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Supplement of

Multifractality of climate networks

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Supplementary Material

S1 Time-Delayed Mutual Information (TDMI) Properties

The TDMI measure has the following interesting properties:

– Symmetry (for wide-sense stationary time-series): $I\left(R_\lambda(\vec{x},t), R_\lambda(\vec{y},t+\tau)\right) = I\left(R_\lambda(\vec{y},t), R_\lambda(\vec{x},t-\tau)\right)$

– It measures the shared or redundant information between two time series, and is a generalization of the cross-correlation which can be used to estimate the time delay between processes (Mars and van Arragon, 1982).

– Non-negativeness $I\left(R_\lambda(\vec{x},t), R_\lambda(\vec{y},t+\tau)\right) \geq 0$, with equality when both time series have no dependence.

– $I\left(R_\lambda(\vec{x},t), R_\lambda(\vec{x},t+\tau)\right)$ is analogous to the auto-correlation function, and equals to the entropy of the system at $\tau = 0$.

The rain rate time series is rank transformed and normalized to the unit interval as a data pre-processing step, since mutual information is invariant under it and this can reduce statistical errors in its estimation (Kraskov et al., 2004). Marginal and joint probability densities are computed using histograms with equiquantal binning (Kraskov et al., 2004, Haas et al., 2023, Cellucci et al., 2005).

S2 Comparison with other mutual information estimators

A simple comparison of the equiquantile binning/histogram (EQ) method with other popular mutual information (MI) estimators, such as the KSG estimator (Kraskov et al., 2004) based on distances from the k-nearest neighbours and the KDE estimator (Moon et al., 1995, Steuer et al., 2002) based on kernel density, is proposed here by inspecting the degree patterns of their generated networks. This helps us better understand if there are any biases specific to these methods, especially the effect of zeros in MI estimation.

The parameters for each estimator is fixed as follows: number of bins for EQ is 16, the number of neighbors for KSG is set to 3, and the gaussian kernel with Silverman's rule for selecting optimal bandwidth for KDE. The TRMM precipitation dataset (described in Sect. 2 of the main text) is used for TDMI analysis and network construction.

The effect of increasing fraction of zeros r on the maximal TDMI is studied for two cases: weakly and strongly interacting time-series, by thresholding time-series pairs. Both EQ and KSG give comparable TDMI results, as shown in Fig. S1. However, KDE results are unreliable since it underestimates densities for heavy-tailed distributions. Figure S1 indicates a decreasing trend in TDMI with increasing r for strongly interacting series, but this decreased dependence is still much larger than the maximal TDMI for poor-/non-interacting series. Therefore, TDMI can still be used to test the independence between two variables.

The similarity measure proposed in Sect. 2.2 of the main text is defined as the ratio of the maximal TDMI to the maximal self-information, which is analogous to the auto-correlation between time-series. Figure S2 displays the effect of r on strong and weak coupled series. The decrease in self-dependence in the denominator acts to partially remove the effect of zeros (for $r \ll 1$) from the similarity measure, and at the same time increases the difference between the similarities of strong and weak cases. Please note this is an observation and is not proved rigorously here.

Figure S3 shows respectively the degree patterns of networks generated by EQ, KSG, and KDE, from the coarse-grained dataset at resolutions of 2 days in time and 83 km in space. The KSG and EQ degree patterns are quite comparable, while the KDE degree patterns are drastically different, specifically the lowered degree (fewer teleconnections captured) in the North Pakistan region. The degree results for KDE could be improved using parametric corrections for such heavy-tailed distributions, but this is not further explored here. The EQ approach has been shown to be nearly equivalent to the KSG estimator, and sufficient for the construction of TDMI-based climate networks.

