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Diffusion model of interacting gravity waves on the surface of deep fluid

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Abstract. A simple phenomenological model for nonlinear interactions of gravity waves on the surface of deep water is developed. The S_{nl} nonlinear interaction term in the kinetic equation for wave action is replaced by the nonlinear second-order diffusion-type operator.

Analytical and numerical studies show that the new model gives a reasonably good description of a real situation, consuming three order of magnitude less computer time.

1 Introduction

The leading nonlinear interaction of gravity waves on the surface of deep liquid is four-wave interaction (Phillips, 1966) satisfying the resonant conditions

$$\vec{k_1} + \vec{k_2} = \vec{k_3} + \vec{k_4} \tag{1}$$

$$\omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3} \tag{2}$$

where $\omega = \sqrt{gk}$ is the dispersion law.

The four-wave interactions play a very important role in the surface dynamics. They arrest the growth of wave amplitudes, caused by instability of the flat surface in the presence of the wind, redistribute wave energy along the K-plane and form the basic cascades, governing the wave kinetics: direct cascade of the energy to large k and inverse cascade of the wave-action to small k (see Zakharov and Zaslavskii (1982), Zakharov (1992)).

The four-wave interactions are described by the kinetic equation for squared wave amplitudes derived first by Hasselmann (1962). This equation is a natural base for practical models of wave-prediction. Due to this reason many people during last two decades endeavored to develop efficient numerical solvers for this equation (Hasselmann and Hasselmann (1985) , Hasselmann at al. (1985) , Masuda (1980) , Komatsu and Masuda (1996) , Resio and Perrie (1991) , Polnikov (1989) , Lavrenov (1991)).

Due to complexity of the kinetic equation, existing codes are still time-consuming and hardly can be used for practical purposes. The development of a simplified model of four-wave interaction describing in an adequate way the main feature of this process is, therefore, an urgent problem.

There is another reason for development of such a model. The stationary kinetic equation has remarkable exact solution: weak-turbulent Kolmogorov spectra (Zakharov and Filonenko (1966), Zakharov and Zaslavskii (1982)). For energy spectrum they are

$$E_\omega = C_1 \epsilon \frac{gu}{\omega^4}, \quad \epsilon = \frac{\rho_{atm}}{\rho_{water}} \tag{3}$$

$$E_\omega = C_2 \epsilon \frac{g^{\frac{2}{3}} u^{\frac{4}{3}}}{\omega^{\frac{11}{3}}} \tag{4}$$

where u is the wind velocity.

The spectrum (3) describes the transport of energy to small scales, while (4) describes the transport of wave action to large scales. Both spectra are obtained in a very idealistic assumption of isotropy in angles. Real wave spectra both in the ocean and in the laboratory are strongly anisotropic. Meanwhile, there are a lot of evidences, that at least the spectrum (4) fits very well the real situation.

Asymptotic ω^{-4} was observed by many experimentalists since Toba (e.g. Toba (1973), Donelan et al. (1985), Phillips (1966)). This asymptotic appears systematically in numerical experiments (Resio and Perrie (1991), Komatsu and Masuda (1996), Polnikov (1989)). But the complexity of the real kinetic equation does not allow to construct analytical angle – dependent anisotropic spectra. A properly simplified model would serve better this purpose.

In this article we suggest a very simple model of four-wave interaction of the gravity waves. We replace the complex nonlinear integral interaction term by simple nonlinear elliptic differential operator of the second order.

The whole kinetic equation becomes the nonlinear diffusion equation. Its stationary solution can be easily found analytically. They describe not only isotropic Kolmogrov spectra (3) and (4), but also anisotropic spectra corresponding to momentum transport to small scales. We developed the code for numerical solution of new model equation in the presence of a wind and received quite reasonable results. Due to simplicity of the model, it consumes three order of magnitude less computer time than the model using exact kinetic equation. We hope that the new model can be efficiently used in practical programs of wave prediction.

2 General background

Let $\eta(k)$ be a Fourier Transform of the surface elevation and

$$I_k \delta(k+k') = \langle \eta(k)\eta(k') \rangle \quad (5)$$

where I_k is the spatial spectrum of the surface. It contains important, but incomplete information about the surface and cannot satisfy any self-consistent evolutionary equation. More complete information is contained in the distribution of wave action

$$n_k \delta(k-k') = \langle a(k)a(k') \rangle \quad (6)$$

where a_k is the complex normal amplitude (see Zakharov (1968)) , $n_{-k} \neq n_k$ and

$$I_k = \frac{1}{2g}\omega_k(n_k + n_{-k}) \quad (7)$$

Hasselmann showed that n_k satisfies the kinetic equation (Hasselmann, 1962)

$$\frac{\partial n}{\partial t} = S_{nl}(n,n,n) - \beta_k n_k \quad (8)$$

where β_k is the coefficient describing interaction with the wind and wave-breaking and

$$\begin{aligned} S_{nl}(n,n,n) = 4\pi \int & |T_{kk_1k_2k_3}|^2 \delta_{k+k_1-k_2-k_3} \\ & \delta_{\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3}}(n_{k_1}n_{k_2}n_k + n_k n_{k_2} n_{k_3} \\ & -n_k n_{k_1} n_{k_2} - n_k n_{k_1} n_{k_3}) dk_1 dk_2 dk_3 \end{aligned} \quad (9)$$

where $T_{kk_1k_2k_3}$ is the coefficient, describing four-wave interaction. It is a homogeneous function of the third degree

$$T(\epsilon k, \epsilon k_1, \epsilon k_2, \epsilon k_3) = \epsilon^3 T(k, k_1, k_2, k_3) \quad (10)$$

A relatively compact explicit expression for $T_{kk_1k_2k_3}$ can be found in the article of Webb (1978). This function has the following symmetry properties

$$T_{kk_1k_2k_3} = T_{k_1kk_2k_3} = T_{kk_1k_3k_2} = T_{k_2k_3kk_1} \quad (11)$$

Due to these properties, equation (8) formally preserves the following quantities of wave action N, wave energy E and momentum P if $\beta_k = 0$:

$$\begin{aligned} N &= \int n_k dk \\ E &= \int \omega_k n_k dk \\ P &= \int \vec{k} n_k dk \end{aligned} \quad (12)$$

Conservation of these quantities is "formal" because one have to change the order of the integration in four-dimensional integrals to prove it. According to Fubini theorem, this change is permitted if n_k vanishes fast enough at $|k| \to \infty$.

