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Nonlinear quenching of current fluctuations in a self-exciting homopolar dynamo

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Abstract. In the interpretation of geomagnetic polarity reversals with their highly variable frequency over geological time it is necessary, as with other irregularly-fluctuating geophysical phenomena, to consider the relative importance of forced contributions associated with changing boundary conditions and of free contributions characteristic of the behaviour of nonlinear systems operating under fixed boundary conditions. New evidence –albeit indirect– in favour of the likely predominance of forced contributions is provided by the discovery reported here of the possibility of complete quenching by nonlinear effects of current fluctuations in a self-exciting homopolar dynamo with its single Faraday disk driven into rotation with angular speed $y(\tau)$ (where τ denotes time) by a steady applied couple. The armature of an electric motor connected in series with the coil of the dynamo is driven into rotation with angular speed $z(\tau)$ by a torque $xf(x)$ due to Lorentz forces associated with the electric current $x(\tau)$ in the system (just as certain parts of the spectrum of eddies within the liquid outer core are generated largely by Lorentz forces associated with currents generated by the self-exciting magnetohydrodynamic (MHD) geodynamo). The discovery is based on bifurcation analysis supported by computational studies of the following (mathematically novel) autonomous set of nonlinear ordinary differential equations:

$$dx/dt = x(y-1) - \beta z f(x),$$
$$dy/dt = \alpha(1-x^2) - \kappa y,$$
$$dz/dt = xf(x) - \lambda z, \qquad where f(x) = 1 - \epsilon + \epsilon\sigma x,$$

in cases when the dimensionless parameters $(\alpha, \beta, \kappa, \lambda, \sigma)$ are all positive and $0 \leq \epsilon \leq 1$. Within those regions of $(\alpha, \beta, \kappa, \lambda, \sigma)$ parameter space where the applied couple, as measured by α, is strong enough for persistent dynamo action (i.e. $x \neq 0$) to occur at all, there are in general extensive regions where $x(\tau)$ exhibits large amplitude regular or irregular (chaotic) fluctuations. But these fluctuating régimes shrink in size as ϵ increases from zero, and they disappear altogether when $\epsilon = 1$, leaving only steady régimes of dynamo action.

1 Introduction

It has recently been shown (Hide et al., 1996) that the electric current I generated by a self-exciting Faraday-disk homopolar dynamo with a motor (or capacitor, see Paynter, 1982; Hide et al., 1996) placed in series with the coil can exhibit multiply-periodic as well as chaotic persistent temporal fluctuations, even when the applied couple that drives the Faraday disk into rotation is steady, if the torque T on the armature of the motor is proportional to I. Here we report the unexpected finding that persistent fluctuations are completely quenched when T is proportional to I^2, the square of the current. Partial quenching occurs in the intermediate "quadratic" case when T is proportional to

$$(1-\epsilon)I + \epsilon S I^2 \tag{1}$$

where $S(A^{-1})$ is a constant and the value of ϵ ranges from zero to unity.

These results are derivable from the following mathematically-novel set of nonlinear ordinary differential equations that govern the behaviour of the system:

$$\dot{x} = x(y-1) - \beta z f(x) \tag{2a}$$
$$\dot{y} = \alpha(1-x^2) - \kappa y \tag{2b}$$
$$\dot{z} = xf(x) - \lambda z \tag{2c}$$

where the dimensionless dependent variables (x, y, z) are functions of the re-scaled dimensionless independent time variable τ, $\dot{x} = dx/d\tau$ etc., and

$$f(x) = 1 - \epsilon + \epsilon\sigma x, \tag{3}$$

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σ being a dimensionless measure of S. In Eqs. (2), $x(\tau)$ is the re-scaled electric current generated by the dynamo, $y(\tau)$ is the angular speed of rotation of the disk, and $z(\tau)$ is the angular speed of rotation of the armature of the motor.

The rotating Faraday disk is designed so that current flows radially across the disk between the stationary brushes on the rim and axle. Eq. (2a) expresses Kirchhoff's laws applied to the equivalent circuit of the dynamo, with the term $-\beta z f(x)$ representing the back electromotive force in the motor. Eqs. (2b) and (2c) are the equations of motion of the disk and motor respectively, with the $-\alpha x^2$ term in the former and the $xf(x)$ term in the latter representing torques due to the action of Lorentz forces. In addition to ϵ and σ there are four other dimensionless and essentially non-negative parameters namely $(\alpha, \beta, \kappa, \lambda)$ that determine the behaviour of the system. In Eq. (2b), α (which is inversely proportional to the moment of inertia of the disk) measures the strength of the applied couple and κ measures the coefficient of mechanical friction in the disk. β is inversely proportional to the moment of inertia of the armature of the motor, the motion of which is retarded by a frictional torque proportional to λz (see Eq. (2c)).

