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Remarks on rotating shallow-water magnetohydrodynamics

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Abstract. We show how the rotating shallow-water MHD model, which was proposed in the solar tachocline context, may be systematically derived by vertical averaging of the full MHD equations for the rotating magneto fluid under the influence of gravity. The procedure highlights the main approximations and the domain of validity of the model, and allows for multi-layer generalizations and, hence, inclusion of the baroclinic effects. A quasi-geostrophic version of the model, both in barotropic and in baroclinic cases, is derived in the limit of strong rotation. The basic properties of the model(s) are sketched, including the stabilizing role of magnetic fields in the baroclinic version.

1 Introduction

The (rotating) shallow-water magnetohydrodynamics model was introduced on heuristic grounds in Gilman (2000) in the context of the solar tachocline. (We will call it mRSW in what follows; with respect to the originally used acronym sMHD, this one reflects better the nature of the model; see below.) Its applications were further discussed in Dikpati and Gilman (2001a) and Dikpati and Gilman (2001b). The spectrum of linear waves and nonlinear stationary wave solutions were established in Shecter et al. (2001), and Hamiltonian structure and hyperbolicity properties were investigated in Dellar (2002). The primary purpose of the present paper is systematic derivation of the mRSW model from the full MHD equations by vertical averaging, which will (1) clarify the basic hypothesis underlying the model, and (2) immediately give multi-layer generalizations. These generalizations allow for incorporation of the *baroclinic effects* in the model. Another purpose is to establish the quasi-geostrophic (QG) approximation of the (multi-layer) mRSW, arising in the limit of strong rotation, as is traditionally done in geophysical applications of RSW (cf., e.g., Pedlosky, 1982; Zeitlin, 2007). Shallow-water and QG approaches being standard working tools in the atmosphere–ocean community, the present work might help in establishing a common language with the astrophysical community.

2 Derivation of the mRSW equations

2.1 Vertical averaging of the MHD equations

Our starting point is the system of three-dimensional compressible MHD equations on the rotating plane in the presence of gravity:

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} - \mathbf{b}\cdot\nabla\mathbf{b} + g\hat{\mathbf{z}} + f\hat{\mathbf{z}}\wedge\mathbf{v} + \frac{1}{\rho}\nabla p^* = 0, \tag{1}$$

$$\partial_t \mathbf{b} + \mathbf{v}\cdot\nabla\mathbf{b} - \mathbf{b}\cdot\nabla\mathbf{v} + \mathbf{b}\nabla\cdot\mathbf{v} = 0, \tag{2}$$

$$\partial_t \rho + \nabla\cdot(\rho\mathbf{v}) = 0, \quad \nabla\cdot\mathbf{b} = 0, \tag{3}$$

where $\mathbf{v} = (v_1, v_2, v_3)$ is the fluid velocity, $\mathbf{b} = (b_1, b_2, b_3)$ is the magnetic field, g is gravity acceleration, f is the Coriolis parameter, $f = 2\Omega$, and $\Omega\hat{\mathbf{z}}$ is the angular velocity of rotation. We also introduced the magnetic pressure:

$$p^* = p + \rho\frac{\mathbf{b}^2}{2}, \tag{4}$$

and considered, as usual, that the centrifugal effects are hidden in p. If the axis of rotation is supposed to be parallel to the gravity acceleration, then $f = f_0 = \text{const}$, which corresponds to the f-plane approximation for the tangent plane to the rotating planet/star for geo- and astrophysical applications. For scales of the motion sufficiently small with respect to the radius of the planet/star, the non-verticality of the rotation axis may be taken into account in the β-plane approximation: $f = f_0 + \beta y$, where y is the latitudinal coordinate.

For what follows, it is important to note that the scaling for the magnetic field is chosen in Eqs. (1)–(4) in such a way that it is measured in velocity units. Effects of molecular dissipation and diffusion are neglected in Eqs. (1)–(3). If necessary, they may be re-introduced via the standard viscosity, electroconductivity and diffusion terms proportional to the Laplacians of velocity, magnetic field and density, respectively, in the corresponding equations.

Equations (1)–(3) are written in the most general form. We should emphasize that the procedure below is applicable as well to their simplified versions, like the Boussinesq approximation, where velocity is supposed to be divergenceless and density is advected. Thermal effects may be introduced in such models (cf. Rachid, 2008 in the solar tachocline context) by relating variable parts of density to temperature. Finally, potential temperature θ may be used as a thermodynamical variable instead of density.

The horizontal momentum and magnetic field Eqs. (1) and (2) may be rewritten with the help of Eq. (3) in the form of conservation laws:

$$\begin{aligned}&\partial_t(\rho\mathbf{v})+\nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v})-\\&\quad\nabla\cdot(\mathbf{b}\otimes\mathbf{b})+g\hat{\mathbf{z}}+f\hat{\mathbf{z}}\wedge(\rho\mathbf{v})+\nabla p^*=0,\end{aligned}\tag{5}$$

$$\partial_t(\mathbf{b})+\nabla\cdot(\mathbf{v}\otimes\mathbf{b})-\nabla\cdot(\mathbf{b}\otimes\mathbf{v})=0.\tag{6}$$

Here and below we use the shorthand notation $\nabla\cdot(\mathbf{A}\otimes\mathbf{B})\equiv\partial_i(A_iB_k)$, $i,k=1,2,3$ for a pair of vector fields $\mathbf{A},\mathbf{B}$.

We now proceed by vertical integration of the horizontal momentum Eq. (5) between a pair of *material surfaces* $z_{1,2}(x,y,t)$:

$$v_3|_{z_i}=\frac{dz_i}{\mathrm{d}t}=\partial_t z_i+u\partial_x z_i+v\partial_y z_i,\;\; i=1,2,\tag{7}$$

which we suppose to be, at the same time, *magnetic surfaces*:

$$\mathbf{b}\cdot\nabla z_i=0.\tag{8}$$

With the help of the Leibnitz formula, we get:

$$\begin{aligned}&\partial_t\int_{z_1}^{z_2}\mathrm{d}z\rho\mathbf{v}_\mathrm{h}+\nabla_\mathrm{h}\cdot\int_{z_1}^{z_2}\mathrm{d}z\,(\rho\mathbf{v}_\mathrm{h}\otimes\mathbf{v}_\mathrm{h})-\\&\nabla_\mathrm{h}\cdot\int_{z_1}^{z_2}\mathrm{d}z\,(\mathbf{b}_\mathrm{h}\otimes\mathbf{b}_\mathrm{h})+f\hat{\mathbf{z}}\wedge\int_{z_1}^{z_2}\mathrm{d}z\,(\rho\mathbf{v}_\mathrm{h})\\&=-\nabla_\mathrm{h}\int_{z_1}^{z_2}\mathrm{d}zp^*-\nabla_\mathrm{h}z_1\;p^*|_{z_1}+\nabla_\mathrm{h}z_2\;p^*|_{z_2},\end{aligned}\tag{9}$$

where the index h denotes the horizontal part. This derivation follows the standard procedure for obtaining the non-magnetic RSW equations, cf. Zeitlin (2007, Chapter 1). Note that additional surface terms appear if the hypothesis of magnetic surfaces is relaxed.

We now integrate the horizontal magnetic field equations in the same way and get:

$$\partial_t\int_{z_1}^{z_2}\mathrm{d}z\mathbf{b}_\mathrm{h}+\nabla_\mathrm{h}\cdot\int_{z_1}^{z_2}\mathrm{d}z\,(\mathbf{v}_\mathrm{h}\otimes\mathbf{b}_\mathrm{h})-\nabla_\mathrm{h}\cdot\int_{z_1}^{z_2}\mathrm{d}z\,(\mathbf{b}_\mathrm{h}\otimes\mathbf{v}_\mathrm{h})=0.\tag{10}$$

Again, the hypothesis that material surfaces are at the same time magnetic surfaces is crucial for arriving at such simple forms of integrated equations. It should be noted that the vertical component of magnetic field b_3 is not supposed to be zero, but in fact decouples from the mRSW equations to be obtained below, and may be recovered once all other fields are determined, as is the case with v_3 in the ordinary RSW equations.

Finally, we integrate Eq. (3) and get:

$$\partial_t\int_{z_1}^{z_2}\mathrm{d}z\rho+\nabla_\mathrm{h}\cdot\int_{z_1}^{z_2}\mathrm{d}z\rho\mathbf{v}_\mathrm{h}=0,\quad\nabla_\mathrm{h}\cdot\int_{z_1}^{z_2}\mathrm{d}z\mathbf{b}_\mathrm{h}=0.\tag{11}$$

2.2 Mean-field approximation and magnetohydrostatic hypothesis

Up to now, no approximation has been made whatsoever. The only hypothesis was the existence of a pair of material surfaces that are, at the same time, magnetic surfaces. Let us introduce the vertical averages

$$\bar{F}=\frac{\int_{z_1}^{z_2}\mathrm{d}zF}{z_2-z_1},\tag{12}$$

and apply the *mean-field approximation*, i.e., replace $\overline{A\cdot B}$ by $\bar{A}\cdot\bar{B}$ for any A and B. We thus get from Eqs. (10)–(11):

$$\begin{aligned}&\bar{\rho}\,(z_2-z_1)\left(\partial_t\bar{\mathbf{v}}_\mathrm{h}+\bar{\mathbf{v}}\cdot\nabla_\mathrm{h}\bar{\mathbf{v}}_\mathrm{h}+f\hat{\mathbf{z}}\wedge\bar{\mathbf{v}}_\mathrm{h}\right)\\&\quad-\nabla\left[(z_2-z_1)\,\bar{\mathbf{b}}_\mathrm{h}\otimes\bar{\mathbf{b}}_\mathrm{h}\right]=\\&\quad-\nabla_\mathrm{h}\int_{z_1}^{z_2}\mathrm{d}zp^*-\nabla_\mathrm{h}z_1\;p^*\big|_{z_1}+\nabla_\mathrm{h}z_2\;p^*\big|_{z_2},\end{aligned}\tag{13}$$

$$\begin{aligned}&\partial_t\left[(z_2-z_1)\,\bar{\mathbf{b}}_\mathrm{h}\right]+\nabla_\mathrm{h}\cdot\left[(z_2-z_1)\,\bar{\mathbf{v}}_\mathrm{h}\otimes\bar{\mathbf{b}}_\mathrm{h}\right]\\&\quad-\nabla_\mathrm{h}\cdot\left[(z_2-z_1)\,\bar{\mathbf{b}}_\mathrm{h}\otimes\bar{\mathbf{v}}_\mathrm{h}\right]=0,\end{aligned}\tag{14}$$

$$\begin{aligned}&\partial_t\left[\bar{\rho}\,(z_2-z_1)\right]+\nabla_\mathrm{h}\cdot\left[\bar{\rho}\,(z_2-z_1)\,\bar{\mathbf{v}}_\mathrm{h}\right]=0,\\&\nabla_\mathrm{h}\cdot\left[(z_2-z_1)\,\bar{\mathbf{b}}\right]=0.\end{aligned}\tag{15}$$

Detailed discussion of the applicability of the mean-field approximation is out of the scope of the present paper. It is obvious, however, that it requires sufficiently slow variations of all fields in the vertical direction. As usual, corrections to the mean-field theory may be accounted for by parameterizing neglected Reynolds stresses. Thus, following the traditional line of argument, "turbulent" viscosity, diffusivity, and conductivity may be introduced relating the neglected stresses to the mean fields.