For conservation of the wave action N one has to satisfy the condition

$$n_k < Ck^{-(\frac{23}{6}+\epsilon)} \quad (13)$$

Conservation of the wave energy E is guaranteed if

$$n_k < Ck^{-(4+\epsilon)} \quad (14)$$

and conservation of the momentum P takes place if

$$n_k < Ck^{(-\frac{25}{6}+\epsilon)} \quad (15)$$

where $\epsilon > 0$.

Corresponding critical behavior of the energy spectral density

$$\varepsilon_\omega d\omega = \omega_k n_k k dk d\phi \quad (16)$$

is $\varepsilon_\omega < \omega^{-\frac{11}{3}}$ for wave action, $\varepsilon_\omega < \omega^{-4}$ for energy and $\varepsilon_\omega < \omega^{-\frac{13}{3}}$ for momentum.

In reality, typical asymptotic for ε_ω is $\varepsilon_\omega \simeq \omega^{-4}$, and conditions (14) and (15) are not satisfied while the condition (13) is fulfilled. Thus, in the typical situation only wave action N is a real constant of motion. Energy and momentum "leak" to the small-scale region.

Let us consider the equation

$$S_{nl}(n,n,n) = 0 \quad (17)$$

It has obvious thermodynamic solutions

$$n_k = \frac{T}{\mu + \omega_k} \tag{18}$$

where T and μ are the constant temperature and chemical potential. Special cases of this solution are

$$n_k^{(1)} = \frac{T}{\omega_k}, \quad (\mu = 0)$$
$$n_k^{(2)} = n_0, \quad (T \to \infty, \mu \to \infty, \frac{T}{\mu} = n_0) \tag{19}$$

All motion constants on thermodynamic solutions diverge on thermodynamic solutions at $k \to \infty$. This fact makes thermodynamic solutions useless for applications.

Equation (17) has also non-thermodynamic solutions. Looking for the solutions in the form

$$n_k = C|k|^{-x} \tag{20}$$

one can find (Zakharov and Filonenko (1966), Zakharov and Zaslavskii (1982), Zakharov at al. (1992)) that x can take four values

$$x_1 = 0, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{23}{6}, \quad x_4 = 4 \tag{21}$$

Exponents $x_1 = 0$ and $x_2 = \frac{1}{2}$ correspond to thermodynamic solutions (19). Exponents $x_3 = \frac{23}{6}$ and $x_4 = 4$ give the spectra

$$n_k = C_3|k|^{-\frac{23}{6}}, \quad E_\omega = C_3\omega^{-\frac{11}{3}} \tag{22}$$
$$n_k = C_4|k|^{-4}, \quad E_\omega = C_4\omega^{-4} \tag{23}$$

These solutions are Kolmogorov spectra (3), (4).

Since the equation (17) preserves the momentum, it must have Kolmogorov solution, carrying the momentum to small scales. From dimensional consideration it has a form

$$n_k = f(\phi)k^{-\frac{25}{6}} \tag{24}$$

where $f(\phi)$ is some unknown function of the angle , which cannot be found analytically so far. Solution (24) is realized in the case when there is the source of the momentum, but no source of the energy in the region of small k. This situation is non-physical and therefore one can't get much use from the solution (24). The generic Kolmogorov solution, corresponding to given fluxes of both the energy and the momentum, is much more important. This solution is anisotropic and non-power-like. It has the form

$$n_k = Ck^{-4}F(\phi, k) \tag{25}$$

where $F(\phi, k)$ is so far unknown function of ϕ and k. One can guess that its dependence on k is "slow". In the limit of large k this function should has the form (Kats and Kontorovich, 1971) :

$$F(\phi, k) = 1 + a\frac{S}{P}\cos\phi\sqrt{\frac{g}{k}} \tag{26}$$

where S is the momentum flux, P is the energy flux and a is a constant.

Complexity of the equation (9) is the compelling factor for construction of the simplified models of four-wave interaction. The most popular model is the WAM model. In this model, five-dimensional variety of resonances satisfying conditions (1)-(2) is contracted to 2-dimensional manifold describing the resonance of a single type

$$\vec{k_1} = \vec{k}, \quad \vec{k_2} = (1 + \hat{s})\vec{k}, \quad \vec{k_3} = (1 - \hat{s})\vec{k} \tag{27}$$

where $\hat{s}$ is certain linear operator on the k-plane.

In WAM model, integral equation (9) transforms into nonlinear difference equation. The most basic properties of this equation stay the same. In particular, the stationary WAM equation has the same Kolmogorov solutions (22), (23).

The most weak point of the WAM approximation is an ambiguity in the choice of the basic resonance. Actually, there is no particular reasons for preferring of the resonance (27) over others. Some sort of the optimization of the choice of the basic resonance could essentially improve this "difference" model.

3 Differential approximation

In this article we replace the integral equation (9) by nonlinear diffusive equation of the second order. In the contrary to the difference models, the differential model can be constructed in the unique way. Our model is quite convenient for numerical simulation and gives quite reasonable description of the four-wave interaction. Differential approximation in the theory of the four -wave interaction was offered independently in the papers of Iroshnikov (1985) and Hasselmann at al. (1985). More simple derivation of this equation was done in the article of Balk and Zakharov (1988). Later, the differential equation was used in the work of Dyachenko at al. (1992).

Rigorously speaking, the integral operator in (9) can be replaced by the differential operator only when $T_{kk_1k_2k_3} \neq 0$ and the wave vectors k_1, k_2, k_3 are close to k. In this case, the differential approximation can be obtained using the expansion of $n_{k_1}, n_{k_2}, n_{k_3}$ into the Taylor series in the vicinity of the points $k_i = k$. This cumbersome procedure was done by Hasselmann and

Hasselmann (1985) who obtained nonlinear differential equation of the fourth order.

We, following the work of Balk and Zakharov (1988), offer the more simple way of derivation of this equation. First, we put $g = 1$ and introduce the polar coordinates $(\phi, k = \omega^2)$ on K-plane. In these coordinates, eq. (8) reads

$$\frac{\partial n(\phi,t)}{\partial t} = 32 \int |T(\omega,\omega_1,\omega_2,\omega_3,\phi,\phi_1,\phi_2,\phi_3)|^2 \cdot\delta(\omega+\omega_1-\omega_2-\omega_3)\delta(\omega^2\cos\phi+\omega_1^2\cos\phi_1 -\omega_2^2\cos\phi_2-\omega_3^2\cos\phi_3)\delta(\omega^2\sin\phi+\omega_1^2\sin\phi_1 -\omega_2^2\sin\phi_2-\omega_3^2\sin\phi_3)(n_1n_2n_3+nn_2n_3 -nn_1n_2-nn_1n_3)d\omega_1 d\omega_2 d\omega_3 \tag{28}$$

This equation preserves the following quantities

$$N = \int \omega^3 n(\omega,\phi)d\phi d\omega \tag{29}$$

$$E = \int \omega^4 n(\omega,\phi)d\phi d\omega \tag{30}$$

$$R_1 = \int \omega^5 \cos\phi\, n(\omega,\phi)d\phi d\omega \tag{31}$$

$$R_2 = \int \omega^5 \sin\phi\, n(\omega,\phi)d\phi d\omega \tag{32}$$

It has the following stationary thermodynamic solutions

$$n = \frac{1}{C_1 + C_2\omega + C_3\omega^2\cos\phi + C_4\omega^2\sin\phi} \tag{33}$$

where C_1, C_2, C_3, C_4 are the arbitrary constants. Let us introduce the differential operator

$$L = \frac{1}{2}\frac{\partial^2}{\partial\omega^2} + \frac{1}{\omega^2}\frac{\partial^2}{\partial\phi^2} \tag{34}$$

One can see that the equation

$$\frac{\partial n}{\partial t} = \frac{1}{\omega^3} L u \tag{35}$$

preserves all four quantities (29)-(32) if u is periodic function bounded at $\omega = 0$, satisfying the condition $u \to 0$ at $\omega \to \infty$.

From the other hand, one can check that

$$L\frac{1}{n} = 0 \tag{36}$$

if n is given by (33).

Since $T_{kk_1k_2k_3} \sim k^3$, equation (28) can be roughly estimated as

$$\frac{\partial n}{\partial t} \simeq \omega^{12-1-2-2+9} n^3 = \omega^{19} n^3 \tag{37}$$

Let us consider the fours-order differential equation

$$\frac{\partial n}{\partial t} = \frac{C}{\omega^3} L n^4 \omega^{26} L \frac{1}{n} \tag{38}$$

This equation can be treated as a differential model for kinetic equation (28). Indeed, it preserves the same constants of motion, has the same thermodynamic solutions and the same dimensional estimate (37) as the exact equation (8). It is uniquely constructed up to the constant C.

We have to stress that the equation (38) is heuristic. It cannot be derived from the exact kinetic equation (8)-(9) for any realistic $T_{kk_1k_2k_3}$. Therefore, if the differential model (38) (or more simple one) is applied for description of the real situation, there is absolutely no way to find the constant C analytically. It has to be found from the comparison of the physical and numerical experiment. This fact was first mentioned by Hasselmann and Hasselmann (1985).

The equation (38) satisfies the H-theorem. Let us define the entropy as

$$H = \int \ln n_k d\vec{k} = 2\int \omega \ln n d\phi d\omega$$

From (38) one gets

$$\frac{dH}{dt} = 2C \int n^4\omega^{26}\left(L\frac{1}{n}\right)^2 d\omega d\phi > 0$$

This is an additional argument in favor of this equation.

The stationary equation (17) in the simplest axially-symmetric case takes a form

$$\frac{1}{\omega^2}\frac{\partial^2}{\partial\omega^2} n^4\omega^{26}\frac{\partial^2}{\partial\omega^2}\frac{1}{n} = 0 \tag{39}$$

Looking for power-like solutions of the equation (39), one obtains

$$n = \omega^{-y} \qquad y = y_1, y_2, y_3, y_4 \tag{40}$$

$$y_1 = 0, \quad y_2 = 1, \quad y_3 = \frac{23}{2} \quad y_4 = 8 \tag{41}$$

Equation (39) has four power-like solutions

$$n_1 = const, \quad n_2 = \frac{1}{\omega}, \quad n_3 = \omega^{-\frac{23}{3}}, \quad n_4 = \omega^{-8} \tag{42}$$

First two solutions are thermodynamic, the other two are Kolmogorov spectra (22),(23).

Differential equation (38) coincides with the equations obtained earlier by Iroshnikov (1985) and Hasselmann and Hasselmann (1985).

4 Diffusion approximation

Equation (38) is good for the description of both thermodynamic and Kolmogorov solutions. It can be essentially simplified if we are interested only in turbulent solutions of Kolmogorov type. In accordance with (25)

$$n_\omega = \omega^{-8} F(\phi, \omega) \tag{43}$$

Since $F(\phi,\omega)$ is a slow function of ω, one gets approximately

$$L\frac{1}{n} = \left(\frac{1}{2}\frac{\partial^2}{\partial\omega^2} + \frac{1}{\omega^2}\frac{\partial^2}{\partial\phi^2}\right)\frac{\omega^8}{F(\phi,\omega)} \simeq \frac{28}{F}\omega^6 \simeq \frac{28}{n\omega^2} \tag{44}$$

This calculation prompts us to replace (38) by more simple equation

$$\frac{\partial n}{\partial t} = \frac{a}{\omega^3} L n^3 \omega^{24} = \frac{a}{\omega^3}\left[\frac{1}{2}\frac{\partial^2}{\partial\omega^2} + \frac{1}{\omega^2}\frac{\partial^2}{\partial\phi^2}\right] n^3\omega^{24} \tag{45}$$

where a is new indefinite constant. This is a nonlinear diffusion equation. It preserves the integrals (29)-(32) and has the correct dimensional estimate (37). One can easily find its stationary solutions. They are given by the equation

$$LU = \left(\frac{1}{2}\frac{\partial^2}{\partial\omega^2} + \frac{1}{\omega^2}\frac{\partial^2}{\partial\omega^2}\right) U = 0, \quad U = a n_1^3 \omega^{24} \tag{46}$$

A general periodic solution of this equation can be presented in the form

$$\begin{aligned} u &= q_0 + q_1\omega + (\frac{a_1}{\omega} + b_1\omega^2)\cos(\phi - \phi_0) \\ &+ \omega^{\frac{1}{2}} \sum_{n=2}^{\infty} (a_n \omega^{-\lambda_n} + b_n \omega^{\lambda_n}) \cos n(\phi - \phi_n), \\ \lambda &= \sqrt{\frac{1}{4} + 2n^2} \end{aligned} \tag{47}$$

To find a physical interpretation of this solution, one should calculate fluxes of the conservative quantities (29)-(32). Let us denote

$$\begin{aligned} Q &= \int_0^\omega d\omega \int_0^{2\pi} d\phi\, \omega^3 \dot{n}(\omega,\phi) = \int_0^\omega d\omega \int_0^{2\pi} d\phi L u \\ &= \frac{1}{2}\frac{\partial}{\partial\omega}\langle u\rangle|_0^\infty \\ \langle u\rangle &= \int_0^{2\pi} u\, d\phi \end{aligned} \tag{48}$$

From the physical consideration, one has to assume that $u \to 0$ at $\omega \to 0$ together with all its derivatives. Hence

$$Q = \frac{\partial}{\partial\omega}\langle u(\omega,\phi)\rangle = \pi q_1 \tag{49}$$

Thus q_1 is the flux of the wave action coming from infinity.

In the same way one can find

$$P = -\int_0^\omega d\omega \int_0^{2\pi} d\phi\ \omega^4 \dot{n}(\omega,\phi) = \pi q_0 \tag{50}$$

where P is the flux of the energy to infinity.

A little bit more complicated calculation gives

$$S = -\int_0^\omega d\omega \int_0^{2\pi} d\phi\ \omega^5 cos\phi\ \dot{n}(\omega,\phi) = 3\pi a_1 \tag{51}$$

where S is the flux of the longitudinal momentum to infinity.

Thus the special stationary solution

$$U = \frac{1}{\pi}(P + Q\omega + \frac{1}{3}\frac{S}{\omega}\cos\phi) \tag{52}$$

can be easily interpreted. It is the Kolmogorov spectrum, corresponding to constant flux of the wave action from infinity Q and constant fluxes of energy P and momentum S from $\omega = 0$.

A general Kolmogorov solution has a form

$$n_\omega = \frac{1}{(\pi a)^{\frac{1}{3}}\omega^8}\left(P + Q\omega + \frac{1}{3}\frac{S}{\omega}\cos\phi\right)^{\frac{1}{3}} \tag{53}$$

If the flux of wave action from infinity is absent, equation (53) gives the expression for $F(\phi,\omega)$:

$$F(\phi,\omega) = \frac{1}{\pi^{\frac{1}{3}}}\left(P + \frac{1}{3}\frac{S}{\omega}\cos\phi\right)^{\frac{1}{3}} \tag{54}$$

It is the "slow function" in comparison to ω^{-8}.

If $S \neq 0$, Kolmogorov solution (53) becomes negative for small enough ω. It can not be applied in this range of frequencies. Asymptotically, at $\omega \gg \frac{1}{3}\frac{S}{P}$, this solution becomes isotropic. At infinity

$$F(\phi,\omega) \simeq \left(\frac{P}{a\pi}\right)^{\frac{1}{3}}\left(1 + \frac{S}{g\omega}\cos\phi\right) \tag{55}$$

Referring to the formula (26) one can find that in this model $a = \frac{1}{9}$.

The other stationary solutions have no simple physical interpretation.

5 Numeral simulation

We solved the equation

$$\frac{\partial n}{\partial t} = \frac{a}{\omega^3} L n^3 \omega^{24} + \Gamma_\omega n \tag{56}$$

numerically, assuming that the dumping coefficient Γ_ω is the sum of two parts

$$\Gamma_\omega = \Gamma_1 + \Gamma_2 \tag{57}$$

Coefficient Γ_1 was nonzero in all experiments. It consists of high-frequency hyper-viscosity Γ_h and strong damping Γ_l in low frequency region:

$$\Gamma_l(\omega) = -C_l(\omega-\omega_0)^2, \quad \omega_0 = 4, \quad \omega \le \omega_0 \tag{58}$$

$$\Gamma_h(\omega) = -C_h(\frac{\omega}{\omega_1} - 1)^2, \quad \omega_1 = 98, \quad \omega \ge \omega_1 \tag{59}$$

where C_l and C_h are the positive constants. Presence of the low-frequency damping is necessary for the numerical reasons.

Coefficient Γ_2 includes forcing due to external physical mechanisms, and the damping due to the wave-breaking.

We studied the following variants of the forcing:

Case A. Symmetric forcing

$$\Gamma_2(\omega) = C_p \delta(\omega - 8), \quad C_p > 0$$

Case B. Point forcing

$$\Gamma_2(\omega) = C_p \delta(\omega - 8)\delta(\phi), \quad C_p > 0$$

Case C. "Realistic" forcing

This point should be explained. There is no universal agreement in the wave-modeling community about the form of the source of the energy transmitted from wind to the surface waves. One of the commonly used expression for Γ_f is

$$\Gamma_f(\omega) = \begin{cases} \alpha \frac{\rho_{atm}}{\rho_{water}} \omega(\frac{v}{c} - 1)\cos\phi, & c > v \\ 0, & c < v \end{cases} \tag{60}$$

Here v is wind velocity, $c = \frac{\omega}{k} = \frac{g}{\omega}$ is phase velocity, α is dimensionless constant.

Therefore we used the following parameters of forcing corresponding to "realistic" situation:

$$\Gamma_2(\omega, \phi) = \Gamma_f - C_b \omega^2$$

$$\Gamma_f(\omega) = \begin{cases} C_1^f(\omega - 10)\omega\cos\phi, & 10 < \omega < 94 \\ C_2^f e^{-(\frac{\omega-94}{2.5})^2}\cos\phi, & 94 < \omega < 98 \end{cases}$$

where C_b, C_1^f and C_2^f are positive constants.

This means that in the region $10 < \omega < 94$ $\Gamma_f(\omega)$ is chosen to get the accordance with (60), while in the interval $94 < \omega < 98$ $\Gamma_f(\omega)$ is chosen to provide a smooth transition from region of forcing to the region of high-frequency viscosity.

Equation (56) is not convenient for direct numerical simulation due to the presence of the numerical instabilities appearing from "simple-minded" discretizations. It can be regularized and effectively solved by introducing a new variable $y = (\omega^8 n)^3$:

$$\frac{\partial y}{\partial t} = P(\omega, y)\frac{\partial^2 y}{\partial \omega^2} + Q(\omega, y)\frac{\partial^2 y}{\partial \phi^2} + 3\Gamma_\omega y \tag{61}$$

where

$$P(\omega, y) = \frac{3}{2} a\omega^5 y^{\frac{2}{3}}, \quad Q(\omega, y) = 3a\omega^3 y^{\frac{2}{3}}$$

are nonlinear diffusion coefficients.

This "classical" diffusion equation is solved economically with the help of implicit numerical scheme by simple recursion in the direction of ω and cyclical recursion in the direction of ϕ. The efficiency of the algorithm is illustrated by the fact that it takes just a few dozen of minutes to calculate the development of the turbulence from the random noise initial conditions to stationary state using Pentium 133 MHz CPU on the grid of 128×32 nodes of (ω, ϕ) domain.

We started with the "free" case ($\Gamma_\omega = 0$) putting as an initial data the JONSWAP spectrum:

$$n(\omega, \phi) = -\omega^5 e^{-\frac{5}{4}\left(\frac{\omega}{\omega_p}\right)^{-4}} \gamma^{e^{-\frac{(\omega-\omega_p)^2}{2\sigma^2\omega_p^2}}} cos^4\phi$$

where $\omega_p = 2\pi f_m$, $f_m = 0.144 Hz$, $\gamma = 3.3$, $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$, $\sigma = 0.07$ for $\omega < \omega_p$ and $\sigma = 0.09$ for $\omega > \omega_p$

Fig.1 presents $\frac{\partial n}{\partial t}|_{t=0}$ plotted together with the results of Resio and Perrie (RP), Masuda (RIAM) and WAM method. It is seen that the diffusion approximation results are close to the results of first two groups and essentially differ from the DIA WAM results.

In the symmetric *Case A* (see Fig.2, 3) there was an ample range of frequency with $\Gamma = 0$ (transparency window). Stationary isotropic solution in this case is

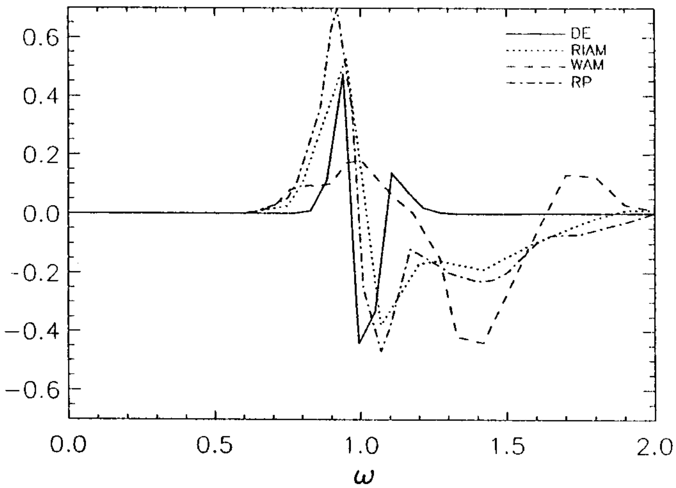

Fig. 1. Comparison of the collision terms for diffusion approximation with RIAM, WAM and RP data taken from Komatsu and Masuda (1996).