2 Self-exciting dynamos

Invented in the last century independently by Varley, Wheatstone and the Siemens brothers following Faraday's discovery of motional induction, homopolar dynamos based on the self-excitation principle were used in the first electric lighting installations, and they played a rôle in the development of more practical systems of public electricity supply (Bowers, 1982; Jeffrey, 1997). Mathematical investigations of the detailed behaviour of self-exciting homopolar dynamos of varying degrees of complexity are more recent, being facilitated by developments in the theory of nonlinear ordinary differential equations and the availability of computers, and motivated in many cases by the general acceptance by geophysicists and astrophysicists that the magnetic fields of planets, stars (and even galaxies) are due to self-exciting MHD dynamo action in their electrically-conducting fluid interiors. This process was first proposed by Larmor (1919) in connection with the Sun and it is now the subject of much modern research (for references see Weiss, 1994; Hollerbach, 1996; Zweibel & Heiles, 1997). The homopolar dynamo is seen as a useful "low-dimensional" analogue from which helpful insights can be obtained into the much more complicated MHD dynamo processes in continuous fluids.

An extensive theoretical literature now exists on homopolar dynamos as intrinsically important nonlinear electromechanical systems of possible geophysical and astrophysical interest (for references see Rikitake, 1966; Moffatt, 1979; Knobloch, 1981; Melchior, 1986; Kono, 1987; Ershov et al, 1989; Moreau, 1990; Jacobs, 1994; Dubois, 1995; Hide et al., 1996; Hide, 1997). It starts with the pioneering study by Bullard (1955) of the simplest imaginable case, namely that of a homopolar dynamo with a single Faraday disk driven into rotation by a steady applied couple when (a) azimuthal currents are prevented from flowing in the disk (cf. Moffatt, 1979; Knobloch, 1981), (b) mechanical friction in the disk can be neglected, and (c) there are no additional elements in the circuit (such as electric motor connected in series with the coil, as described above). Then the governing equations are (2a) and (2b) with the parameters β and κ both set equal to zero.

The nonlinear partial differential equations that govern MHD dynamos express the laws of dynamics, thermodynamics and electrodynamics as applied to a continuous medium. The first step taken by theoretical geophysicists in any discussion of the interpretation of geomagnetic polarity reversals is to note that for every solution $(\mathbf{B}, \mathbf{u})$ (where $\mathbf{B}$ is the magnetic field and $\mathbf{u}$ the Eulerian flow-velocity field) of these equations, reflectional symmetries are such that there is a corresponding solution $(-\mathbf{B}, \mathbf{u})$ in which the magnetic field $\mathbf{B}$ (but not $\mathbf{u}$) everywhere has the opposite sign (provided of course that the boundary conditions are independent of the sign of $\mathbf{B}$). The magnetic field is proportional to x in the homopolar dynamos here discussed, and we note that for every solution (x, y, z) of the governing Eqs. (2) & (3) in the case when $\epsilon = 1$, there is a magnetically-reversed solution $(-x, y, z)$ with no corresponding change in the motions, as represented by y and z. This case is geophysically more relevant than the case when $\epsilon = 0$, which – though interesting and important in its own right as a nonlinear electromechanical system exhibiting rich and varied behaviour (see Hide et al., 1996) – is characterised by solutions (x, y, z) for which there are always corresponding solutions $(-x, y, -z)$, implying that the rotation of the armature of the motor z as well as the current x differ in sign, leaving just the rotation of the disk y unchanged.

3 Bifurcation analysis

Solutions of Eqs. (2) can be classified on the basis of a régime diagram with the (essentially non-negative) dimensionless parameters $\bar{\beta} = \beta/\lambda$ as abscissa and $\bar{\alpha} = \alpha/\kappa$ as ordinate. There are two equilibrium solutions, the first being

$$(x_0, y_0, z_0) = (0, \bar{\alpha}, 0) \tag{4}$$

irrespective of the value of ϵ. This solution corresponds to no dynamo action and it is stable when $\bar{\alpha}$ falls below a certain critical value, which in general depends on $\bar{\beta}$ and the other parameters (κ, λ, σ and ϵ, see below), but not otherwise.

