

Supplementary Material 1 : Ornstein-Uhlenbeck processes example in detail Correcting for Model Changes in Statistical Post-Processing

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In this supplementary note, we detail the computations that lead to the main Section 2 results.

I. ORNSTEIN-UHLENBECK PROCESSES AS A FORECAST MODEL CHANGE

We consider two Ornstein-Uhlenbeck processes supposedly representing the reality $x(\tau)$ and its forecast model $y(\tau)$ obeying the following equations:

$$\dot{x} = -\lambda_x x + K_x + Q_x \xi_x(\tau) \quad (\text{S1})$$

$$\dot{y} = -\lambda_y y + K_y + Q_y \xi_y(\tau) \quad (\text{S2})$$

where ξ_x and ξ_y are white noise processes such that

$$\begin{aligned} \langle \xi_x(\tau) \rangle &= \langle \xi_y(\tau) \rangle = 0 \\ \langle \xi_x(\tau) \xi_x(\tau') \rangle &= \delta(\tau - \tau') \\ \langle \xi_y(\tau) \xi_y(\tau') \rangle &= \delta(\tau - \tau') \\ \langle \xi_x(\tau) \xi_y(\tau') \rangle &= 0 \end{aligned}$$

These dynamics are thus uncorrelated Ornstein-Uhlenbeck processes with noise amplitudes Q_x and Q_y .

A change of the model $y(\tau)$ toward a model $\hat{y}(\tau)$ is then considered, possibly improving or degrading the forecast performances:

$$\dot{\hat{y}} = -\lambda_y \hat{y} + K_y + Q_y \xi_y(\tau) + \Psi_y(\tau) \quad (\text{S3})$$

where

$$\Psi_y(\tau) = -\kappa (\delta K + \delta Q \xi_y(\tau)) \quad (\text{S4})$$

with $\delta K = K_y - K_x$ and $\delta Q = Q_y - Q_x$. It can represent for example a better parameterization of subgrid-scale processes. Note that if $\kappa = 1$, the correction is perfect.

A post-processing scheme constructed before the model change, and then applied after it, would give a deteriorated correction. Therefore, a post-processing scheme is in general recomputed after such a change. The object of the present note is to show that response theory can correct the prior post-processing scheme and gives thus an approximation to the one computed after the change. Moreover, in the present case involving Ornstein-Uhlenbeck processes, this correction is exact. First, this post-processing scheme is described.

A. The post-processing of x

We consider an EVMOS [4] post-processing of the variable x :

$$x_C(\tau) = \alpha(\tau) + \beta(\tau) \cdot y(\tau) \quad (\text{S5})$$

The coefficient α and β are given by the equations:

$$\alpha(\tau) = \langle x(\tau) \rangle - \beta(\tau) \cdot \langle y(\tau) \rangle \quad (\text{S6})$$

$$\beta(\tau) = \sqrt{\frac{\sigma_x^2(\tau)}{\sigma_y^2(\tau)}} \quad (\text{S7})$$

with

$$\sigma_x^2(\tau) = \left\langle (x(\tau) - \langle x(\tau) \rangle)^2 \right\rangle \quad (\text{S8})$$

$$\sigma_y^2(\tau) = \left\langle (y(\tau) - \langle y(\tau) \rangle)^2 \right\rangle \quad (\text{S9})$$

where the average is taken over initial conditions of the x process. Indeed, both the forecast models y and \hat{y} are initialized with the initial conditions of x . The corrected quantity x_C depends thus on the lead time τ .

B. Post-processing before and after the model change

Before the model change, $\Psi_y(\tau) = 0$, and a direct computation gives:

$$\langle x(\tau) \rangle = \langle x(0) \rangle e^{-\lambda_x \tau} + \frac{K_x}{\lambda_x} (1 - e^{-\lambda_x \tau}) \quad (\text{S10})$$

$$\langle y(\tau) \rangle = \langle x(0) \rangle e^{-\lambda_y \tau} + \frac{K_y}{\lambda_y} (1 - e^{-\lambda_y \tau}) \quad (\text{S11})$$

and

$$\sigma_x^2(\tau) = \sigma_x^2(0) e^{-2\lambda_x \tau} + \frac{Q_x^2}{2\lambda_x} (1 - e^{-2\lambda_x \tau}) \quad (\text{S12})$$

$$\sigma_y^2(\tau) = \sigma_x^2(0) e^{-2\lambda_y \tau} + \frac{Q_y^2}{2\lambda_y} (1 - e^{-2\lambda_y \tau}) \quad (\text{S13})$$

since the model is initialized with the same initial conditions as the reality:

$$\langle y(0) \rangle = \langle x(0) \rangle \quad , \quad \sigma_y^2(0) = \sigma_x^2(0) \quad . \quad (\text{S14})$$

