Large-scale linear instability of the Kolmogorov flow with modified viscosity

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Smagorinsky-Clark mixed model + friction
Fluid mechanics streamfunction
Calculation are done in terms of U
Here we modify the scaling to have a temporal term at order 3
U is supposed NOT to have the SinI symmetry

Settings

Last use

In[12]:=
```
DateList[]
```

Out[12]=
```
{2009, 1, 22, 16, 55, 12.800087}
```

General

In[2]:=
```
Off[General::spell1]
```

Laplacian

In[3]:=

$$\Delta = -\left(\epsilon^2 \partial_{x,x} \# + \partial_{y,y} \#\right) \&$$

Out[3]=

$$-\left(\epsilon^2 \partial_{x,x} \#1 + \partial_{y,y} \#1\right) \&$$

Definition of the turbulent viscosity, Smagorinsky-Clark mixed model

Standard version with a heuristic component accounting for the vertical shear

In[4]:=
```
S̄ = √(4 (ϵ ∂x,y #)^2 + (ϵ^2 ∂x,x # - ∂y,y #)^2 + μ (ϵ ∂x #)^2 + μ (∂y #)^2) &
```

Out[4]=

$$\sqrt{4\,(\epsilon\,\partial_{x,y}\#1)^2+\left(\epsilon^2\,\partial_{x,x}\#1-\partial_{y,y}\#1\right)^2+\mu\,(\epsilon\,\partial_x\#1)^2+\mu\,(\partial_y\#1)^2}\ \&$$

Dissipation: C in front of the Smagorinsky component, D in front of the Clark component
In standard notations $C=\Delta^2/12$ and $D = C_S\Delta^2$

In[5]:=
```
DS = C (∂y,y (S̄[#] (-∂y,y # + ϵ^2 ∂x,x #)) -
      ϵ^2 ∂x,x (S̄[#] ( - ∂y,y # + ϵ^2 ∂x,x #)) - 4 ϵ^2 ∂x,y (S̄[#] ∂x,y #)) +
   D ( ϵ ∂x,y # ( ϵ^4 ∂x,x,x,x # - ∂y,y,y,y #) + ( ϵ ∂x,y,y,y # + ϵ^3 ∂x,x,x,y #) (∂y,y # - ϵ^2 ∂x,x #) ) &
```

Out[5]=

$$C\left(\partial_{y,y}\left(\bar{S}[\#1]\left(-\partial_{y,y}\#1+\epsilon^2\,\partial_{x,x}\#1\right)\right)-\epsilon^2\,\partial_{x,x}\left(\bar{S}[\#1]\left(-\partial_{y,y}\#1+\epsilon^2\,\partial_{x,x}\#1\right)\right)-4\,\epsilon^2\,\partial_{x,y}\left(\bar{S}[\#1]\,\partial_{x,y}\#1\right)\right)+D\left(\epsilon\,\partial_{x,y}\#1\left(\epsilon^4\,\partial_{x,x,x,x}\#1-\partial_{y,y,y,y}\#1\right)+\left(\epsilon\,\partial_{x,y,y,y}\#1+\epsilon^3\,\partial_{x,x,x,y}\#1\right)\left(\partial_{y,y}\#1-\epsilon^2\,\partial_{x,x}\#1\right)\right)\ \&$$

Reference
Pope, S.B. Turbulent flows, C.U.P., 2000

Definition of the modified Kolmogorov Flow

Velocity and streamfunction

U[y_] := A Sin[θ [y]]

In[6]:=
```
Φ[y_] := Integrate[U[s], {s, 0., y}]
```

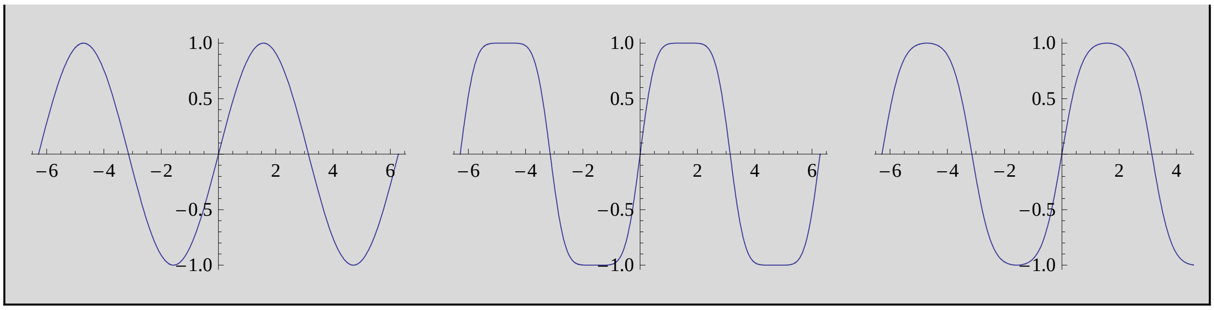
Exemples of flows

Standard Kolmogorov flow in Flow1, "flat" flow in Flow2, intermediate case in Flow3