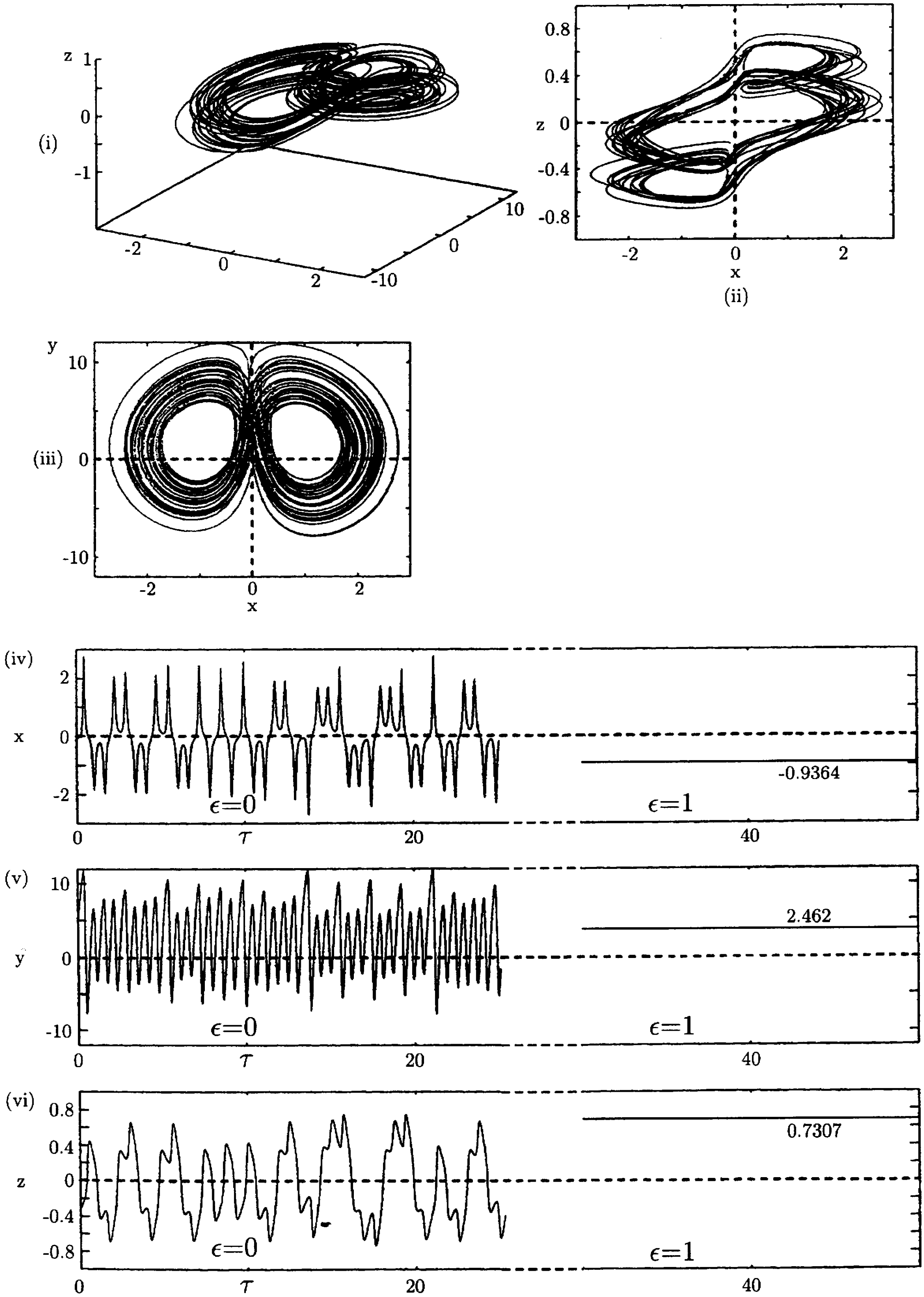

Fig. 1. Illustrating the quenching of chaotic solutions $x(\tau)$, $y(\tau)$ and $z(\tau)$ obtained in a case when ϵ is zero, namely when $(\alpha, \beta, \kappa, \lambda) = (20.0, 2.0, 1.0, 1.2)$ (see Figure 9 of Hide et al., 1996), by changing the value of ϵ to unity. Diagrams (i)-(iii) refer to the timeseries from (arbitrary) $\tau = 0$ to $\tau = 24$ units shown in diagrams (iv)-(vi) for the case when $\epsilon = 0$, which also show the steady behaviour found (after $\tau = 30$) when $\epsilon = 1$ (and $\sigma = 1$). The measured values are consistent with the stable equilibrium solution $(x_0, y_0, z_0) = (\pm 0.9364, 2.462, 0.7307)$ given by Eq. (6). For comparison, note that the corresponding unstable equilibrium solution given by Eq. (6) for the case $\epsilon = 0$ is $(x_0, y_0, z_0) = (\pm 0.9309, 2.667, \pm 0.7758)$.