These equations allow us to compute the post-processing coefficients at every lead time τ :

$$\alpha(\tau) = \langle x(\tau) \rangle - \beta(\tau) \langle y(\tau) \rangle \quad (\text{S15})$$

$$\beta(\tau) = \sqrt{\frac{\sigma_x^2(\tau)}{\sigma_y^2(\tau)}} \quad (\text{S16})$$

It is easy to extend these results when model change Ψ is incorporated:

$$\langle \hat{y}(\tau) \rangle = \langle x(0) \rangle e^{-\lambda_y \tau} + \frac{K_y - \kappa \delta K}{\lambda_y} (1 - e^{-\lambda_y \tau}) \quad (\text{S17})$$

and

$$\sigma_{\hat{y}}^2(\tau) = \sigma_x^2(0) e^{-2\lambda_y \tau} + \frac{(Q_y - \kappa \delta Q)^2}{2\lambda_y} (1 - e^{-2\lambda_y \tau}) \quad (\text{S18})$$

with

$$\hat{\alpha}(\tau) = \langle x(\tau) \rangle - \hat{\beta}(\tau) \langle \hat{y}(\tau) \rangle \quad (\text{S19})$$

$$\hat{\beta}(\tau) = \sqrt{\frac{\sigma_x^2(\tau)}{\sigma_{\hat{y}}^2(\tau)}} \quad (\text{S20})$$

The variation of the moments of the dynamics are then

$$\langle \hat{y}(\tau) \rangle = \langle y(\tau) \rangle - \frac{\kappa \delta K}{\lambda_y} (1 - e^{-\lambda_y \tau}) \quad (\text{S21})$$

$$\sigma_{\hat{y}}^2(\tau) = \sigma_y^2(\tau) + \frac{1}{2\lambda_y} (\kappa^2 \delta Q^2 - 2\kappa Q_y \delta Q) (1 - e^{-2\lambda_y \tau}) \quad (\text{S22})$$

and it implies the following variation of the bias α :

$$\delta\alpha(\tau) = \hat{\alpha}(\tau) - \alpha(\tau) = \beta(\tau) \langle y(\tau) \rangle - \hat{\beta}(\tau) \langle \hat{y}(\tau) \rangle \quad (\text{S23})$$

The ratio between the biases β is given by

$$\frac{\hat{\beta}(\tau)}{\beta(\tau)} = \sqrt{\frac{\sigma_y^2(\tau)}{\sigma_{\hat{y}}^2(\tau)}} \quad (\text{S24})$$

For $\tau \gg \max(1/\lambda_x, 1/\lambda_y)$, we note that this ratio tend to

$$\frac{\hat{\beta}(\tau)}{\beta(\tau)} = \frac{1}{1 - \kappa \delta Q / Q_y} \quad (\text{S25})$$

and the asymptotic limit for the variation of α is given by

$$\delta\alpha(\tau) = -\beta(\tau) \frac{K_y}{\lambda_y} \left[\frac{1 - \kappa \delta K / K_y}{1 - \kappa \delta Q / Q_y} - 1 \right] \quad (\text{S26})$$

C. (Non-Stationary) Response theory

The post-processing problem is typically a non-stationary initial value problem, since the initial conditions of the model equations (S2) and (S3) are typically chosen close to the reality (S1). As a consequence, the model actual averages relax toward the stationary response in the long-time limit, but the stationary response theory [3, 5] cannot provide us the short-time relaxation behaviors. We thus consider the short-time evolution of the averages. Therefore, the Ruelle time-dependent response theory should be used [2].