In[150]:=
```
Flow2 = U → Evaluate[Sin[# + 1/2 Sin[ 2 #]] &];
Flow1  = U → Evaluate[Sin[# ] &];
Flow3 = U → Evaluate[Sin[# + 1/4 Sin[ 2 #]] &];
```

Plots of the flow

```
p1 = Plot[U[y]  /. {Flow1, A → 1}, {y, -2 π,  2 π}, DisplayFunction → Identity];
p2 = Plot[U[y]  /. {Flow2, A → 1}, {y, -2 π,  2 π}, DisplayFunction → Identity];
p3 = Plot[U[y]  /. {Flow3, A → 1}, {y, -2 π,  2 π}, DisplayFunction → Identity];
Show[GraphicsRow[{p1, p2, p3}]]
```



Plots of the streamfunction

Plots of the forcing (or dissipation)

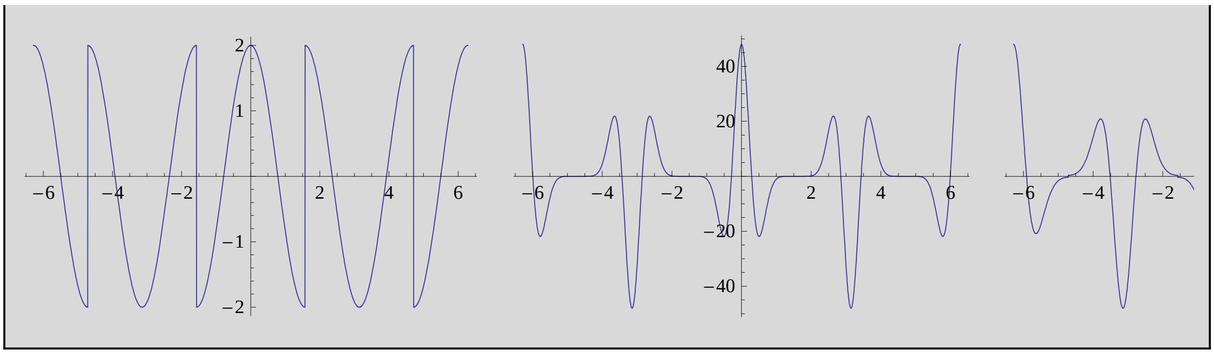
Forcing is calculated to equilibrate the dissipation since there is no advection for a flow that depends only on y
Some arbitrary values are assumed for the parameters

```
Forcing = Assuming[A > 0 && θ′[y] > 0, Simplify[ DS[ Φ[y]]]]
```

$$-\frac{1}{\left(\mu\, U[y]^2 + U'[y]^2\right)^{3/2}} C \left(\mu\, U'[y]^5 + 3\,\mu\, U[y]^2\, U'[y]\, U''[y]\, (\mu\, U[y] + U''[y]) + U'[y]^3\, U''[y]\, (\mu\, U[y] + 2\, U''[y]) + \mu^2\, U[y]^4\, U^{(3)}[y] + 3\,\mu\, U[y]^2\, U'[y]^2\, U^{(3)}[y] + 2\, U'[y]^4\, U^{(3)}[y]\right)$$

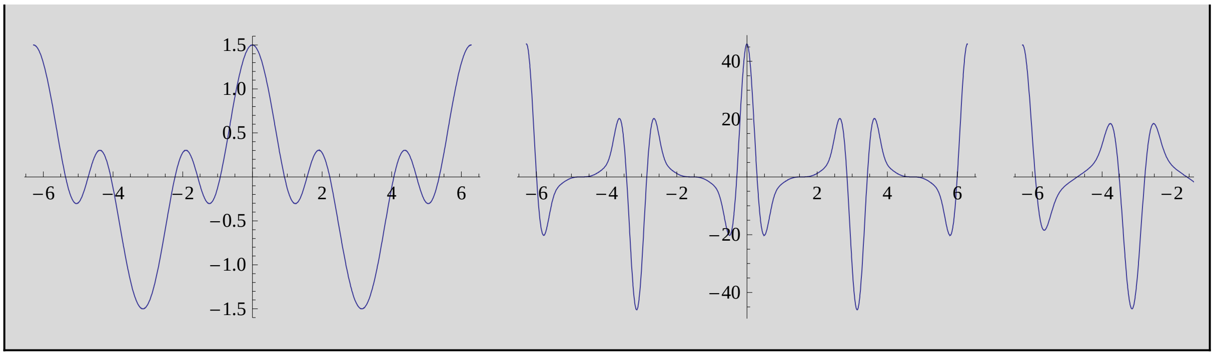
Setting μ -> 0. shows that Flow1 requires a forcing with discontinuities that disappear in Flow2 and Flow3.

```
p1 = Plot[ Forcing //. {Flow1,  A → 1,  C → 1,  μ → 0.},
   {y, -2 π,  2 π}, PlotRange → All, DisplayFunction → Identity];
p2 = Plot[ Forcing //. {Flow2,  A → 1,  C → 1,  μ → 0.}, {y, -2 π,  2 π},
   PlotRange → All, DisplayFunction → Identity];
p3 = Plot[ Forcing //. {Flow3,  A → 1,  C → 1,  μ → 0.}, {y, -2 π,  2 π},
   PlotRange → All, DisplayFunction → Identity];
Show[GraphicsRow[{p1, p2, p3}]]
```