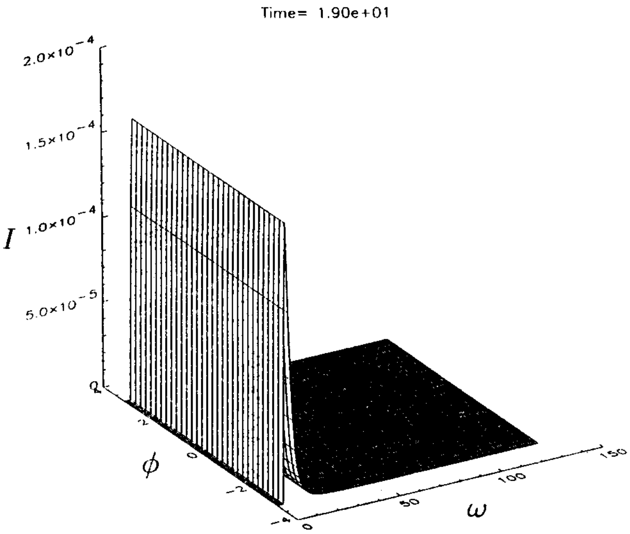

Fig. 2. Energy density $I = \omega^4 n(\omega, \phi)$ for symmetrical forcing at $\omega = 8$

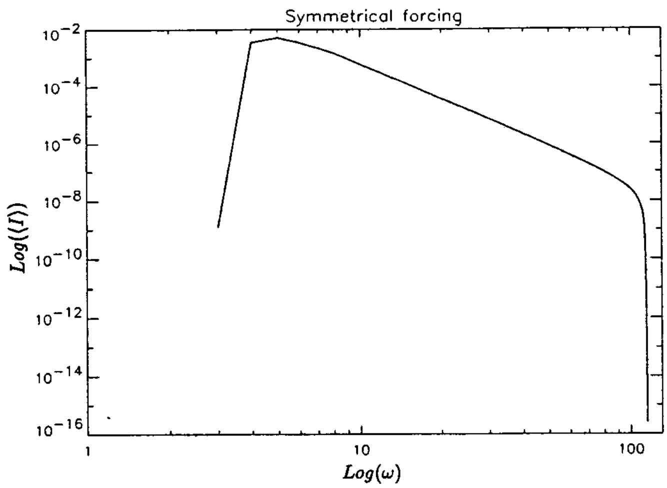

Fig. 3. $Log(\langle I(\log(\omega))\rangle)$ for symmetrical forcing at $\omega = 8$

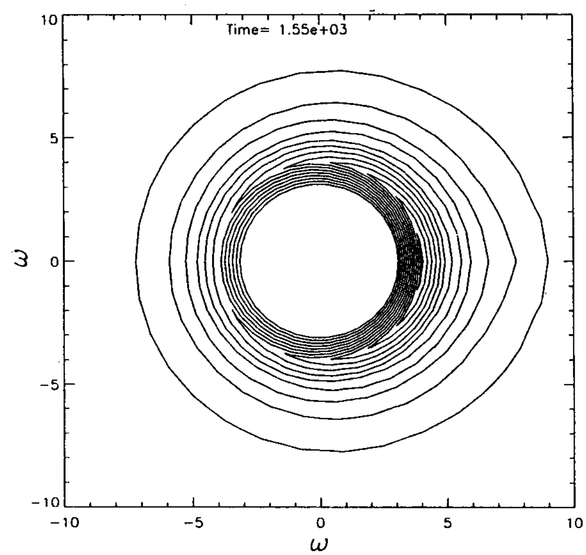

Fig. 4. Line levels for $I(\omega, \phi)$ - point forcing at $\omega = 8$, $\phi = 0$

$$y = A\omega + B \tag{62}$$

where $A < 0$ is the flux of the wave action to large frequencies. The stationary spectrum was established rather soon.

In the case of "point forcing" (see Fig.4, Fig.5 and Fig.6; $\langle\rangle$ means angle averaging), the stationary spectrum is essentially anisotropic. It was reached very soon as well.

An essential anisotropy also exists in the case of "realistic" forcing (see Fig.7, Fig.8).

It is important to note that in all three abovementioned cases the angle-averaged spectrum exhibits ω^{-4} Kolmogorov law despite angular dependence in the last two cases. Temporal evolution of the "realistic" spectrum in the form of the wave propagating toward low ω is presented on Fig.9.

Fig.10, Fig.11 and Fig.12 show temporal behavior of the integrals N,E and average frequency $\omega = \frac{E}{N}$ in the "realistic" case.

Fig.13, Fig.14, Fig.15 show temporal behavior of the same functions for the case $\Gamma_2 = 0$. We used as an initial condition the stationary spectrum obtained for $\Gamma_2 \neq 0$ case.

6 Conclusion

The diffusion model of four-wave interaction is the most simple model presenting the major feature of the physical phenomenon under investigation – conservation of the constants of motion and righteous scaling. It is very convenient and effective for numerical simulation. The numerical experiments show that this model describes

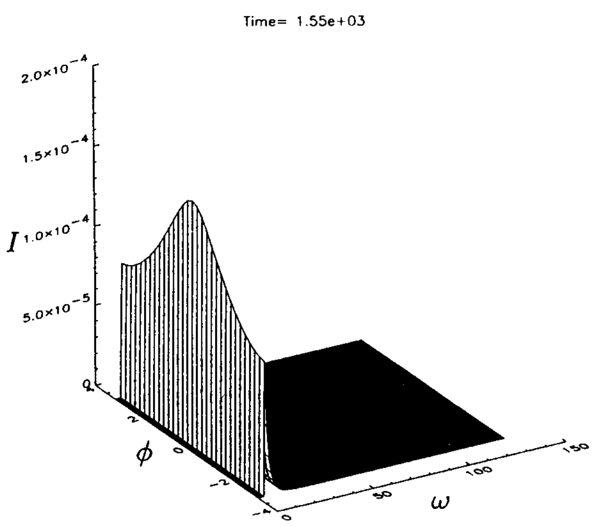

Fig. 5. Energy density $I = \omega^4 n(\omega, \phi)$ for point forcing at $\omega = 8$, $\phi = 0$