After the model change, the model is ruled by Eq. (S3):

$$\dot{\hat{y}} = -\lambda_y \hat{y} + K_y + Q_y \xi_y(\tau) + \Psi_y(\tau) \quad (\text{S27})$$

with

$$\Psi_y(\tau) = -\kappa \left(\delta K + \delta Q \xi_y(\tau) \right) \quad (\text{S28})$$

Given an observable A , its average after the model change can then be related to it's average before by

$$\langle A(\tau) \rangle_{\hat{y}} = \langle A(\tau) \rangle_y + \delta \langle A(\tau) \rangle_y + \delta^2 \langle A(\tau) \rangle_y + \dots \quad (\text{S29})$$

If the perturbations (S28) is small, then the response theory states that the first order is given by:

$$\delta \langle A(\tau) \rangle_y = \int_0^\tau d\tau' \int dy \rho_{y,0}(y) \left\langle \Psi_y(\tau') \cdot \nabla_{f^{\tau'}(y)} A(f^\tau(y)) \right\rangle \quad (\text{S30})$$

where $\rho_{y,0}$ is the initial distribution with which the model is initialized. As indicated by Eq. (S14), in the post-processing framework, it is typically taken as the initial distribution of the reality $\rho_{x,0}$. We have also $\langle \cdot \rangle$ which denotes the average over the realization of the stochastic process [1], and also the mapping f^τ is the stochastic “flow” of the unperturbed system (S2):

$$f^\tau(y) = y e^{-\lambda_y \tau} + \int_0^\tau d\tau' e^{-\lambda_y(\tau-\tau')} \left[Q_y \xi_y(\tau') + K_y \right] \quad (\text{S31})$$

1. First order

a. First moment: First the response for the shift of the mean of $y(\tau)$ due to the perturbation is evaluated:

$$\delta \langle y(\tau) \rangle = -\kappa \int_0^\tau d\tau' \int dy \rho_{y,0}(y) \left\langle \left[\delta K + \delta Q \xi_y(\tau') \right] \cdot \nabla_{f^{\tau'}(y)} f^\tau(y) \right\rangle \quad (\text{S32})$$

With

$$\nabla_{f^{\tau'}(y)} f^\tau(y) = \nabla_{f^{\tau'}(y)} f^{\tau-\tau'} \circ f^{\tau'}(y) = e^{-\lambda_y(\tau-\tau')} \quad (\text{S33})$$

$$\int dy \rho_{y,0}(y) = 1 \quad , \quad (\text{S34})$$

and $\langle \xi_y(\tau') \rangle = 0$ for all τ' , we get:

$$\delta \langle y(\tau) \rangle = -\kappa \int_0^\tau d\tau' \delta K e^{-\lambda_y(\tau-\tau')} \quad (\text{S35})$$

and thus:

$$\langle \hat{y}(\tau) \rangle = \langle y(\tau) \rangle - \frac{\kappa}{\lambda_y} \delta K (1 - e^{-\lambda_y \tau}) \quad (\text{S36})$$

and the exact relation (S21) is recovered.

b. Second moment: We now turn to the computation of

$$\delta \langle y(\tau)^2 \rangle = -\kappa \int_0^\tau d\tau' \int dy \rho_{y,0}(y) \left\langle \left[\delta K + \delta Q \xi_y(\tau') \right] \cdot \nabla_{f^{\tau'}(y)} (f^\tau(y))^2 \right\rangle \quad (\text{S37})$$

$$= -\kappa \int_0^\tau d\tau' \int dy \rho_{y,0}(y) \left\langle \left[\delta K + \delta Q \xi_y(\tau') \right] \cdot \nabla_{f^{\tau'}(y)} f^\tau(y) \cdot \nabla_{f^\tau(y)} (f^\tau(y))^2 \right\rangle \quad (\text{S38})$$

$$= -2\kappa \int_0^\tau d\tau' \int dy \rho_{y,0}(y) \left\langle \left[\delta K + \delta Q \xi_y(\tau') \right] \cdot f^\tau(y) \cdot \nabla_{f^{\tau'}(y)} f^\tau(y) \right\rangle \quad (\text{S39})$$

which gives, using the formula (S31):