Setting μ -> 0.5 shows that the discontinuities disappear in Flow1, while the forcings in Flow2 and Flow3 are hardly modified

```
p1 = Plot[ Forcing //. {Flow1, A → 1, C → 1, μ → 0.5},
   {y, -2 π, 2 π}, PlotRange → All, DisplayFunction → Identity];
p2 = Plot[ Forcing //. {Flow2, A → 1, C → 1, μ → 0.5}, {y, -2 π, 2 π},
   PlotRange → All, DisplayFunction → Identity];
p3 = Plot[ Forcing //. {Flow3, A → 1, C → 1, μ → 0.5}, {y, -2 π, 2 π},
   PlotRange → All, DisplayFunction → Identity];
Show[GraphicsRow[{p1, p2, p3}]]
```



Definition of the pertubation problem

2D Navier-Stokes with scaling

We have taken u = $\partial_y \psi$ and $v = -\partial_x \psi$ (definition used in fluid mechanics, opposite to the convention in GFD)
The temporal term has its order lowered with respect to the standard calculation.

In[7]:=
```
NS = ϵ ∂_T Δ[ψ[T, X, y]] - ϵ ∂_X ψ[T, X, y] ∂_y Δ[ψ[T, X, y]] +
   ϵ ∂_y ψ[T, X, y] ∂_X Δ[ψ[T, X, y]] - DS[ψ[T, X, y]] + DS[Φ[y]];
```

Expansion

The streamfunction is expanded up to the fourth order in ϵ

In[8]:=
```
ψ[T_, X_, y_] :=
 Φ[y] + ψ0[T, X, y] + ϵ ψ1[T, X, y] + ϵ^2 ψ2[T, X, y] + ϵ^3 ψ3[T, X, y] + ϵ^4 ψ4[T, X, y]
```

In[9]:=
```
NS2 = Collect[Normal[Series[NS, {ϵ, 0, 4}]], ϵ];
```

Order 0

Extract order 0 terms from the general expansion and perform algebraic simplication.
Fairly lengthy expression anyway.

In[13]:=
```
a0 = Simplify[Coefficient[NS2, ϵ, 0] ]
```

Symbolic solution

The 0 th order solution satisfies a0 = 0 as soon as it does not depend on the fast variable y

In[32]:=
```
Rule0 = ψ0 → (φ0[#1, #2] &)
```

Out[32]=
```
ψ0 → (φ0[#1, #2] &)
```

Verification

Check that φ 0[X,T] satisfies the 0 th order solution

In[15]:=
```
FullSimplify[a0 /. {Rule0}]
```

Out[15]=
```
0
```

Useful rules for simplifications

In[16]:=
```
Improve = Simplify[Together[#1]] &
```

Out[16]=
```
Simplify[Together[#1]] &
```

Periodicity in space

In[17]:=
```
Solva[expr_] := ((expr /. y → 2 π) - (expr /. y → 0)) /. g___[2 π] → g[0]
```

Order 1

Extract order 1 terms from the general expansion, use formal solution to order 0 equation and perform algebraic simplication. Need to be put under the form $\mathcal{L}$[ψ1[T, X, y]] = second member

In[20]:=
```
a1 = Simplify[Expand[Coefficient[NS2, ϵ, 1]] /. Rule0 ];
```

Extraction of the linear operator and verification

First derivative of ψ 1in y

In[21]:=
```
a1F1 = FullSimplify[Coefficient[a1, ψ1^(0,0,1)[T, X, y]] , Trig → True]
```

Out[21]=

$$\frac{1}{\left(\mu U[y]^2 + U'[y]^2\right)^{5/2}} C \mu \left(-3 \mu U[y] U'[y]^5 + 7 \mu U[y]^2 U'[y]^3 U''[y] + U'[y]^5 U''[y] + \mu^2 U[y]^5 U^{(3)}[y] + \mu U[y]^3 U'[y] \left(-3 U''[y]^2 + U'[y] U^{(3)}[y]\right)\right)$$

Second derivative of ψ 1in y

In[22]:=
```
a1F2 = FullSimplify[Coefficient[a1, ψ1^(0,0,2)[T, X, y]] , