

Non-linear quenching of current fluctuations in a self-exciting homopolar dynamo, proved by feedback system theory

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Abstract. Hide (Nonlinear Processes in Geophysics, 1998) has produced a new mathematical model of a self-exciting homopolar dynamo driving a series-wound motor, as a continuing contribution to the theory of the geomagnetic field. By a process of exact perturbation analysis, followed by combination and partial solution of differential equations, the complete nonlinear quenching of current fluctuations reported by Hide in the case that a parameter ε has the value 1 is proved via the Popov theorem from feedback system stability theory.

1. Introduction

In a recent continuing advance to the self-exciting dynamo theory of the earth's magnetic field, Hide (1998) has presented a mathematically novel set of three non-linear first order differential equations. There is a particular function involved in these which, in the slightly modified notation used here, may be written

$$f(X) = 1 - \varepsilon + \varepsilon\sigma X \quad (1).$$

In eq.(1) we have $\sigma > 0$ and $0 \leq \varepsilon \leq 1$. If the armature current of the series-wound motor driven by the Faraday disk dynamo is I , the torque developed by that motor is taken to be proportional to $(1 - \varepsilon)I + \varepsilon\sigma I^2$. Hide (1998) has found, by numerical and analogue electronic circuit experiments, and by bifurcation analysis, that for $0 = \varepsilon$ a rich dynamical behaviour, including multiply-periodic as well as chaotic (Acheson, 1997) persistent temporal fluctuations ensues; that for $0 < \varepsilon < 1$ the state variables settle at steady equilibrium values over a volume of the system parameter space which increases with ε , giving "partial quenching" of the non-stationary behaviour; and that for $\varepsilon = 1$ the fluctuations are completely suppressed. He refers to the complete suppression of fluctuations for $\varepsilon = 1$ as "non-linear quenching," because the linear term in the torque expression vanishes under this condition. Since $\varepsilon = 1$ is geophysically very relevant, Hide views the non-linear quenching phenomenon as indirect evidence in favour of the likely predominance of *forced* as opposed to *free* contributions in inducing geomagnetic polarity reversals.

The purpose of this note, prepared after correspondence with Professor Hide, is to give a mathematical proof of the non-linear quenching phenomenon, based on a theorem of V. M. Popov (Willems, 1970), concerned with the stability of non-linear feedback systems. It is hoped that this illustration of the way of thinking of an electrical engineer engrossed in the study of automatic control may prove helpful to workers in the field of geophysics.

As hinted above, it is convenient (but done with apologies) to use upper case letters for the state variables in the Hide equations, reserving lower case for perturbations from equilibrium values. In addition, we mention at the outset that, as confirmed with Professor Hide, appropriate scaling of variables and of another system parameter allows σ , without any loss of generality, to be assigned the value 1, and this choice is made here. Thus, *confining attention to the case $\varepsilon = 1$* , the Hide equations, for the present purpose, become

$$\begin{aligned} \dot{X} &= X(Y - 1) - \beta ZX \\ \dot{Y} &= \alpha(1 - X^2) - \kappa Y \\ \dot{Z} &= X^2 - \lambda Z \end{aligned} \quad (2).$$

The dot denotes notes differentiation with respect to dimensionless time τ .

If (X_0, Y_0, Z_0) is any equilibrium state for eq.(2) and the state variables are expressed as

$$X = X_0 + x$$

$$Y = Y_0 + y$$

$$Z = Z_0 + z,$$

the *exact* perturbation equations are

$$\begin{aligned} \dot{x} &= Y_0 x + x(y - 1) + X_0 y - \beta Z_0 x - \beta X_0 z - \beta z x \end{aligned}$$

$$\begin{aligned}\dot{y} &= -\alpha x^2 - 2\alpha X_0 x - \kappa y \\ \dot{z} &= 2X_0 x + x^2 - \lambda z\end{aligned}\quad (3)$$

The null state $x = 0, y = 0, z = 0$ of eq.(3) corresponds to the appropriate equilibrium state of eq.(2). We now examine the stability of the possible equilibrium states of eq.(2) through the medium of eq.(3).