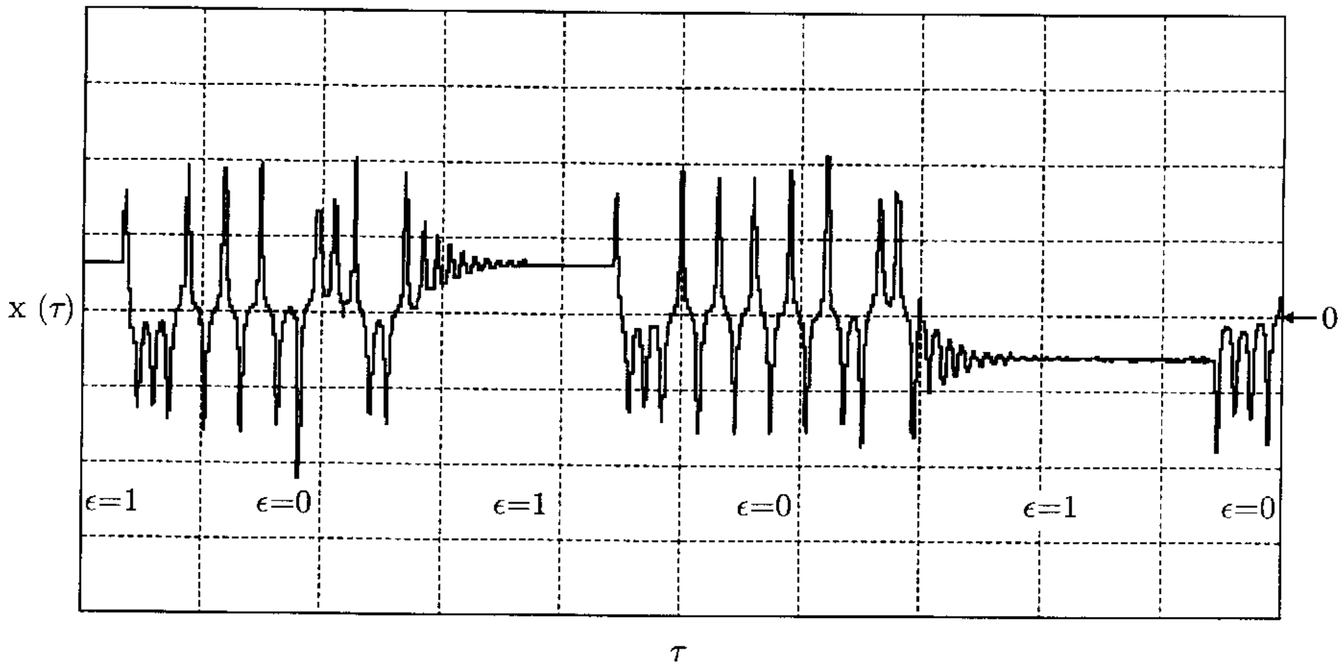

Fig. 2. More detailed version of an $x(\tau)$ timeseries with ($\epsilon = 1$) and without ($\epsilon = 0$) quenching.

The second equilibrium solution, which does depend on the value of ϵ, corresponds to steady (but not necessarily stable) dynamo action, which gives way through Hopf bifurcations (see e.g. Guckenheimer & Holmes, 1986; Thompson & Stewart, 1986; Mullin, 1993) to fluctuating dynamo action when the parameters ($\bar{\alpha}$, $\bar{\beta}$, κ, λ, σ, ϵ) are such that the solution is unstable to infinitesimal oscillatory disturbances. Thus, it is possible in general to divide the $(\bar{\beta}, \bar{\alpha})$ régime diagram into three types of region, where, respectively, there is no dynamo action, where there is steady dynamo action, and where there is fluctuating dynamo action, which turns out to be highly chaotic in certain sub-regions and more regular in others.

