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Supplement of

Superstatistical analysis of sea surface currents in the Gulf of Trieste, measured by high-frequency radar, and its relation to wind regimes using the maximum-entropy principle

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S1 On the $\Gamma_{n,\lambda}$ distributions from Gaussians with $\sigma \neq 1$

We want to fit the $\sigma^2 = \frac{1}{\beta}$ distributions as a convolution of exponential distributions. Let's first consider which is the general expression of a $\Gamma_{n,\lambda}$ coming from gaussian distributions with $\sigma \neq 1$ in general.

Let x_i be independent gaussian distributed random variables with zero mean ($\mu_i = 0$) and same variance ($\sigma_i = \sigma$):

$$f(x_i) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x_i^2}{2\sigma^2}} \tag{S1}$$

Let's define:

$$Q = \sum_{i=1}^{2n} x_i^2 \tag{S2}$$

So, calling V the elemental shell volume at $Q(x)$, which is proportional to the $(2n-1)$-dimensional surface in $2n$-space for which $Q = \sum_{i=1}^{2n} x_i^2$,

$$\begin{aligned} P(Q)dQ &= \int_V \frac{e^{-\frac{1}{2\sigma^2}\left(x_1^2+x_2^2+\ldots+x_{2n}^2\right)}}{(2\pi)^{2n/2}\sigma^{2n}} dx_1 dx_2 \cdots dx_{2n} \\ &= \frac{e^{-\frac{Q}{2\sigma^2}}}{(2\pi)^n \sigma^{2n}} \int_V dx_1 dx_2 \cdots dx_{2n} \\ &= \frac{e^{-\frac{Q}{2\sigma^2}}}{(2\pi)^n \sigma^{2n}} \cdot \frac{2Q^{\frac{2n-1}{2}}\pi^n}{\Gamma(n)} \cdot \frac{dQ}{2Q^{1/2}} \\ &= \frac{Q^{n-1}e^{-\frac{Q}{2\sigma^2}}}{2^n \Gamma(n) \sigma^{2n}} \cdot dQ \end{aligned} \tag{S3}$$

The resulting normalized $\Gamma_{n,\lambda}$ distribution (where $\lambda = 1/2\sigma^2$) with $2n$ degrees of freedom (the number of the added gaussian variables) is:

$$\Gamma_{n,\lambda}(Q) = \frac{Q^{n-1}e^{-\frac{Q}{2\sigma^2}}}{(2\sigma^2)^n \Gamma(n)} = \frac{\lambda^n Q^{n-1} e^{-\lambda Q}}{\Gamma(n)} \tag{S4}$$

Imponing $2n = 2$ we obtain the exponential distribution:

$$\Gamma_{1,\lambda}(Q) = \frac{1}{2\sigma^2} \cdot e^{-\frac{Q}{2\sigma^2}} = \lambda e^{-\lambda Q} \tag{S5}$$

The mean value of a gamma distributed variable is:

$$\mu_{\Gamma,n} = \int_0^\infty Q \Gamma_{n,\lambda}(Q) dQ = \frac{n}{\lambda} \tag{S6}$$

The variance of a gamma distributed variable is:

$$\sigma^2_{\Gamma,n} = \int_0^\infty Q^2 \Gamma_{n,\lambda}(Q) dQ - \mu^2_{\Gamma,n} = \frac{n}{\lambda^2} \tag{S7}$$

Imponing $n = 1$ (the case of the exponential distribution) we obtain:

$$\sigma^2_{\exp} := \sigma^2_{\Gamma,1} = \frac{1}{\lambda^2} \tag{S8}$$

S2 On the convolution of $\Gamma_{1,\lambda}$ distributions

Let X and Y be two independent random variables distributed as $\Gamma_{1,\lambda}$ and $\Gamma_{1,k}$ respectively, see Equation (S5), where $\lambda = \frac{1}{2\sigma_X^2} > 0$ and $k = \frac{1}{2\sigma_Y^2} > 0$:

$$\begin{aligned} f_X(x) &= \lambda e^{-\lambda x}\Theta_{(0,\infty)}(x) \\ f_Y(y) &= k e^{-ky}\Theta_{(0,\infty)}(y) \end{aligned} \tag{S9}$$

where

$$\Theta_{(a,b)}(t) = \begin{cases} 1 & \text{if } a \leq t \leq b \\ 0 & \text{otherwise} \end{cases} \tag{S10}$$