2 The process of self-excitation

The equilibrium states (X_0, Y_0, Z_0) of eq.(2) are readily found and are denoted as follows:

$$E1 = (0, \alpha/\kappa, 0) \quad (4)$$

$$E2,3 = \left(\pm \sqrt{\frac{\lambda(\alpha - k)}{\alpha\lambda + \beta k}}, \frac{\alpha(\beta + \lambda)}{\alpha\lambda + \beta k}, \frac{\alpha - k}{\alpha\lambda + \beta k} \right) \quad (5)$$

In eq.(5), we associate the positive sign on the square root with E2 and the negative with E3.

It is seen that E2 and E3 exist as *real* equilibria only for $\alpha \geq \kappa$.

For perturbations from E1, eq.(3) becomes

$$\begin{aligned}\dot{X} &= ((\alpha - \kappa)/\kappa)X + XY - \beta ZX \\ \dot{Y} &= -\alpha X^2 - \kappa Y \\ \dot{Z} &= X^2 - \lambda Z\end{aligned}\quad (6)$$

We propose the Liapunov function (Willems, 1970)

$$V = X^2 + (1/\alpha)Y^2 + \beta Z^2 \quad (7)$$

to investigate the stability of the null state of eq.(6). There results

$$\begin{aligned}\dot{V} &= (\partial V/\partial X) \dot{X} + (\partial V/\partial Y) \dot{Y} + (\partial V/\partial Z) \dot{Z} \\ &= 2((\alpha - \kappa)/\kappa)X^2 - 2(\kappa/\alpha)Y^2 - 2\beta\lambda Z^2\end{aligned}\quad (8)$$

Since all parameters $\alpha, \beta, \kappa, \lambda$, are positive, then for $\alpha < \kappa$ (which corresponds to too low a driving couple on the Faraday disk), \dot{V} is *negative definite*, and the null state of eq.(3) has *global asymptotic stability*. This means that all perturbations from E1, the non-excited state, collapse for $\alpha < \kappa$.

For $\alpha > \kappa$, however, \dot{V} becomes sign indefinite in such a way that E1 becomes a point of *unstable* equilibrium. Trajectories diverge from E1, but what happens to them? We shall prove that they are attracted either to E2 or E3, depending on the sign of the initial value of x . We shall perform the analysis only for E2: that for E3 follows immediately.

3 Stability of the equilibrium state E2

Substituting E2 into eq.(3) we get, after some algebra,

$$\begin{aligned}\dot{x} &= (y - \beta z) \left(x + \sqrt{\frac{(\alpha - \kappa)\lambda}{\alpha\lambda + \beta\kappa}} \right) \\ \dot{y} &= -\alpha x \left(x + 2\sqrt{\frac{(\alpha - \kappa)\lambda}{\alpha\lambda + \beta\kappa}} \right) - \kappa y \\ \dot{z} &= x \left(x + 2\sqrt{\frac{(\alpha - \kappa)\lambda}{\alpha\lambda + \beta\kappa}} \right) - \lambda z\end{aligned}\quad (9)$$

It is convenient to introduce compact notations for quantities occurring in eq.(9):

$$\begin{aligned}g &= x \left(x + 2\sqrt{\frac{(\alpha - \kappa)\lambda}{\alpha\lambda + \beta\kappa}} \right) \\ w &= y - \beta z\end{aligned}\quad (10)$$

In terms of the operator

$$D = d/dt,$$

the last two members of eq.(9) become

$$y = -\alpha g/(D + \kappa)$$

$$z = g/(D + \lambda),$$

which combine to give

$$w = - \frac{((\alpha + \beta)D + (\alpha\lambda + \beta\kappa))}{(D + \lambda)(D + \kappa)} g \quad (11)$$

Thus, w is generated by a linear differential operator acting on g . This relationship is portrayed by the block diagram representation on Fig. 1.

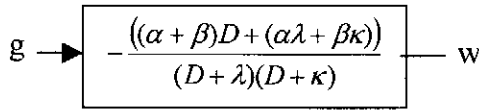


Fig. 1 Block diagram representation of the relationship between g and w .