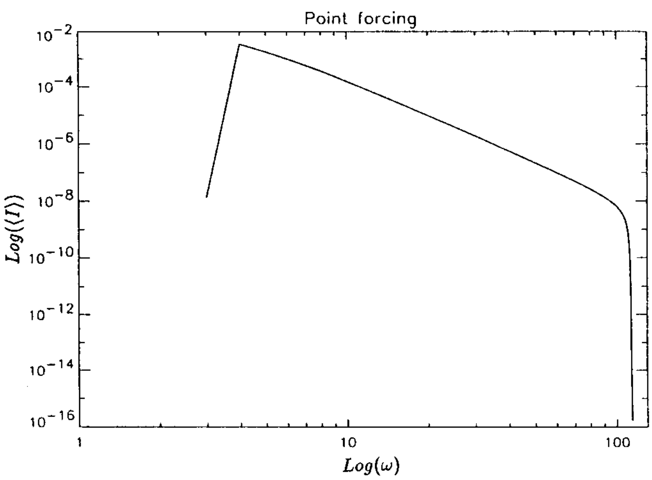

Fig. 6. $\log(\langle I(\log(\omega))\rangle)$ for point forcing at $\omega = 8$, $\phi = 0$

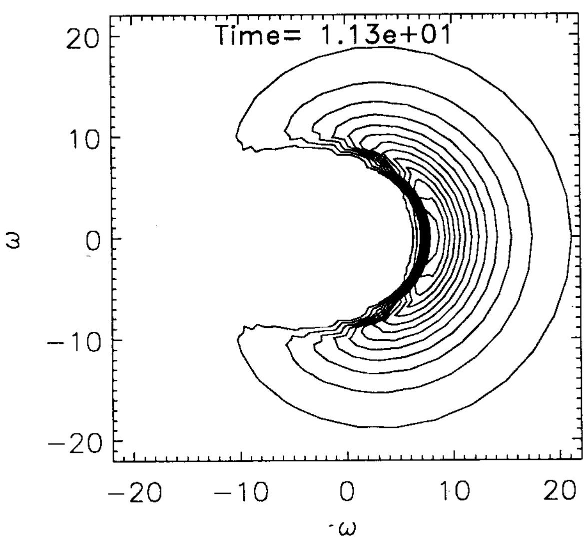

Fig. 7. Same as Fig.4 for "realistic" case

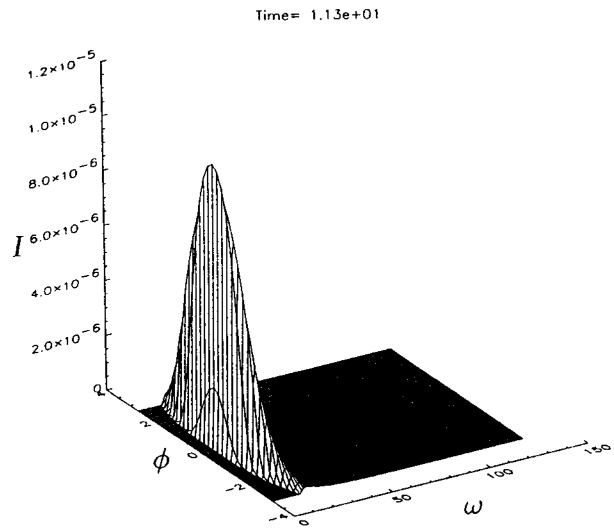

Fig. 8. Same as Fig.5 for "realistic" case

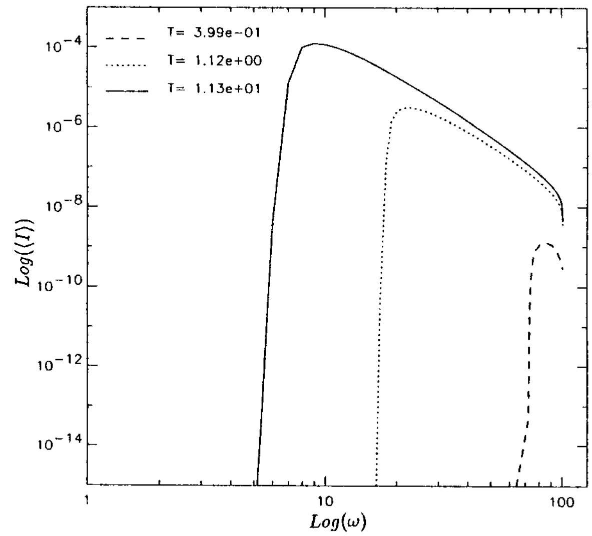

Fig. 9. $\log(\langle I(\log(\omega))\rangle)$ for "realistic" forcing for three different time moments

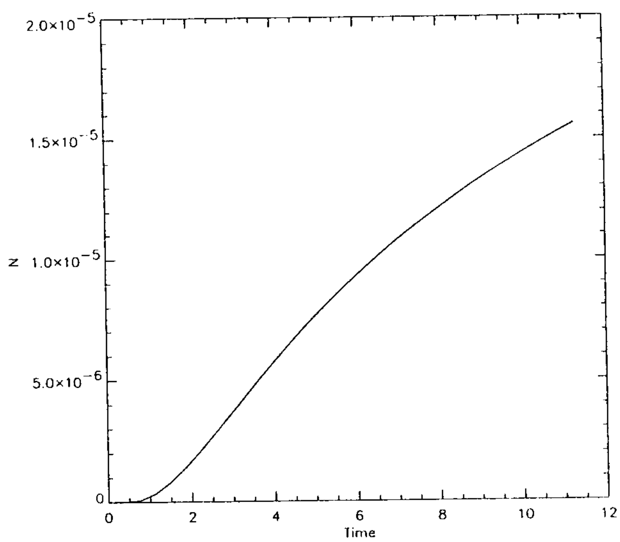

Fig. 10. Temporal behavior of integral N in the "realistic case"