We will now make a crucial *magnetohydrostatics hypothesis* that will allow one to get the mRSW equations from the vertically averaged MHD in the mean-field approximation. It consists of supposing vertical accelerations to be small, as well as the term $\mathbf{b}\cdot\nabla b_3$, and of replacing the vertical momentum equation with the magnetohydrostatic balance relation:

$$\rho g=-\partial zp^*.\tag{16}$$

Thus

$$p^* = -g \int_{z_1}^{z} \mathrm{d}z\rho + p^*\big|_{z_1} \approx -g\bar{\rho}\,(z - z_1) + p^*\big|_{z_1}, \tag{17}$$

or

$$p^* = +g \int_{z}^{z_2} \mathrm{d}z\rho + p^*\big|_{z_2} \approx g\bar{\rho}\,(z_2 - z) + p^*\big|_{z_2}. \tag{18}$$

Traditionally, the mean density $\bar{\rho}$ is considered to be constant in the RSW context, which will be our hypothesis below. Yet this hypothesis may be relaxed in the case of pure RSW, leading to the so-called Ripa's equations Ripa (1995), i.e., shallow-water equations with variable mean density. mRSW equations including thermal effects may be obtained along the same lines – see below.[1]

2.3 Boundary conditions – multi-layer configurations

The final step consists of imposing boundary conditions at $z_{1,2}$. In the simplest configuration, one of the material surfaces is fixed to be constant and the other is free with constant magnetic pressure above it, which leads to the mRSW equations as proposed in Gilman (2000):

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + f\hat{\mathbf{z}}\wedge\mathbf{v} + g\nabla h = \frac{1}{h}\left[\nabla\left(h\,\mathbf{b}\otimes\mathbf{b}\right)\right], \tag{19}$$

$$\partial_t h + \nabla\cdot(\mathbf{v}h) = 0, \tag{20}$$

$$\nabla\cdot(h\mathbf{b}) = 0 \tag{21}$$

$$\partial_t \mathbf{b} + \mathbf{v}\cdot\nabla\mathbf{b} = \frac{1}{h}\left[\nabla\left(h\,\mathbf{v}\otimes\mathbf{b}\right)\right], \tag{22}$$

where $h = z_2 - z_1,\ z_1 = \text{const}$, and we omit the bars and the index h. Note that if for some reason the fixed material surface is not flat: $z_1 = \eta(x, y)$, "magnetic topography" η will enter the equations with a replacement $h \to h - \eta(x, y)$ everywhere except for the gravity term. Note also that in the MHD context, unlike the standard RSW equations, the fixed surface may be the upper one as well, with corresponding changes in the mRSW equations.

We should emphasize that the mRSW equations (19)–(22) are obtained under the hypothesis of no dissipation and strict mean-field approximation. If molecular viscosity and magnetic diffusivity are kept in the original equations (1)–(3), it is easy to see that, through the above-described vertical averaging procedure, they would result in terms proportional to $\nabla^2\mathbf{v}$ and $\nabla^2\mathbf{b}$ (with two-dimensional ∇) in the r.h.s. of the Eqs. (19) and (22). As was already mentioned, the deviations from the strict mean-field theory would result in similar terms (with possibly non-constant "phenomenological" coefficients) if the hypotheses of turbulent viscosity and turbulent magnetic diffusivity are applied to parameterize the stresses.

As already mentioned, thermal effects may be introduced in the mRSW equations, following Ripa (1995) (cf. also Dellar, 2003). If an additional equation of advection of temperature (or potential temperature θ) is added to the original set of 3-D equations, it will give the 2-D advection equation by vertical averaging:

$$\partial_t\theta + \mathbf{v}\cdot\nabla\theta = 0, \tag{23}$$

to be added to the system (19)–(22). Variable potential temperature in the buoyancy term in (1) will lead to the replacement of the term $g\nabla h$ by $\theta\nabla h + \frac{1}{2}h\nabla\theta$ after the vertical averaging and proper renormalizations.

One may extend the simplest mRSW system (19)–(22) by superimposing N layers of different mean density, still under the magnetohydrostatic hypothesis, ending up (at the top and at the bottom) either with a fixed (flat or not) or with a free material surface. As a result, multi-layer mRSW models arise, allowing one to include the *baroclinic phenomena* in consideration. The structure of the multi-layer mRSW equations is clear from the above construction: they will inherit the velocity and (magnetic) pressure terms from the multi-layer RSW equations, with the same addition of a magnetic field in each layer as in Eq. (19). Mass is conserved layer-wise. As to the magnetic field equations, they will be the same as in Eqs. (21) and (22) for each layer. We give as an example the equations of the two-layer mRSW with fixed flat upper and lower boundaries at a distance $H = \text{const}$:

$$\partial_t \mathbf{v}_i + \mathbf{v}_i\cdot\nabla\mathbf{v}_i + f\hat{\mathbf{z}}\bar{\rho}_i + \frac{1}{\bar{\rho}_i}\nabla\pi_i^* = \frac{1}{h_i}\left[\nabla\left(h_i\,\mathbf{b}_i\otimes\mathbf{b}_i\right)\right],\ i = 1, 2; \tag{24}$$

$$\partial_t h_i + \nabla\cdot(\mathbf{v}_i h_i) = 0, \quad h_1 + h_2 = H, \tag{25}$$

$$\pi_1^* = (\bar{\rho}_1 - \bar{\rho}_2) g h_1 + \pi_2^*, \tag{26}$$

$$\nabla\cdot(h_i\mathbf{b}_i) = 0 \tag{27}$$

$$\partial_t \mathbf{b}_i + \mathbf{v}_i\cdot\nabla\mathbf{b}_i = \frac{1}{h_i}\left[\nabla\left(h_i\,\mathbf{v}_i\otimes\mathbf{b}_i\right)\right], \tag{28}$$

where the subscripts $i = 1, 2$ denote the lower and the upper layer, respectively, no summation over repeated indices is supposed, π_i^* are magnetic pressures in the respective layers, h_i – thicknesses of the layers, the bottom topography is not introduced, for simplicity, and the subscript h is omitted. The one-layer RSW model is recovered in the limit $\bar{\rho}_2 \to 0$.

[1]The relation between mRSW and Ripa's equations, in particular their common Hamiltonian structure, was discussed in Dellar (2003).

3 Properties of the barotropic mRSW system and the QG limit

3.1 General properties

The one-layer mRSW equations (19)–(22) can be rewritten in conservative form:

$$\partial_t (h\,\mathbf{v}) + \nabla\left[h\,(\mathbf{v}\otimes\mathbf{v} - \mathbf{b}\otimes\mathbf{b}) + \frac{1}{2}gh^2\right] + f\,h\,\hat{\mathbf{z}}\wedge\mathbf{v} = 0, \tag{29}$$

$$\partial_t (h\,\mathbf{b}) + \hat{\mathbf{z}}\wedge\nabla\left(h\,\hat{\mathbf{z}}\cdot(\mathbf{b}\wedge\mathbf{v})\right) = 0, \tag{30}$$

$$\partial_t h + \nabla\cdot(\mathbf{v}h) = 0, \tag{31}$$

$$\nabla\cdot(h\,\mathbf{b}) = 0, \tag{32}$$

useful for numerical simulations.

Magnetic potential can be introduced for vertically integrated horizontal magnetic fields, thus resolving the constraint (32):

$$h\,\mathbf{b} = \hat{\mathbf{z}}\wedge\nabla A, \Rightarrow \partial_t A + \mathbf{v}\cdot\nabla A = 0. \tag{33}$$

The magnetic field therefore may be eliminated in favor of A in Eqs. (19) and (29). The system thus becomes a RSW system with additional forcing in the momentum equations, due to the magnetic field that is determined from the passively advected magnetic potential:

$$\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + f\hat{\mathbf{z}}\wedge\mathbf{v} + g\nabla h = -\frac{1}{h}\hat{\mathbf{z}}\wedge\mathcal{J}\left(A, \frac{\nabla A}{h}\right), \tag{34}$$

to be completed with Eqs. (31) and (33). Here and below $\mathcal{J}$ denotes the Jacobian.

As was repeatedly mentioned in the literature, cf. Dellar (2002), the main difference between the RSW and mRSW systems is non-conservation of potential vorticity (PV) in the latter. The only Lagrangian invariant, therefore, is the magnetic potential A, cf. Eq. (33).