$$\begin{aligned} \delta \langle y(\tau)^2 \rangle &= -2\kappa \delta K \int_0^\tau d\tau' \int dy \rho_{y,0}(y) \left[y e^{-\lambda_y \tau} e^{-\lambda_y(\tau-\tau')} + \frac{K_y}{\lambda_y} e^{-\lambda_y(\tau-\tau')} (1 - e^{-\lambda_y \tau}) \right] \\ &\quad - 2\kappa Q_y \delta Q \left\langle \int_0^\tau d\tau' e^{-\lambda_y(\tau-\tau')} \xi_y(\tau') \int_0^\tau d\tau'' e^{-\lambda_y(\tau-\tau'')} \xi_y(\tau'') \right\rangle \end{aligned} \quad (\text{S40})$$

$$\begin{aligned} &= -2\kappa \delta K \left[\frac{1}{\lambda_y} \langle y(0) \rangle e^{-\lambda_y \tau} (1 - e^{-\lambda_y \tau}) + \frac{K_y}{\lambda_y^2} (1 - e^{-\lambda_y \tau})^2 \right] \\ &\quad - 2\kappa \delta Q Q_y \int_0^\tau d\tau' e^{-2\lambda_y(\tau-\tau')} \end{aligned} \quad (\text{S41})$$

where we have used $\langle \xi_y(\tau) \xi_y(\tau') \rangle = \delta(\tau - \tau')$. Therefore, we finally get:

$$\delta_t \langle y(\tau)^2 \rangle = -2\kappa \frac{\delta K}{\lambda_y} \langle y(\tau) \rangle (1 - e^{-\lambda_y \tau}) - \frac{\kappa}{\lambda_y} \delta Q Q_y (1 - e^{-2\lambda_y \tau}) \quad (\text{S42})$$

which gives

$$\begin{aligned} \sigma_{y,\Psi}^2(\tau) &\approx \langle y(\tau)^2 \rangle + \delta \langle y(\tau)^2 \rangle - (\langle y(\tau) \rangle + \delta \langle y(\tau) \rangle)^2 \\ &= \sigma_y^2(\tau) + \delta \langle y(\tau)^2 \rangle - 2\langle y(\tau) \rangle \delta \langle y(\tau) \rangle - (\delta \langle y(\tau) \rangle)^2 \\ &= \sigma_y^2(\tau) - 2\kappa \frac{\delta K}{\lambda_y} \langle y(\tau) \rangle (1 - e^{-\lambda_y \tau}) - \frac{\kappa}{\lambda_y} \delta Q Q_y (1 - e^{-2\lambda_y \tau}) \\ &\quad + 2\langle y(\tau) \rangle_0 \frac{\kappa}{\lambda_y} \delta K (1 - e^{-\lambda_y \tau}) - \frac{\kappa^2}{\lambda_y^2} \delta K^2 (1 - e^{-\lambda_y \tau})^2 \end{aligned} \quad (\text{S43})$$

$$= \sigma_y^2(\tau) - \frac{\kappa}{\lambda_y} \delta Q Q_y (1 - e^{-2\lambda_y \tau}) - \frac{\kappa^2}{\lambda_y^2} \delta K^2 (1 - e^{-\lambda_y \tau})^2 \quad (\text{S44})$$

which does not match Eq. (S22) and thus the second order terms should be computed.

2. Second order

The second order response for an observable A is given by:

$$\delta^2 \langle A(\tau) \rangle = \int_0^\tau d\tau' \int_{\tau'}^\tau d\tau'' \int dy \rho_{y,0}(y) \left\langle \Psi_x(\tau') \cdot \nabla_{f^{\tau'}(y)} \Psi_y(\tau'') \cdot \nabla_{f^{\tau''}(y)} A(f^\tau(y)) \right\rangle \quad (\text{S45})$$

