Supplement of

Aggregation of slightly buoyant microplastics in 3D vortex flows

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Begin with equations (2) and (3), written in the form

\[ \frac{dx_i}{dt} = v_i \quad (S1a) \]

\[ \varepsilon \frac{dv}{dt} = \varepsilon \frac{3R Du}{2D} + (v - u) + \varepsilon \left( 1 - \frac{3R}{2} \right) g_r + 3 \varepsilon R \Omega \times (u - v) + 2 \varepsilon \left( \frac{3R}{2} - 1 \right) \Omega \times v \quad (S1b) \]

Let \( \tilde{t} = (t - t_o)/\varepsilon \) represent a fast time variable, and consider the particle position and velocity to be functions of \( t \) and \( \tilde{t} \), which are formally treated as separate variables. Thus \( d/dt \) is replaced by \( \varepsilon^{-1} \partial/\partial \tilde{t} + \partial/\partial t \), where it is understood that both \( \partial/\partial t \) and \( \partial/\partial \tilde{t} \) are to be interpreted as particle-following derivatives (and not derivatives with position held constant). Note that the background flow does not depend on \( \tau \), and thus the substantial derivative \( Du/Dt \) is with respect to \( t \) alone. However, dependence on \( \tilde{t} \) is introduced when the derivative is evaluated at the position \( \tilde{x}(\tilde{t}, t) \).

Expanding both variables in a power series in \( \varepsilon \) leads to

\[ \tilde{x} = \tilde{x}^{(0)}(\tilde{t}, t) + \varepsilon \tilde{x}^{(1)}(\tilde{t}, t) + \ldots \]

\[ \tilde{v} = \tilde{v}^{(0)}(\tilde{t}, t) + \varepsilon \tilde{v}^{(1)}(\tilde{t}, t) + \ldots \]

Substitution into Eqs. (S1a,b) leads to

\[ \left( \varepsilon^{-1} \frac{\partial \tilde{x}^{(0)}}{\partial \tilde{t}} + \frac{\partial \tilde{x}^{(0)}}{\partial t} \right) + \varepsilon \left( \varepsilon^{-1} \frac{\partial \tilde{x}^{(1)}}{\partial \tilde{t}} + \frac{\partial \tilde{x}^{(1)}}{\partial t} \right) + \ldots = \tilde{v}^{(0)}(\tilde{t}) + \varepsilon \tilde{v}^{(1)}(\tilde{t}) + \ldots \quad (S2a) \]

and

\[ \varepsilon \left( \varepsilon^{-1} \frac{\partial \tilde{v}^{(0)}}{\partial \tilde{t}} + \frac{\partial \tilde{v}^{(0)}}{\partial t} \right) + \varepsilon^2 \left( \varepsilon^{-1} \frac{\partial \tilde{v}^{(1)}}{\partial \tilde{t}} + \frac{\partial \tilde{v}^{(1)}}{\partial t} \right) + \ldots = \tilde{u}(\tilde{x}(\tilde{t}, t), t) - \tilde{v}^{(0)}(\tilde{t}) + \varepsilon \frac{\partial \tilde{u}}{\partial x_j} x_j^{(1)} \]
\[-\varepsilon v_i^{(1)} + \varepsilon \left[ \frac{3R}{2} \frac{D\tilde{u}}{Dt} + 3R \tilde{\Omega} \times (\tilde{u} - \tilde{v}^{(0)}) + 2 \left( \frac{3R}{2} - 1 \right) \tilde{\Omega} \times \tilde{v}^{(0)} + \left( 1 - \frac{3R}{2} \right) \tilde{g}_r \right] + \ldots, \tag{S2b}\]

where again, the derivatives of \(\tilde{u}\) are evaluated at \(\tilde{x}^{(0)}\). To lowest order, we have

\[
\frac{\partial \tilde{x}^{(0)}}{\partial \tilde{t}} = 0 \quad \text{and} \quad \frac{\partial \tilde{v}^{(0)}}{\partial \tilde{t}} = \tilde{u}(\tilde{x}^{(0)}(\tilde{t}, t), t) - \tilde{v}^{(0)}. \tag{S3a,b}\]

Thus \(\tilde{x}^{(0)} = \tilde{x}^{(0)}(t)\), and since the right-hand side of Eq. (S3b) is then independent of \(\tilde{t}\), it follows that

\[
\tilde{v}^{(0)} = \tilde{u}(\tilde{x}^{(0)}(t), t) + \tilde{c}^{(0)}(t)e^{-\tilde{\tau}}. \tag{S4}\]

If a particle is initiated with a velocity that is differs from the local fluid velocity by more than \(O(\varepsilon)\), then the drag on the particle brings it \(O(\varepsilon)\) close to the fluid velocity over a time scale of \(O(\varepsilon^{-1})\). This behavior is consistent with the requirement in Fenichel theory that the background flow is a normally attracting manifold.

At the next order of approximation \([O(\theta)\) in Eq. (S2a)], we have

\[
\frac{\partial \tilde{x}^{(1)}}{\partial \tilde{t}} = - \frac{\partial \tilde{x}^{(0)}}{\partial \tilde{t}} + \tilde{v}^{(1)} = - \frac{\partial \tilde{x}^{(0)}}{\partial \tilde{t}} + \tilde{u}(\tilde{x}^{(0)}(t), t) + \tilde{c}^{(0)}(t)e^{-\tilde{\tau}}.
\]

After decay of the final term, the remaining terms on the right-hand side depend only on \(t\) and therefore lead to secular growth in \(\tau\) of \(x_i^{(1)}\). To prevent this growth we must set these terms to zero:

\[
\frac{\partial \tilde{x}^{(0)}}{\partial \tilde{t}} = \tilde{u}(\tilde{x}^{(0)}(t), t) \tag{S5}\]

Which indicates simply that following the decay from the initial velocity, the particle follows the flow at leading order. Solving the remaining equation for \(\tilde{x}^{(1)}\) then gives
\[ \tilde{x}^{(1)} = x_o^{(1)}(t) - \bar{c}^{(1)}(t)e^{-\bar{\ell}} \quad (S6) \]

Proceeding to \( O(\varepsilon) \) in \((S2b)\) then gives

\[
\frac{\partial \tilde{v}^{(1)}}{\partial \tilde{t}} + \tilde{v}^{(1)} = -\frac{\partial \tilde{v}^{(0)}}{\partial t} + \left( \frac{\partial \tilde{u}}{\partial x_j} \right)_{x_j=x_i^{(0)}} x_j^{(1)} + \frac{3R \, D\tilde{u}}{2} + 3R\bar{\Omega} \times (\tilde{u} - \tilde{v}^{(0)}) 
+ 2 \left( \frac{3R}{2} - 1 \right) \bar{\Omega} \times \tilde{v}^{(0)} + \left( 1 - \frac{3R}{2} \right) \bar{g}_r
\]

Using Eq. \((S4)\) to substitute for \( v_t^{(0)} \) on the right-hand side, and keeping in mind that \( \frac{\partial}{\partial t} \) represents not a local time derivative but a time derivative with \( \tilde{t} \) held constant, we have

\[
\frac{\partial \tilde{v}^{(0)}}{\partial t} = \frac{\partial}{\partial t} \tilde{u}(x^{(0)}(t), t) + \frac{\partial \bar{c}^{(0)}}{\partial t} \, e^{-\bar{\ell}} = \frac{D\tilde{u}}{D\ell} + \frac{\partial \bar{c}^{(0)}}{\partial t} \, e^{-\bar{\ell}}
\]

Using this expression as well as Eq. \((S6)\) to substitute for and \( x_j^{(1)} \) leads to, after some regrouping of terms, to

\[
\frac{\partial \tilde{v}^{(1)}}{\partial \tilde{t}} + \tilde{v}^{(1)} = \tilde{a}^{(1)}(t) - \tilde{b}^{(1)}(t)e^{-\bar{\ell}}, \quad (S7)
\]

where

\[
\tilde{a}^{(1)}(t) = \frac{\partial \tilde{u}}{\partial x_j} x_j^{(1)}(t) + \left( \frac{3R}{2} - 1 \right) \left[ \frac{D\tilde{u}}{D\ell} - \bar{g}_r + 2\bar{\Omega} \times \tilde{u}^{(0)} \right],
\]

and

\[
\tilde{b}^{(1)}(t) = \frac{\partial \bar{c}^{(0)}}{\partial t} + \frac{\partial \tilde{u}}{\partial x_j} c_j^{(0)} + 3R\bar{\Omega} \times \bar{c}^{(0)}.
\]

The solution to \((S7)\) is
\[ \ddot{v}(1) = \ddot{a}(1)(t) - \ddot{b}(1)(t)\ddot{\ell}e^{-\ddot{\ell}} \]  

(S8)

We can now write down an expression for the particle velocity on the slow manifold, obtained by taking the limit \( \ddot{\ell} \to \infty \) in Eqs. (S4) and (S8):

\[ \ddot{v}(0) + \varepsilon \ddot{v}(1) = \ddot{u}(\ddot{x}(0)(t), t) + \varepsilon \frac{\partial \ddot{u}}{\partial \ddot{x}_j} x^{(1)}_j(t) + \varepsilon \left( \frac{3R}{2} - 1 \right) \left[ \frac{D\ddot{u}}{Dt} - \dddot{g}_r + 2\dddot{\Omega} \times \dddot{u}(0) \right] \]

or, noting that \( \dddot{x} = \dddot{x}(0) + \varepsilon \dddot{x}^{(1)}_o(t) + O(\varepsilon^2) \) on the slow manifold:

\[ \frac{d\dddot{x}}{dt} = \dddot{u}(\dddot{x}(t), t) + \varepsilon \left( \frac{3R}{2} - 1 \right) \left[ \frac{D\dddot{u}}{Dt} - \dddot{g}_r + 2\dddot{\Omega} \times \dddot{u}(0) \right] + O(\varepsilon^2). \]  

(S9)