3.2 Quasi-geostrophic approximation

The limit of *strong rotation* is characterized by the smallness of both Rossby number $Ro = \frac{U}{f_0 L}$, and magnetic Rossby number $Ro_\mathrm{m} = \frac{B}{f_0 L}$, where U and B are the typical averaged velocity and magnetic field in the fluid layer, and L is a typical horizontal scale. Supposing that both Rossby numbers are of the same order of magnitude, and applying the standard straightforward expansion

$$\mathbf{v} = \mathbf{v}^{(0)}(x,y,T) + Ro\,\mathbf{v}^{(1)}(x,y,T) + \mathcal{O}(Ro^2) \tag{35}$$

for motions of the scale $L \propto R_\mathrm{d} = \frac{\sqrt{gH}}{f_0}$ depending only on slow time $T \propto Ro\,t$, cf. Zeitlin (2007), we get that the leading order velocity field is, as it should be, geostrophic:

$$v_2^{(0)} = \partial_x\eta, \quad v_1^{(0)} = -\partial_y\eta, \tag{36}$$

while the first ageostrophic correction acquires a magnetic addition:

$$v_2^{(1)} = \left(\partial_t + \mathbf{v}^{(0)}\cdot\nabla\right)v_1^{(0)} + \mathcal{J}\left(A, \partial_y A\right)$$
$$v_1^{(1)} = -\left(\partial_t + \mathbf{v}^{(0)}\cdot\nabla\right)v_2^{(0)} + \mathcal{J}\left(A, \partial x A\right). \tag{37}$$

Here H is the mean depth of the layer, and $h = H(1 + Ro\,\eta)$.

When plugged into (31), together with (33) this gives the *quasi-geostrophic* MHD equations (QG MHD):

$$\partial_t\nabla^2\eta + \mathcal{J}(\eta, \nabla^2\eta) - \frac{1}{R_\mathrm{d}^2}\partial_t\eta - \mathcal{J}(A, \nabla^2 A) = 0,$$
$$\partial_t A + \mathcal{J}(\eta, A) = 0. \tag{38}$$

In the limit of infinite deformation radius $R_\mathrm{d}\to\infty$, the equations in (38) become the standard 2d MHD. It is worth noting that, in non-magnetic RSW, the corresponding classical QG equation may be derived straightforwardly from the PV equation, and expresses the Lagrangian conservation of PV. In Eulerian terms this is translated in terms of conservation of the Casimir functionals – any function of quasi-geostrophic PV defined as $\nabla^2\eta - \frac{1}{R_\mathrm{d}^2}\eta$, if integrated over the domain of the flow, is conserved. Here, although PV is not conserved, we still have QG MHD equations. Instead of a family of PV Casimirs, we now have two families of Casimirs (Zeitlin, 1992): integrated functions of the magnetic potential, and integrated functions of magnetic potential times quasi-geostrophic PV.

If the β effect is introduced, with β of the order of Ro, as usual, the system (38) becomes

$$\partial_t\nabla^2\eta + \mathcal{J}(\eta, \nabla^2\eta) - \frac{1}{R_\mathrm{d}^2}\partial_t\eta - \mathcal{J}(A, \nabla^2 A) + \beta\partial_x\eta = 0,$$
$$\partial_t A + \mathcal{J}(\eta, A) = 0. \tag{39}$$

The system (39) with infinite deformation radius (and with addition of turbulent viscosity and conductivity) was recently introduced heuristically and was studied numerically in Tobias et al. (2007) in the context of solar tachocline. The same system, or rather its two-layer counterpart following from the system (24)–(28) and used below in Sect. 4, were recently derived directly from the full MHD equations by Umurhan (2013).

3.3 Linear wave spectrum on the f- and β-planes, and the role of external magnetic fields

The QG version of the mRSW equations was derived above in a semi-heuristic manner, by using the slow time scale, and thus filtering "by hand" the fast inertia–gravity waves. A more formal justification of this limit via the separation of fast and slow variables and systematic fast-time averaging may be achieved, following the lines of Reznik et al. (1992) and Zeitlin et al. (1992). The situation, however, will be more tricky in the presence of the external magnetic field. Indeed, it it easy to see that, due to its Lagrangian conservation character, the magnetic field equation (33) is "slow" and does not change the linear, i.e., inertia–gravity, wave spectrum of the RSW system in the absence of background magnetic fields. Thus, the procedure of Reznik et al. (1992) and Zeitlin et al. (1992) should work for perturbation about the rest state *without background magnetic fields*. However, the Alfvén waves that should appear if a background magnetic field is present, do not have a spectral gap, see below, and may invalidate the slow-fast separation in this case. This issue will be addressed in detail elsewhere.