We now consider the first member of eq.(9), rearranging and integrating it to get

$$\begin{aligned} \ln \left[x(\tau) + \sqrt{\frac{(\alpha - \kappa)\lambda}{\alpha\lambda + \beta\kappa}} \right] - \ln \left[x(0) + \sqrt{\frac{(\alpha - \kappa)\lambda}{\alpha\lambda + \beta\kappa}} \right] \\ = \int_0^\tau w(\gamma) d\gamma \end{aligned} \quad (12)$$

with γ simply a dummy variable of integration. The \ln terms are real only for

$$x(\tau), x(0) > -\sqrt{\frac{(\alpha - \kappa)\lambda}{\alpha\lambda + \beta\kappa}},$$

i.e., for $X > 0$. Thus, this step confines us to the half space in which trajectories are—as we shall see—attracted to E2.

We now introduce the variable p , where

$$\frac{dp}{d\tau} = w \quad (13)$$

Integrating eq.(13), there results

$$p(\tau) - p(0) = \int_0^\tau w(\gamma) d\gamma \quad (14)$$

Eqs.(14) and (12) are *identical* if we make the identification

$$p = \ln \left[x + \sqrt{\frac{(\alpha - \kappa)\lambda}{\alpha\lambda + \beta\kappa}} \right] \quad (15)$$

where the arguments τ and 0 have been dropped for convenience.

Inverting eq.(15) gives

$$x = e^p - \sqrt{\frac{(\alpha - \kappa)\lambda}{\alpha\lambda + \beta\kappa}},$$

which leads, via the first member of eq.(10) to

$$g = e^{2p} - \frac{(\alpha - \kappa)\lambda}{(\alpha\lambda + \beta\kappa)} \quad (16)$$

The graph of g vs. p is sketched on Fig.2, with some significant values indicated.

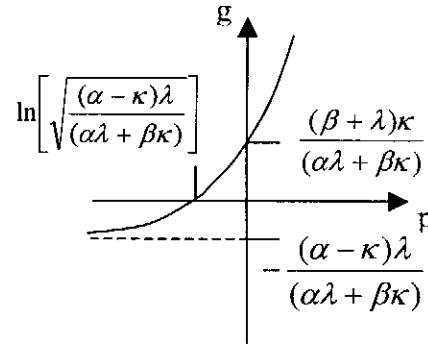


Fig. 2 The graph of g vs. p .

Writing eq. (13) in the operator form

$$p = \frac{1}{D} w \quad (17)$$

and taking eq.(17) in conjunction with eqs.(11) and (16), the non-linear feedback system shown on Fig.3 results.

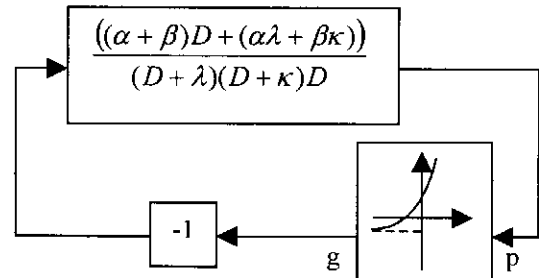


Fig. 3 Representation of the perturbation equations for E2.

Just one step remains to cast the problem into a form suitable for application of the Popov Theorem. We note that if the system shown on Fig.3 settles to a static equilibrium state, it must do so with $w = 0$, otherwise p would still be changing. The static gain of the process relating w to g is $-(\alpha\lambda + \beta\kappa)/(\kappa\lambda) \neq 0$ so that g must also settle at zero. This means that p must settle at the value

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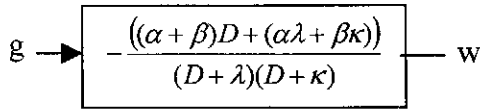


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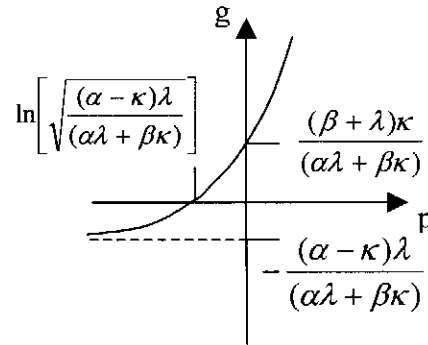


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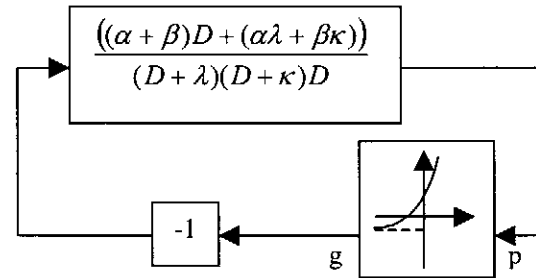