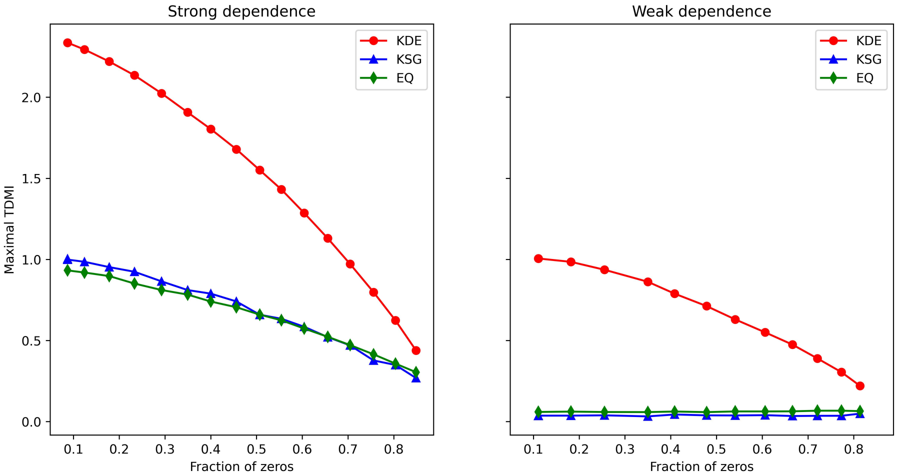

Fig. S1: Effect of zeros on maximal TDMI: Strong dependence (left), and Weak-/Zero-dependence (right) between rainfall time-series.

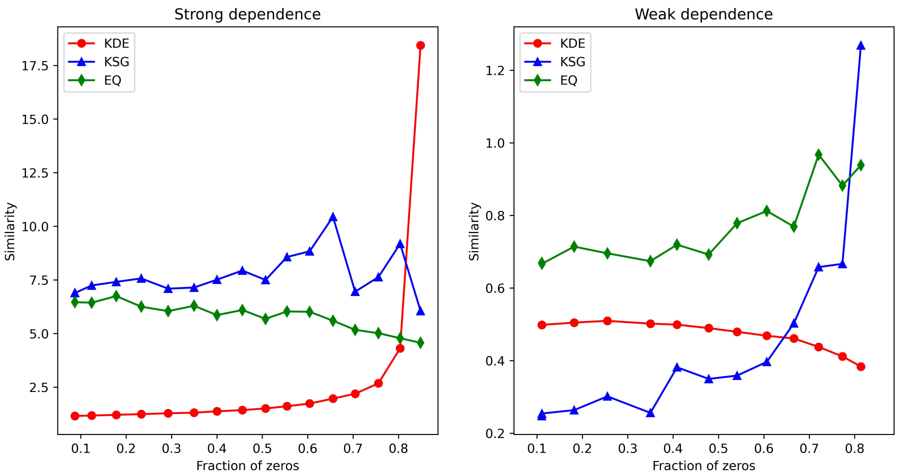

Fig. S2: Effect of zeros on Similarity: Strong dependence (left), and Weak-/Zero-dependence (right) between rainfall time-series.

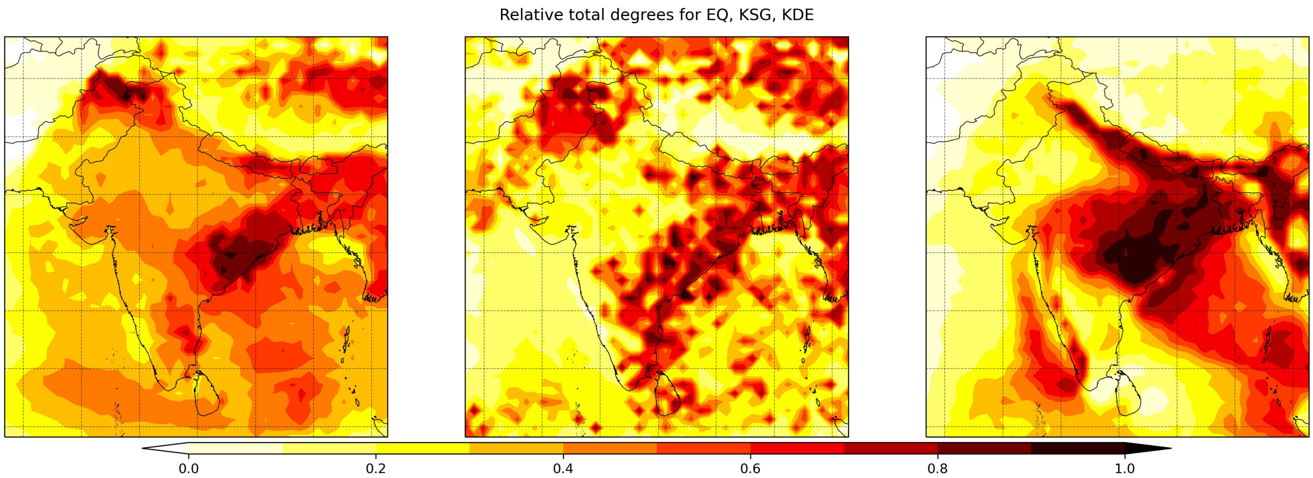

Fig. S3: Climate Networks using TDMI estimators: EQ (left col), KSG (middle col), KDE (right col)

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