Supplement of Nonlin. Processes Geophys., 22, 749–764, 2015
http://www.nonlin-processes-geophys.net/22/749/2015/
doi:10.5194/npg-22-749-2015-supplement

Nonlinear Processes in Geophysics
Open Access

Supplement of

Nonlinear feedback in a six-dimensional Lorenz model: impact of an additional heating term

B.-W. Shen

Correspondence to: B.-W. Shen (bshen@mail.sdsu.edu, bowen.shen@gmail.com)

1 Introduction

This report, which documents the mathematical analysis on the extensions of the nonlinear feedback loop in the 5DLM and 6DLM as well as higher-dimensional Lorenz models, is provided as supplementary materials to the manuscript entitled " Nonlinear feedback in the six-dimensional Lorenz model: impact of an additional heating term. by Shen (2015)." In the following, we briefly introduce the three-dimensional (3D) Lorenz model (3DLM, Lorenz, 1963) and its Fourier modes, and identify the nonlinear feedback loop of the 3DLM by analyzing the nonlinear Jacobian term $J(\psi,\theta)$. We then discuss how the analysis of $J(\psi,\theta)$ can help select new modes to extend the nonlinear feedback loop in higher-dimensional LMs. Our approach, using incremental changes in the number of Fourier modes, can help trace their individual and/or collective impact on the solution stability as well as the extension of the nonlinear feedback loop. To avoid repeated definitions, we use the same symbols as those in Shen (2014) and Shen (2015).

2 The Nonlinear Feedback Loop and its Extensions in the Lorenz Models

To derive the 3DLM, we use the following three Fourier modes:

$$M_1 = \sqrt{2}sin(lx)sin(mz), \quad M_2 = \sqrt{2}cos(lx)sin(mz), \quad M_3 = sin(2mz), \tag{1}$$

here l and m are defined as $\pi a/H$ and π/H, representing the horizontal and vertical wavenumbers, respectively. And, a is a ratio of the vertical scale of the convection cell to its horizontal scale, i.e., $a = l/m$. H is the domain height, and $2H/a$ represents the domain width. With the three modes in Eq. (1), the streamfunction ψ and the temperature perturbation θ can be represented as:

$$\psi = C_1 \left(X M_1 \right), \tag{2}$$

$$\theta = C_2 \left(Y M_2 - Z M_3 \right), \tag{3}$$

here, C_1 and C_2 are constants (Shen 2014). (X, Y, Z) represent the amplitudes of (M_1, M_2, M_3), respectively. The modes in the 3DLM include one horizontal wavenumber (i.e., l) and two vertical wavenumbers (i.e., m and $2m$). After the derivations, the 3DLM is written as:

$$\frac{dX}{d\tau} = -\sigma X + \sigma Y, \tag{4}$$

$$\frac{dY}{d\tau} = -XZ + rX - Y, \tag{5}$$

$$\frac{dZ}{d\tau} = XY - bZ. \tag{6}$$

In the following, we will show that the two nonlinear terms, $-XZ$ and XY, appear in association with the nonlinear advection of temperature ($J(\psi,\theta)$), and illustrate that these two terms form a nonlinear feedback loop in the 3DLM. Then, we discuss how new modes are selected to extend the nonlinear feedback loop in the higher-dimensional LMs. To facilitate discussions below, the additional modes that have been used in the higher-dimensional LMs (Shen, 2014, 2015; Yoo and Shen, 2015) are defined as follows:

$$M_4 = \sqrt{2}sin(lx)sin(3mz), \quad M_7 = \sqrt{2}sin(lx)sin(5mz), \tag{7}$$

$$M_5 = \sqrt{2}cos(lx)sin(3mz), \quad M_6 = sin(4mz), \tag{8}$$

$$M_8 = \sqrt{2}cos(lx)sin(5mz), \quad M_9 = sin(6mz). \tag{9}$$

2.1 The nonlinear feedback loop in the 3DLM

In this section, we first discuss the characteristics of nonlinearity associated with the Jacobian term represented by a finite number of Fourier modes. With Eqs. (2-3), we have