We recall that the linearization of Eqs. (19)–(22) about the state of rest $h = H$ on the f-plane $f = \text{const}$, with constant magnetic field $\mathbf{b} = \mathbf{B}$, results in harmonic wave solutions with wavefrequency ω and wavevector $\boldsymbol{k}$ satisfying the following dispersion relation, cf. Shecter et al. (2001):

$$\omega^2 = (\mathbf{B}\cdot\boldsymbol{k})^2 + \frac{gH\boldsymbol{k}^2 + f^2}{2} \pm \frac{1}{2}\sqrt{\left(gH\boldsymbol{k}^2 + f^2\right)^2 + 4f^2(\mathbf{B}\cdot\boldsymbol{k})^2}, \quad (40)$$

which gives, in the limit of no rotation, $\omega^2 = (\mathbf{B}\cdot\boldsymbol{k})^2$, $\omega^2 = (\mathbf{B}\cdot\boldsymbol{k})^2 + gH\boldsymbol{k}^2$, i.e., Alfvén and mixed Alfvén–gravity waves.

Likewise the formal linearization of the system (39) over the rest state with constant background magnetic field $\mathbf{B}$: $\eta = 0$, $A = B_1 y - B_2 x$ results in harmonic solutions with wavefrequency ω and wavevector $\boldsymbol{k} = (k_1, k_2)$ with the following dispersion relation:

$$\omega = -\frac{\beta k_1}{2(\boldsymbol{k}^2 + R_\mathrm{d}^{-2})} \pm \sqrt{\left(\frac{\beta k_1}{2(\boldsymbol{k}^2 + R_\mathrm{d}^{-2})}\right)^2 + (\mathbf{B}\cdot\boldsymbol{k})^2}, \quad (41)$$

corresponding to mixed Alfvén–Rossby waves. Indeed, in the limit of vanishing magnetic field (41) gives the usual Rossby waves, and in the opposite limit of vanishing β it gives the Alfvén waves.

It is clear from Eq. (41) that, in the presence of the background magnetic field, the frequency spectrum is not bounded from above anymore, and the formal validity of the system (39) as “slow” limiting equations of the full mRSW equations remains to be proved.

In any case, the known effect of the “elasticity” of the magnetic field leading to Alfvén waves is well represented in the barotropic mRSW and its “slow” version. We will see that this effect will play a stabilizing role when baroclinic effects are included.

4 Effects of baroclinicity

The multi-layer mRSW can be treated in a similar way as the one-layer case, by introducing magnetic potentials for each layer. The QG approximation may be as well developed, again along the standard lines, giving the following 2-layer QG MHD equations for (magnetic) pressures and magnetic potentials in the layers in the f-plane approximation:

$$\frac{d_i^{(0)}}{\mathrm{d}t}\left[\nabla^2\pi_i^* - (-1)^i D_i^{-1}\eta\right] - \mathcal{J}(A_i, \nabla^2 A_i) = 0,$$
$$\frac{d_i^{(0)}}{\mathrm{d}t} A_i = 0, \qquad i = 1, 2, \quad (42)$$

where

$$\frac{d_i^{(0)}}{\mathrm{d}t}(\ldots) := \partial_t(\ldots) + \mathcal{J}\left(\pi_i^*, \ldots\right), \; i = 1, 2. \quad (43)$$

Here D_i are nondimensional thicknesses of the layers, and η denotes a nondimensional interface deviation. In the standard in GFD limit $\rho_2 - \rho_1 \to 0$ it is simply expressed in terms of the pressure difference: $\eta = \pi_2 - \pi_1$. No summation over repeated indices is understood, and stars at π_i are omitted. The details of the scaling and the procedure may be found in Zeitlin (2007, Chapters 1 and 2). Their extension to the MHD case is straightforward.