The two random variables X and Y are independent, so their joint probability distribution is:

$$f_{XY}(x,y) = f_X(x)\cdot f_Y(y) \tag{S11}$$

If we define a new random variable $Z = X + Y$, we will obtain:

$$\begin{aligned} f_Z(x,z) &= f_X(x)\cdot f_Y(z-x) \\ &= \lambda k e^{-\lambda x} e^{-k(z-x)}\Theta_{(0,\infty)}(x)\Theta_{(0,\infty)}(z-x) \\ &= \lambda k e^{-kz} e^{-(\lambda-k)x}\Theta_{(0,z)}(x) \end{aligned} \tag{S12}$$

So, we obtain the convolution:

$$\begin{aligned} f_Z(z) &= \int_0^\infty f_Z(x,z)dx \\ &= \lambda k e^{-kz}\int_0^z e^{-(\lambda-k)x}dx \\ &= \frac{\lambda k}{\lambda - k}(e^{-kz} - e^{-\lambda z}) \end{aligned} \tag{S13}$$

The normalization can be easily checked.

S2.1 The case where $\lambda = k$

In the case where $\lambda = k$, it means that X and Y are identically distributed, so Z is the sum of $n = 2$ exponential processes (positive variables with maximum entropy distribution). We can interpret Z as the sum of $2n = 4$ indipendent gaussian random variables with $\sigma_X^2 = \frac{1}{2\lambda}$:

$$Z \sim \Gamma_{2,\lambda}(z) = \lambda^2 z e^{-\lambda z} \tag{S14}$$

In this case the variance results:

$$\sigma_{\Gamma,2}^2 = \frac{2}{\lambda^2} \tag{S15}$$

In the following and in the paper, when we omit the pedix $n = 2$, we refer to the case $n = 2$:

$$\sigma_\Gamma^2 := \sigma_{\Gamma,2}^2 \tag{S16}$$

S3 The general case: the superstatistical combination of a gaussian distribution and a $\Gamma_{n,\lambda}$ distribution

The PDF of the total signal x is locally (in time) gaussian $N_{(0,z)}(x)$ with zero mean and the variance z varying (on longer time scales) following a $\Gamma_{n,\lambda}(z)$ distribution. The PDF $p(x)$ of the total signal can be obtained in the following way (look at integral 16 page 369 of [**1**]):

$$\begin{aligned} p(x) &= \int_0^\infty \Gamma_{n,\lambda}(z) \cdot N_{(0,z)}(x) dz \\ &= \int_0^\infty \frac{\lambda^n}{\Gamma(n)} z^{n-1} e^{-\lambda z} \cdot \frac{1}{\sqrt{2\pi}\sqrt{z}} e^{-x^2/2z} dz \\ &= \frac{\lambda^n}{\Gamma(n)\sqrt{2\pi}} \int_0^\infty z^{(n-1)-\frac{1}{2}} \cdot e^{-\lambda z - \frac{x^2/2}{z}} dz \\ &= \frac{\lambda^n}{\Gamma(n)\sqrt{2\pi}} (-1)^{n-1} \sqrt{\pi} \frac{\partial^{n-1}}{\partial \lambda^{n-1}} \left(\frac{e^{-\sqrt{2\lambda x^2}}}{\sqrt{\lambda}} \right) \\ &= \frac{(-1)^{n-1}\lambda^n}{\Gamma(n)\sqrt{2}} \cdot \frac{\partial^{n-1}}{\partial \lambda^{n-1}} \left(\frac{e^{-\sqrt{2\lambda}\cdot|x|}}{\sqrt{\lambda}} \right) \end{aligned} \tag{S17}$$

with zero mean and variance s^2:

$$\begin{aligned} s^2 &:= var(x) = E[x^2] \\ &= 2 \int_0^\infty \frac{(-1)^{n-1}\lambda^n}{\Gamma(n)\sqrt{2}} \cdot x^2 \cdot \frac{\partial^{n-1}}{\partial \lambda^{n-1}} \left(\frac{e^{-\sqrt{2\lambda}\cdot x}}{\sqrt{\lambda}} \right) dx \\ &= \frac{(-1)^{n-1}\lambda^n \sqrt{2}}{\Gamma(n)} \cdot \frac{d^{n-1}}{d\lambda^{n-1}} \left(\frac{1}{\sqrt{\lambda}} \int_0^\infty x^2 e^{-\sqrt{2\lambda}\cdot x} dx \right) \\ &= \frac{(-1)^{n-1}\lambda^n \sqrt{2}}{\Gamma(n)} \cdot \frac{d^{n-1}}{d\lambda^{n-1}} \left(\frac{1}{\sqrt{\lambda}} \cdot \frac{1}{2\lambda} \cdot \frac{1}{\sqrt{2\lambda}} \underbrace{\int_0^\infty y^2 e^{-y} dy}_{=2} \right) \\ &= \frac{(-1)^{n-1}\lambda^n}{\Gamma(n)} \cdot \frac{d^{n-1}}{d\lambda^{n-1}} \left(\frac{1}{\lambda^2} \right) \\ &= \frac{(-1)^{n-1}\lambda^n}{\Gamma(n)} \cdot \frac{(-1)^{n-1} n!}{\lambda^{n+1}} \\ &= \frac{n}{\lambda} \end{aligned} \tag{S18}$$

The normalization of $p(x)$ can be easily checked.
We note that:

$$s^2 = \frac{n}{\lambda} = \sqrt{n} \cdot \sigma_{\Gamma,n} = n \cdot \sigma_{\mathrm{exp}} \tag{S19}$$

S4 Our case

In our case:

$$\sigma_\star^2 \sim f(\sigma_\star^2) = \Gamma_{2,\lambda_\star}(\sigma_\star^2) = \lambda_\star^2 \sigma_\star^2 e^{-\lambda_\star \sigma_\star^2} \tag{S20}$$

Since our focus is on the variances (positive values), we can interpret $n = 2$ as the effective degree of freedom.

The velocity increment PDF is (see Eq. (S17) with $n = 2$):

$$\begin{aligned} p(\delta u) &= \int_0^\infty f(\sigma_{\delta u}^2) p(\delta u|\sigma_{\delta u}^2) d(\sigma_{\delta u}^2) \\ &= \int_0^\infty \lambda_{\delta u}^2 \sigma_{\delta u}^2 e^{-\lambda_{\delta u}\sigma_{\delta u}^2} \cdot \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sigma_{\delta u}^2}} e^{-\delta u^2/2\sigma_{\delta u}^2} \cdot d(\sigma_{\delta u}^2) \\ &= -\frac{\lambda_{\delta u}^2}{\sqrt{2}} \cdot \frac{\partial}{\partial \lambda_{\delta u}} \left(\frac{e^{-\sqrt{2\lambda_{\delta u}}\cdot|\delta u|}}{\sqrt{\lambda_{\delta u}}} \right) \\ &= \frac{\sqrt{2\lambda_{\delta u}} e^{-\sqrt{2\lambda_{\delta u}}|\delta u|} (\sqrt{2\lambda_{\delta u}}|\delta u| + 1)}{4} \end{aligned} \tag{S21}$$

and analogously for δv. The normalization can be easily checked. The PDF is symmetric.

The analytical velocity increment variance $s_\star^2$ is (see Eq. (S18) with $n = 2$):

$$s_{\delta u}^2 = \int_{-\infty}^{+\infty} \delta u^2 p(\delta u) d(\delta u) = \frac{2}{\lambda_{\delta u}} \tag{S22}$$

and analogously for δv.

In the case $n = 2$, Eq. (S19) reduces to:

$$\left.\begin{aligned} \sigma_{\exp}^2 &= \frac{1}{\lambda_\star^2} \\ \sigma_\Gamma^2 &= \frac{2}{\lambda_\star^2} \end{aligned}\right\} s_\star^2 = \frac{2}{\lambda_\star} = \sqrt{2} \cdot \sigma_\Gamma = 2 \cdot \sigma_{\exp}. \tag{S23}$$

S5 Maximised entropy PDFs

Given a continuous random variable X defined in the domain $\mathbb{D}$ and its PDF $p_X(x)$ such that:

$$\int_{\mathbb{D}} p_X(x)dx = 1, \tag{S24}$$

so it is possible to define the continuous Shannon entropy (called differential entropy) H as:

$$H(p_X) = -\int_{\mathbb{D}} p_X(x)\log(p_X(x))dx. \tag{S25}$$

We use the Lagrangian multipliers method in order to maximise the entropy. Consider the following $m+1$ constrains ($m \in \mathbb{N}$):

$$\int_{\mathbb{D}} f_m(x)dx = a_m \tag{S26}$$

where $f_m(x)$ is any integrable function of x in $\mathbb{D}$ and where, in particular, the case $m = 0$ is the normalization constrain: $f_0(x) = p_X(x)$ and $a_0 = 1$. So we can define the Lagrangian functional $\mathcal{L}(p_X, \vec{\lambda})$:

$$\mathcal{L}(p_X, \vec{\lambda}) = H(p_X) + \sum_{k=0}^{m} \lambda_k \left[a_m - \int_{\mathbb{D}} f_m(x)dx\right] \tag{S27}$$

where $\vec{\lambda}$ is the set of the $m+1$ Lagrangian multipliers.