$$J(\psi,\theta) = C_1C_2\left(XYJ(M_1,M_2) - XZJ(M_1,M_3)\right). \tag{10}$$

$J(\psi,\theta)$ is now expressed in terms of the summation of two nonlinear terms, $J(M_1,M_2)$ and $J(M_1,M_3)$ whose coefficients are XY and $-XZ$, respectively. Through straightforward derivations, we obtain

$$J(M_1,M_2) \approx 2mlsin(mz)cos(mz) = mlM_3, \tag{11}$$

and

$$J(M_1,M_3) \approx \sqrt{2}mlcos(lx)\left(sin(3mz) + sin(-mz)\right). \tag{12}$$

The vertical wave number of 3m is not used in the 3DLM, so the sin(3mz) is neglected. Thus, Eq. (12) becomes

$$J(M_1,M_3) \approx \sqrt{2}mlcos(lx)sin(-mz) = -mlM_2. \tag{13}$$

From Eqs. (11) and (13), a loop can be identified as follows. As Eq. (13) gives $M_2 \approx -J(M_1,M_3)/(ml)$, we can plug the M_2 into Eq. (11) to have

$$J(M_1,J(M_1,M_3)) = -(ml)^2M_3.$$

Similarly, we can derive

$$J(M_1,J(M_1,M_2) = -(ml)^2M_2.$$

Therefore, with the inclusion of the M_3, a loop with $M_2 \to M_3 \to M_2$ is introduced in the 3DLM. More importantly, downscale and upscale transfer processes can be identified using Eqs. (11) and (13). M_2 and M_3 have vertical wave numbers of m and $2m$, respectively. Eq. (11) suggests that the nonlinear interaction between M_1 and M_2 leads to a downscale transfer (to the M_3 mode), while Eq. (13) suggests that the nonlinear interaction between M_1 and M_3 leads to a upscale transfer (to the M_2). However, as $sin(3mz)$ is not included, the approximation using Eq. (13) neglects a downscale transfer (from the M_5 mode with sin(2mz) to the mode with sin(3mz), which will discussed in details in section 2.2.

Next, we illustrate the role of the nonlinear feedback loop in the "nonlinear" 3DLM. Without the inclusion of the nonlinear terms $-XZ$ and XY, Eqs. (4-6) of the 3DLM reduce to

$$\frac{dX}{d\tau} = -\sigma X + \sigma Y, \tag{14}$$

$$\frac{dY}{d\tau} = rX - Y, \tag{15}$$

$$\frac{dZ}{d\tau} = -bZ. \tag{16}$$

Equations (14-15), which are decoupled with Eq. (16), form a forced dissipative system with only linear terms. The system has only a trivial critical point ($X = Y = 0$) and produces unstable normal-mode solutions (i.e., exponentially growing with time) as $r > 1$. Therefore, our analysis indicates that the inclusion of M_3 introduces Eq. (16) and the enabled feedback loop (i.e., Eqs. 11 and 13) couples Eq. (16) with Eqs. (14-15) to form the (nonlinear) 3DLM (Eqs. 4-6) which enables the appearance of convection solutions. From a perspective of total energy conservation, the inclusion of the M_3 mode can help conserve the total energy in the dissipationless limit, which is discussed in Appendix A of Shen (2014). Mathematically, the feedback loop with the nonlinear terms in Eqs. 5 and 6 (i.e., $-XZ$ and XY) leads to the change in the behavior of the system's solutions; the (nonlinear) 3DLM system produces non-trivial critical points, which may be stable (e.g., for $1 < r < 24.74$) or "unstable" (chaotic) (e.g., for $r > 25$). In the next sections, we discuss how the nonlinear feedback loop in the 3DLM can be extended through proper selections of new modes.