It is readily shown that the second equilibrium solution is

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} \pm\left(1 - \frac{1}{\bar{\alpha}}(1+\bar{\beta})\right)^{1/2} \\ 1+\bar{\beta} \\ \pm\frac{1}{\lambda}\left(1 - \frac{1}{\bar{\alpha}}(1+\bar{\beta})\right)^{1/2} \end{pmatrix} \qquad (5)$$

in the case when $\epsilon = 0$, and

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} \pm\left(\frac{\bar{\alpha}-1}{\bar{\alpha}+\sigma^2\bar{\beta}}\right)^{1/2} \\ \frac{\bar{\alpha}(1+\sigma^2\bar{\beta})}{\bar{\alpha}+\sigma^2\bar{\beta}} \\ \frac{\sigma}{\lambda}\left(\frac{\bar{\alpha}-1}{\bar{\alpha}+\sigma^2\bar{\beta}}\right) \end{pmatrix} \qquad (6)$$

in the case when $\epsilon = 1$ (see Eqs. (2) & (3)).

According to a detailed analytical and numerical study of the case when $\epsilon = 0$ (Hide et al., 1996), the region of no dynamo action then occurs where

$$\bar{\alpha} < min(1+\bar{\beta}, 1+\lambda), \qquad (\epsilon = 0) \qquad (7)$$

in the $(\bar{\beta}, \bar{\alpha})$ régime diagram. Steady dynamo action is found in that part of the régime diagram lying above the straight line $\bar{\alpha} = 1 + \bar{\beta}$ and to the left of the curved line

$$\bar{\alpha} = \lambda(2\bar{\beta} - \kappa - \lambda)/2(\kappa - \bar{\beta}) + 3\bar{\beta}/2 + 1 \qquad (8)$$

extending from the point where $(\bar{\beta}, \bar{\alpha}) = (\lambda, 1+\lambda)$ at its lowest end and tending asymptotically to the line where $\bar{\beta} = \kappa$ when $\bar{\alpha} = \infty$. Throughout the extensive region lying to the right of the curved line of Hopf bifurcations given by Eq. (8) and above the line $\bar{\alpha} = 1+\lambda$ the values of $\bar{\alpha}$ and $\bar{\beta}$ are such that steady equilibrium solutions are unstable.and fluctuating dynamo action occurs. The fluctuating solutions exhibit varying degrees of complexity, including multiple periodicity and chaos, behaviour qualitatively similar to that found in other systems with more than two dependent variables governed by autonomous nonlinear ordinary differential equations, (see e.g. Guckenheimer & Holmes, 1986; Thompson & Stewart, 1986; Mullin, 1993).

Now consider the case when $\epsilon = 1$. In contrast to the case when $\epsilon = 0$, the second equilibrium solution – which is given by Eq. (6) and is non-existent when $\bar{\alpha} < 1$ – may be shown by the technique of bifurcation analysis to be stable for all values of $\bar{\alpha}$ that are greater than unity, irrespective of the values of $\bar{\beta}$ and the other dimensionless parameters (κ, λ and σ). This remarkable theoretical result gains support from subsequent numerical calculations in a few representative cases (kindly carried out by Dr. David Acheson), where only steady persistent solutions were found after initial transients had died away. Further successful tests of the theoretical prediction of complete quenching when $\epsilon = 1$ were carried out using an analogue computer based on an electronic circuit (kindly designed and constructed at my request by the late Dr. Neville Robinson and Dr. Guy Peskett) for solving Eqs. (2) (see Figs. 1 & 2).

4 Concluding remarks

Other physical systems in which nonlinear processes promote order rather than disorder are not, of course, unknown. But at the start of the present study it was thought on general grounds that taking ϵ to be non-zero, thereby increasing the number of nonlinear terms in Eqs. (2) from two to four, might serve to increase the complexity of the fluctuations, possibly even enlarging the regions of $(\bar{\beta}, \bar{\alpha})$ parameter space where chaos occurs. So the discovery that fluctuations are strongly inhibited by the processes represented by these extra nonlinear terms and that all fluctuations seem to be completely quenched when $\epsilon = 1$ was most certainly not anticipated.

Some progress is however now being made towards a physical interpretation of the quenching phenomenon, which in retrospect seems less surprising than when it was first discovered. But its discussion lies beyond the scope of this short report, as do details of (a) the analysis of the more complicated general case when ϵ takes values between 0 and 1, and (b) any implications of the quenching phenomenon in self-exciting homopolar Faraday-disk dynamos for future theoretical investigations of self-exciting magnetohydrodynamic dynamos, in which dynamical effects of Lorentz forces must, of course, be simulated correctly. These matters will have to be treated elsewhere, and it should also be profitable to subject the novel autonomous set of nonlinear ordinary differential equations given by Eqs. (2) to detailed mathematical scrutiny without in the first instance placing restrictions on the signs of the parameters $\alpha, \beta, \kappa, \lambda, \sigma$ and ϵ.

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