S5.1 The gaussian PDF

If we consider a continuous real random variable $X \in \mathbb{R}$ with the following 3 constrains (normalization with the first $m = 1$ and $m = 2$ moments):

$$\begin{cases} \int_{\mathbb{R}} p_X(x)dx = 1 \\ \int_{\mathbb{R}} x \cdot p_X(x)dx = \mu \\ \int_{\mathbb{R}} (x-\mu)^2 \cdot p_X(x)dx = \sigma^2, \end{cases} \tag{S28}$$

so we can define the Lagrangian functional:

$$\begin{aligned} \mathcal{L}(p_X, \lambda_0, \lambda_1, \lambda_2) = &-\int_{\mathbb{R}} p_X(x)\log(p_X(x))dx+ \\ &+\lambda_0\left[1 - \int_{\mathbb{R}} p_X(x)dx\right] + \\ &+\lambda_1\left[\mu - \int_{\mathbb{R}} x \cdot p_X(x)dx\right] + \\ &+\lambda_2\left[\sigma^2 - \int_{\mathbb{R}} (x-\mu)^2 \cdot p_X(x)dx\right]. \end{aligned} \tag{S29}$$

Maximising entropy means:

$$\frac{\delta\mathcal{L}}{\delta p_X} = -\int_{\mathbb{R}} \left[\log(p_X(x)) + 1 + \lambda_0 + \lambda_1 x + \lambda_2(x-\mu)^2\right] dx = 0 \tag{S30}$$

that gives:

$$\log(p_X(x)) = -\left(1 + \lambda_0 + \lambda_1 x + \lambda_2(x-\mu)^2\right) \tag{S31}$$

$$p_X(x) = e^{-(1+\lambda_0)} \cdot e^{-\lambda_1 x - \lambda_2(x-\mu)^2}. \tag{S32}$$

The normalization and moment constrains give:

$$\begin{cases} e^{-(1+\lambda_0)} = \frac{\sqrt{\lambda_2}}{\sqrt{\pi}\exp(\lambda_1(\lambda_1-4\lambda_2\mu)/4\lambda_2)} \\ \lambda_1 = 0 \\ \lambda_2 = \frac{1}{2\sigma^2} \end{cases} \tag{S33}$$

leading to a gaussian distribution:

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{S34}$$

S5.2 The exponential PDF

Let's consider a continuous positive real random variable $X \in \mathbb{R}_+$ with the following 2 constrains (normalization with the first $m = 1$ moment):

$$\begin{cases} \int_{\mathbb{R}_+} p_X(x)dx = 1 \\ \int_{\mathbb{R}_+} x \cdot p_X(x)dx = \mu. \end{cases} \tag{S35}$$

We can reproduce the calculations as done before where:

$$\begin{aligned} \mathcal{L}(p_X, \lambda_0, \lambda_1, \lambda_2) = &- \int_{\mathbb{R}_+} p_X(x)\log(p_X(x))dx + \\ &+ \lambda_0 \left[1 - \int_{\mathbb{R}_+} p_X(x)dx\right] + \\ &+ \lambda_1 \left[\mu - \int_{\mathbb{R}_+} x \cdot p_X(x)dx\right]. \end{aligned} \tag{S36}$$

Maximising entropy gives:

$$p_X(x) = e^{-(1+\lambda_0)} \cdot e^{-\lambda_1 x}. \tag{S37}$$

The constrains give finally to the following exponential distribution:

$$p_X(x) = \frac{1}{\mu} e^{-x/\mu}. \tag{S38}$$

where $x \in \mathbb{R}_+$.

References

[1] Gradshteyn, Izrail Solomonovich, and Iosif Moiseevich Ryzhik. Table of integrals, series, and products. Academic press, 2014.