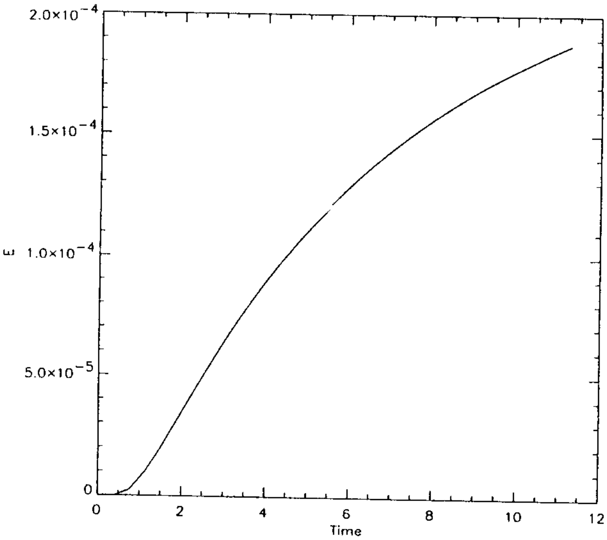

Fig. 11. Temporal behavior of integral E in the "realistic case"

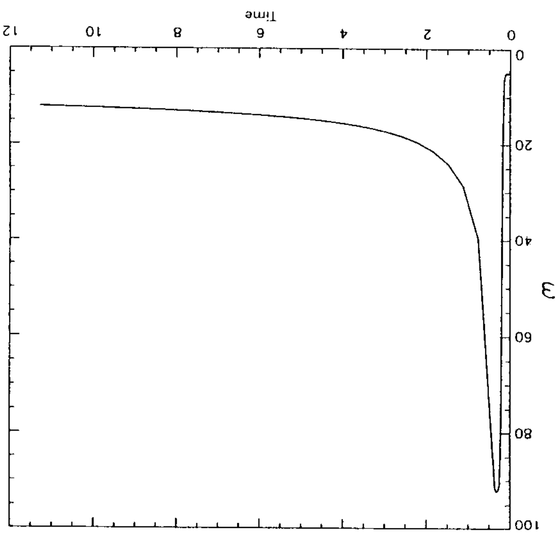

Fig. 12. Temporal behavior of the averaged frequency in the "realistic case"

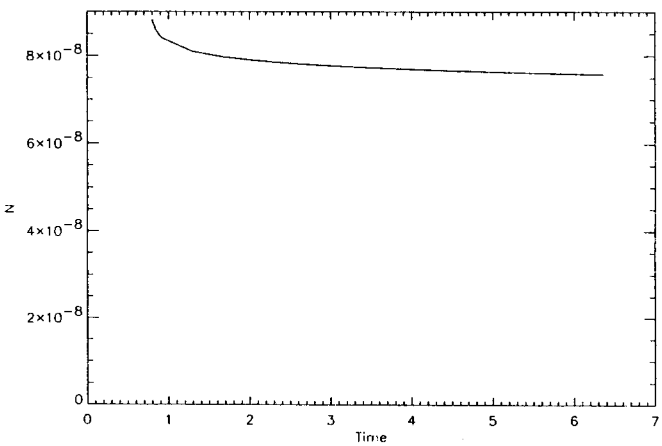

Fig. 13. Temporal behavior of integral N in the $\Gamma_2 \neq 0$ case

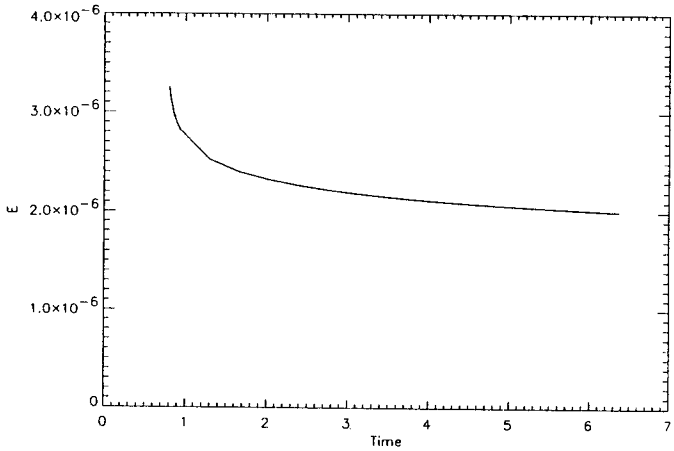

Fig. 14. Temporal behavior of integral E in the $\Gamma_2 \neq 0$ case

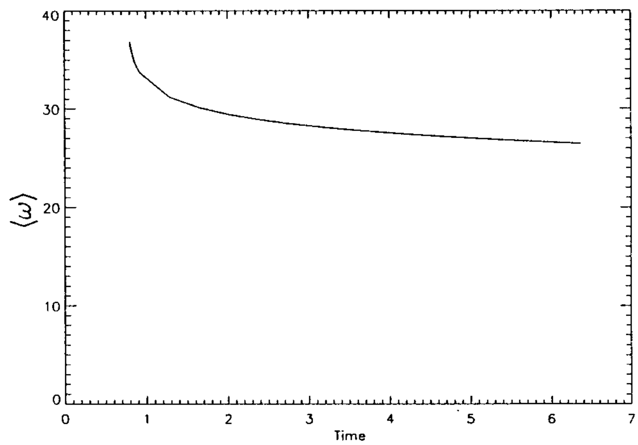

Fig. 15. Temporal behavior of the averaged frequency in the $\Gamma_2 \neq 0$ case

evolution of the wave spectra reasonably well and can be used for development of the new generation of wave-prediction models. It would be desirable to compare the new model with numerous accumulated data of the field observations and laboratory experiments. To do this we should include in the equation (35) the dependence on the spatial coordinate x (the fetch). The new equation is

$$\frac{\partial n}{\partial t} + \frac{\cos\phi}{2\omega}\frac{\partial n}{\partial x} = \frac{1}{\omega^3}Lu + \Gamma(\omega,\phi)n \qquad (63)$$

The numerical simulation of the equation (63) is separate and nontrivial problem. We hope to present the results of the simulation of this equation in the next article.

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