFN produced smaller differences between the perturbed and non-perturbed experiences than the 4Dvar for the entire domain. A remarkable difference between the errors produced by the 4Dvar and the DBFN is the error structure in the western boundary that is produced by the DBFN, which is positive northward $34^{\circ}N$ and negative southward $34^{\circ}N$. The presence of this structure is related to the fact that the DBFN analysis is the final condition produced by the backward model. The same pattern was also observed in the Fig. 3 that shows the backward error for the SSH variable. Since this region is a stable region, e.g. there are no meanders and vortices produced there, this suggests that the remaining errors produced by the DBFN project mostly onto the stable manifold as suggested by Aurox (2009). This should partially explain the other differences between the remaining errors produced by both methods as well as the better performance of the DBFN in the first assimilation cycle since the DBFN naturally corrects the forward unstable errors during the backward integration. The rms error of the identified trajectory for the perturbed experiment may be

seen in Fig. 13 as the green (4Dvar) and black (DBFN) dashed curves. The results clearly show that for the configured experiments the DBFN is much less sensitive to the first guess than the 4Dvar.

The small sensitivity of the DBFN to the first guess is in accordance with the theoretical result about the BFN presented by Auroux and Blum (2005) that states that for a linear system and under complete observation condition the identified trajectory is independent of the first guess. To what extent this theoretical result may be extended to nonlinear systems assimilating an incomplete set of observations, as the one studied in this article, we do not know. The results presented here suggest that the use of the DBFN may be advantageous in situations in which the system passes by strong changes resulting in a background (first guess) that is far from the true state.

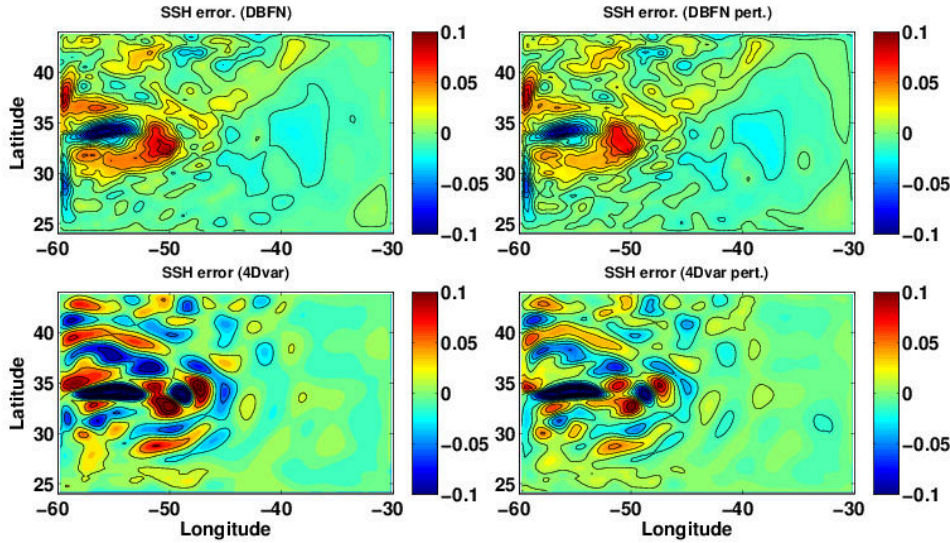


Fig. 15. SSH error (analysis - truth) from DBFN experiments (top panels) and 4Dvar experiments (bottom panels). Right panels show the results obtained from the perturbed experiment.

7 Conclusions and perspectives

This study used the NEMO general circulation model in a double gyre configuration to investigate the Diffusive Back and Forth Nudging performance under different configurations of the data assimilation window, observation network and initial conditions, and to compare it with 4Dvar.

It has been shown that the reliability of the backward integration should be carefully examined when the BFN/DBFN is applied to non-reversible systems. This should support the choice of the assimilation window and identify whether the available observations are sufficient to control the errors induced by the non-reversible terms of the model equations. In this article we have shown that the DBFN might be used for the assimilation of realistically distributed ocean observations, despite the limited accuracy of the backward integration. Improving the backward integration would further improve the DBFN performance and make possible the use of longer assimilation windows.

Our results show that the DBFN can produce results comparable to 4Dvar using lower computa-

626 tional power. This is because DBFN demands less iterations to converge and because one iteration
627 of 4Dvar corresponds to one integration of the tangent linear model, one integration of the adjoint
628 model, which costs four times more than one standard model integration, plus the cost of minimizing
629 the cost function, while the DBFN costs twice the integration of the nonlinear model.

630 The sensitivity tests show that for the 4Dvar long assimilation windows should be preferably used
631 because it favors the propagation of the sea surface height information to the deep layers. For the
632 DBFN, short windows are preferable because it reduces the effect of the diffusion-induced errors. In
633 future works it would be beneficial to account for this errors when constructing the nudging gain.

634 Moreover, the results show that for assimilation systems assimilating a much reduced number of
635 observations with respect to the size of the state space, such as ocean data assimilation systems usu-
636 ally do, the set-up of the covariance matrix is a key step since this matrix propagates the information
637 from the observed variables to the non-observed variables. In addition, although in this study the
638 methods have been configured with different covariance matrices, the results show that the DBFN is
639 less sensitive to the background field than the 4Dvar.

640 Finally, it appears that the DBFN algorithm is worth being further explored both on theoretical
641 and practical aspects, especially those related to the optimization of the matrix \mathbf{K} and applications
642 to a more realistic configuration.

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