On the β-plane, the terms $\beta\partial_x\pi_i$ should be added in Eq. (42), and formal linearization of Eq. (42) will give baroclinic (and barotropic) mixed Alfvén–Rossby waves in the presence of a background magnetic field. It is well known, however, that in the baroclinic system Rossby waves can also propagate due to the velocity shear between the layers even in the absence of β. For QG motions described by Eq. (42), a shear corresponds to an interface inclination via the geostrophic balance. It is also well known that on the f-plane, any shear is unstable for sufficiently long wave perturbations due to the *baroclinic instability* (on the β-plane a threshold for the instability exists), cf. Pedlosky (1982). Let us see how incorporation of the magnetic field influences the baroclinic instability. By linearizing about the state

$$\pi_i^* = -U_i y, \quad A_i = B_i y, \quad (44)$$

with constant U_i, B_i, and looking for harmonic solutions with wavefrequency ω and wavenumber $\boldsymbol{k}$ in the strongly degenerate, but simplest to analyze case $D_1 = D_2 = D$, $U_1 = -U_2 = U$, we arrive at the following dispersion relation:

$$\omega = \pm U k_1 \sqrt{\frac{(1 + B_1^2 + B_2^2)\boldsymbol{k}^2 + D^{-1}(B_1^2 + B_2^2 - 2)}{\boldsymbol{k}^2 + 2D^{-1}}}. \quad (45)$$

The standard baroclinic instability result Pedlosky (1982) is recovered in the limit $B_{1,2} \to 0$.

Thus, if the magnetic field is strong enough in any layer, the baroclinic instability disappears. Such a stabilizing effect of the magnetic field could be anticipated due to its “elasticity” mentioned above. It is characteristic, in fact, not only of the large-scale slow-evolving geostrophic baroclinic instability, but also of the rapid ageostrophic instabilities such as the Kelvin–Helmholtz one, which can be easily studied in the framework of the non-rotating 2-layer equations (24)–(28) by linearizing about the basic state with horizontal velocity shear between the layers. Although the magnetic field does not cure the instability, its influence is stabilizing, adding up with gravity (not presented).

5 Concluding remarks

Thus, we have shown how the mRSW model of Gilman (2000) arises from vertical averaging of the MHD equations for the rotating fluid in the gravity field, and application of the mean-field and magnetohydrostatic hypotheses. Multi-layer generalizations easily follow from this construction. Although it is important to keep in mind the limits of applicability of the mRSW related to these hypotheses, the example of non-magnetic RSW and its applications in geophysical fluid dynamics show that such models remain useful far beyond their formal validity range. The added value, with respect to the pioneering work of Gilman (2000), is that we show that the mRSW equations arise universally from the vertical averaging of basic MHD and independently of the details of stratification and compressibility. The method also gives the possibility to include thermal effects and shows how the deviations from the strict mean-field approximation may be accounted for with turbulent viscosity and diffusivity hypotheses.

The "balanced" quasi-geostrophic limit of the mRSW with filtered inertia–gravity waves was established in a semi-heuristic manner and is closely related to 2d MHD. On the β-plane it gives a framework for studying mixed Alfvén–Rossby waves, although the formal validity of the QG MHD as the slow limit of the mRSW in the presence of constant magnetic field remains to be proved. As follows from the results of Sect. 3.3, sufficient smallness of the magnetic field may be necessary for such a proof.

The parallels between RSW and mRSW may be developed further. For instance, the existence of mixed Alfvén–equatorial waves may be straightforwardly established for the mRSW equations on the equatorial β-plane where $f_0 = 0$. These waves may be important in astrophysical applications. Mixed Alfvén–Kelvin waves can be found propagating along the boundary parallel to the magnetic field, a configuration that may exist in the laboratory, and so on.

Let us finally mention that an important advantage of mRSW equations is that, in their conservative form (29)–(32) (an analogous form may be easily established for a multi-layer mRSW with free upper boundary), they may be efficiently treated numerically with the help of modern finite-volume methods, cf. Bouchet (2007). This work is in progress.

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