Since here the perturbations do not depend on the state variable y

$$\delta^2 \langle A(\tau) \rangle = \int_0^\tau d\tau' \int_{\tau'}^\tau d\tau'' \int dy \rho_{y,0}(y) \left\langle \Psi_x(\tau') \Psi_y(\tau'') \nabla_{f^{\tau'}(y)} \nabla_{f^{\tau''}(y)} A(f^\tau(y)) \right\rangle \quad (\text{S46})$$

and applied to the first moment of the $y(\tau)$ dynamics, it leads to

$$\delta^2 \langle y(\tau) \rangle = 0 \quad (\text{S47})$$

since thanks to the Eq. (S33)

$$\nabla_{f^{\tau'}(y)} \nabla_{f^{\tau''}(y)} f^\tau(y) = 0 \quad . \quad (\text{S48})$$

The computation of the derivatives for the second moments leads to

$$\begin{aligned} \nabla_{f^{\tau'}(y)} \nabla_{f^{\tau''}(y)} (f^\tau(y))^2 &= 2 \nabla_{f^{\tau'}(y)} f^\tau(y) \nabla_{f^{\tau''}(y)} f^\tau(y) \\ &= 2 e^{-\lambda_y(\tau-\tau')} e^{-\lambda_y(\tau-\tau'')} , \end{aligned} \quad (\text{S49})$$

and thus the correction to the second moment of $y(\tau)$ can be obtained by computing

$$\delta^2 \langle y(\tau)^2 \rangle = 2 \kappa^2 \int_0^\tau d\tau' \int_{\tau'}^\tau d\tau'' \left\langle (\delta K + \delta Q \xi_y(\tau')) e^{-\lambda_y(\tau-\tau')} (\delta K + \delta Q \xi_y(\tau'')) e^{-\lambda_y(\tau-\tau'')} \right\rangle \quad (\text{S50})$$

which thanks to the averaging properties of ξ_y gives

$$\begin{aligned} \delta^2 \langle y(\tau)^2 \rangle &= \kappa^2 \delta K^2 \left[\int_0^\tau d\tau' e^{-\lambda_y(\tau-\tau')} \right]^2 + 2 \kappa^2 \delta Q^2 \int_0^\tau d\tau' \int_{\tau'}^\tau d\tau'' \left\langle \xi_y(\tau') e^{-\lambda_y(\tau-\tau')} \xi_y(\tau'') e^{-\lambda_y(\tau-\tau'')} \right\rangle \\ &= \kappa^2 \delta K^2 \left[\int_0^\tau d\tau' e^{-\lambda_y(\tau-\tau')} \right]^2 + \kappa^2 \delta Q^2 \int_0^\tau d\tau' e^{-2\lambda_y(\tau-\tau')} \end{aligned} \quad (\text{S51})$$

$$= \frac{\kappa^2 \delta K^2}{\lambda_y^2} (1 - e^{-\lambda_y \tau})^2 + \frac{\kappa^2 \delta Q^2}{2\lambda_y} (1 - e^{-2\lambda_y \tau}) \quad (\text{S52})$$

3. Final result

By aggregating the two orders, we get:

$$\langle y(\tau) \rangle_{\hat{y}} \approx \langle y(\tau) \rangle + \delta \langle y(\tau) \rangle = \langle y(\tau) \rangle - \frac{\kappa}{\lambda_y} \delta K (1 - e^{-\lambda_y \tau}) \quad (\text{S53})$$

$$\begin{aligned} \sigma_y^2(\tau) &\approx \sigma_y^2(\tau) - \frac{\kappa}{\lambda_y} \delta Q Q_y (1 - e^{-2\lambda_y \tau}) - \frac{\kappa^2}{\lambda_y^2} \delta K^2 (1 - e^{-\lambda_y \tau})^2 \\ &\quad + \frac{\kappa^2 \delta K^2}{\lambda_y^2} (1 - e^{-\lambda_y \tau})^2 + \frac{\kappa^2 \delta Q^2}{2\lambda_y} (1 - e^{-2\lambda_y \tau}) \end{aligned} \quad (\text{S54})$$

$$= \sigma_y^2(\tau) - \frac{\kappa}{\lambda_y} \delta Q Q_y (1 - e^{-2\lambda_y \tau}) + \frac{\kappa^2 \delta Q^2}{2\lambda_y} (1 - e^{-2\lambda_y \tau}) \quad (\text{S55})$$

These approximations are in fact exact for any time τ , even for the transient short-time $\tau < 1/\lambda_y$ and gives back the post-processing results of the section IB of the present note.

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