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For the problem in hand, it is readily calculated that the coordinates of the modified polar plot of $G(s)$ are

$$x = \text{Re}G(j\omega) = \frac{-(\alpha + \beta) \left[\omega^2 + \frac{\alpha\lambda^2 + \beta\kappa^2}{(\alpha + \beta)} \right]}{(\omega^2 + \lambda^2)(\omega^2 + \kappa^2)} \quad (24)$$

and

$$y = \omega \text{Im}G(j\omega) = \frac{-(\alpha\kappa + \beta\lambda) \left[\omega^2 + \frac{\alpha\kappa\lambda^2 + \beta\lambda\kappa^2}{(\alpha\kappa + \beta\lambda)} \right]}{(\omega^2 + \lambda^2)(\omega^2 + \kappa^2)} \quad (25)$$

From these expressions, it is possible to deduce by straightforward geometrical reasoning that (x, y) always lies in the third quadrant ; that the magnitudes of x and y decrease monotonically with ω^2 ; and that the ratio y/x increases monotonically with ω^2 except in the single case $\kappa/\lambda = 1$, which gives y/x constant. Thus, the modified polar plot is either a straight line through the origin ($\kappa/\lambda = 1$) or a curve concave on the upper side, as sketched on Fig. 6.

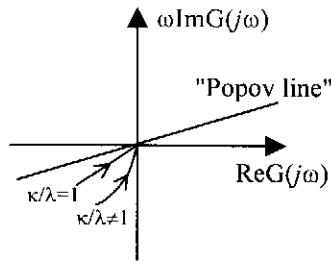


Fig. 6 The modified polar plot of the system shown on Fig. 4.

Our argument reaches its climax in Fig. 6. It shows that the modified polar plot can be fitted with an infinite number of Popov lines *passing through the origin*, i.e., corresponding to $k_m = \infty$. Thus, global asymptotic stability of the null state of the system shown on Fig. 4 is proved and, retracing the development, we appreciate that this proves that $E2$ attracts all trajectories starting in the half space $X > 0$.

4 Concluding remarks

The non-linear quenching effect reported by Professor Hide is now proved, and that is all that this note set out to do. It seemed interesting, however, to explore whether any manoeuvre could be discovered which would result in flipping of the magnetic field i.e., in changing the sign of x in the case $\epsilon = 1$. Various strategies have been tried for

pumping the parameter α , which is a scaled version of the couple applied to the Faraday disk, but so far all have failed. It seems that the plane $X = 0$ represents an impenetrable barrier for trajectories of motion. However, a possibly suggestive experiment is illustrated in Fig. 7.

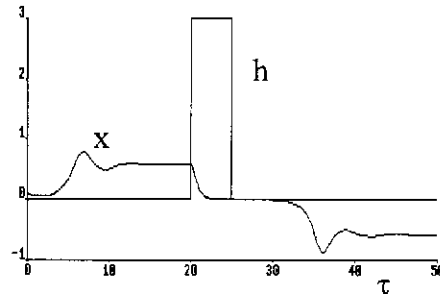


Fig. 7 An experiment to induce current—and therefore magnetic field—flipping in the case $\epsilon = .9999$, $\alpha = 1$, $\beta = 1$, $\kappa = 0.5$, $\lambda = 1$; $h = 3$ for $20 < \tau < 25$.

Here we have set $\epsilon = .9999$ --not quite 1, but close enough to quench fluctuations throughout almost the whole $(\alpha, \beta, \kappa, \lambda)$ space-- in the full Hide equations and have augmented his equation for $dz/d\tau$ with an input function, h , to represent a transient driving couple applied to the motor shaft. Flipping is readily induced by this expedient. Whether this experiment can be termed physically reasonable is beyond the author's competence to tell, but it certainly represents field reversal by a *forced* contribution. The physical significance of not setting ϵ exactly to the value 1 in this experiment is that an ever-so-slight permanent component is necessary in the motor flux to prevent the back electromotive force from collapsing completely as the armature current passes through zero, thus leaving a non-zero driving force to penetrate the wall $X = 0$.

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