2.2 An extension of the nonlinear feedback loop in the 5DLM

The increased degree of nonlinearity in the 5DLM, which has been discussed in Fig. 1 of Shen (2014), is briefly summarized below. In the derivation of the 3DLM, the mode with $sin(3mz)$ in Eq. (12) was neglected. Therefore, it is natural to include $\sqrt{2}cos(lx)sin(3mz)$

as the M_5 mode (Eq. 8). Thus, Eq. (12) can be written as

$$J(M_1, M_3) \approx \sqrt{2}mlcos(lx)\left(sin(3mz) + sin(-mz)\right) = ml(M_5 - M_2). \quad (17)$$

From a perspective of nonlinear interaction, the above mode-mode interaction in Eq. (17) indicates the route of the downscale and upscale energy transfer to the M_5 and M_2 modes, respectively. The M_5 mode can further interact with the M_1 mode to provide feedback to the M_3 mode through

$$J(M_1, M_5) \approx ml\left(2sim(4mz) - sin(2mz)\right) = 2mlM_6 - mlM_3. \quad (18)$$

The processes in Eqs. (17-18) add a new loop (e.g., $M_3 \to M_5 \to M_3$) which is connected to the (existing) feedback loop (e.g., $M_2 \to M_3 \to M_2$) of the 3DLM. Therefore, the feedback loop in the 3DLM is extended with the inclusion of the M_5 mode in the 5DLM. The original feedback loop and new feedback loop may be viewed as the main trunk and branch, respectively. *Note that the term "extension of the nonlinear feedback loop" indicates the linkage between the existing loop and the new loop.* It was reported that inclusion of new modes could produce additional equations that are not coupled with the 3DLM, leading to a generalized LM with the same stability as the 3DLM (e.g., Eqs. 11-16 of Roy and Musielak (2007a)). In this case, the original nonlinear feedback loop (of the 3DLM) is not extended with the new modes.

With the inclusion of M_5, $J(M_1, M_5)$ provides not only upscaling feedback to the M_3 mode but also a downscale energy transfer to a smaller-scale wave mode that, in turn, requires the inclusion of the $sin(4mz)$ mode (i.e., M_6 mode) (Eq. 18). As discussed in Appendix A of Shen (2014), the M_6 mode is required to conserve the total energy in the dissipationless limit. The feedback loop is further extended to $M_5 \to M_6 \to M_5$ through $J(M_1, M_5)$ and $J(M_1, M_6)$, as shown in Table 2 of Shen (2014) and discussed in section 3.1 of Shen (2015).

In summary, the two modes (M_5 and M_6) with higher vertical wavenumbers are added to improve the presentation of vertical temperature, and, therefore, the accuracy of the vertical advection of temperature, as shown:

$$\theta = C_2\left(YM_2 - ZM_3 + Y_1M_5 - Z_1M_6\right), \quad (19)$$

$$J(\psi, \theta) = C_1C_2\left(XYJ(M_1, M_2) - XZJ(M_1, M_3) + XY_1J(M_1, M_5) - XZ_1J(M_1, M_6)\right). \quad (20)$$

While the inclusion of M_3 forms a feedback loop in the 3DLM, the inclusion of M_5 and M_6 in the 5DLM extends the original feedback loop.

2.3 An extended nonlinear feedback loop in the 6DLM

As discussed in the previous sections, the inclusion of M_5 and M_6 modes is not only to improve the representations of the temperature perturbation and the nonlinear advection of temperature, but also to to extend the original nonlinear feedback loop. In this section, we discuss the selection of M_4 that is in association with the M_5 mode. The appearance of $\partial M_5/\partial x$ associated with the linear term $\partial\theta/\partial x$ of Eq. (1) of Shen (2014,2015) requires the inclusion of an M_4 mode and the $\partial M_4/\partial x$ associated with $\triangle T \partial\psi/\partial x$ of Eq. (2) of Shen (2014,2015) provides feedback to the M_5 mode (in Table 1 of Shen, 2014). The M_4 mode shares the same horizontal and vertical wave numbers as the M_5 but has a different phase (i.e., $sin(lx)$ vs. $cos(lx)$ in Eqs. 7-8 or in Eq. 4 of Shen 2015). Alternatively, via the $\partial\theta/\partial x$ and $\triangle T \partial\psi/\partial x$, the M_4 and M_5 modes are linked, as discussed in section 3.1 in the submitted manuscript (Shen 2015).

When M_4 is included, it improves the representation of the streamfunction and thus the advection of temperature, as shown:

$$\psi = C_1 \Bigg(XM_1 + X_1 M_4 \Bigg), \tag{21}$$

$$J(\psi,\theta) = C_1 C_2 \Bigg(J(XM_1 + X_1 M_4,\ YM_2 + Y_1 M_5 - ZM_3 - Z_1 M_6) \Bigg), \tag{22}$$

here X_1 represents the amplitude of the mode M_4. Now, the Jacobian term includes $J(XM_1, YM_2 + Y_1 M_5 - ZM_3 - Z_1 M_6)$ and $J(X_1 M_4, YM_2 + Y_1 M_5 - ZM_3 - Z_1 M_6)$. The former was first discussed in the 5DLM by Shen (2014), while the latter is discussed using the 6DLM in this study. While the M_4 mode introduces linear forcing term (e.g., rX_1), it also extends the nonlinear feedback loop with $J(X_1M_4, YM_2)$, $J(X_1M_4, Y_1M_5)$, $J(X_1M_4, ZM_3)$, and $J(X_1M_4, Z_1M_6)$. The outcome of each of these Jacobian terms can be found in the Table 2 of Shen (2014), and the impact of M_4 is discussed in Shen (2015).

2.4 Further extensions of the nonlinear feedback loop in Higher-order LMs

To examine the role of the nonlinear feedback loop in the solution stability of higher-order LMs, we have derived the following higher-dimensional Lorenz models, including 7D, 8D and 9D LMs. These models give a larger critical value of the normalized Rayleigh parameter for the onset of chaos, as compared to the 3D, 5D and 6D Lorenz models. A manuscript is being prepared for publication (Yoo and Shen, 2015). Here, a brief description for the higher-order LMs is given as follows:

1. 7DLM includes all modes in the 5DLM and the M_8 and M_9 modes (Eq. 9) that can improve the representation of θ and $J(\psi,\theta)$ and to extend the nonlinear feedback loop to provide negative nonlinear feedback;

2. 8DLM contains all modes in the 7DLM and the M_4 mode (Eq. 7) that can improve the representation of ψ and $J(\psi,\theta)$;

3. 9DLM includes all modes in the 8DLM and an additional mode M_7 (Eq. 7) to improve the representation of ψ and $J(\psi,\theta)$.

Note that M_8 with $sin(5mz)$ is selected based on the analysis of $J(M_1, M_6)$ as shown in the Table 2 of Shen (2014). M_9 is added to enable the downscale transfer from $J(M_1, M_8)$. Similar to the inclusion of M_4, M_7 is introduced to have a different phase to that of M_8.

References

Lorenz, E., 1963: Deterministic nonperiodic flow. *J. Atmos. Sci.*, 20, 130-141.

Roy, D. and Z.E. Musielak, 2007a: Generalized Lorenz models and their routes to chaos. I. energy-conserving vertical mode truncations. *Chaos, solitons and Fractals*, **32**, 1038-1052.

Shen, B.-W., 2014: Nonlinear Feedback in a Five-dimensional Lorenz Model. *J. of Atmos. Sci.*, **71**, 1701–1723. doi: http://dx.doi.org/10.1175/JAS-D-13-0223.1

Shen, B.-W., 2015: Nonlinear Feedback in a Six-dimensional Lorenz Model: Impact of an Additional Heating Term. Submitted to NPGD.

Yoo, E. and B.-W. Shen, 2015: On the extension of the nonlinear feedback loop in 7D, 8D and 9D Lorenz models. (In preparation).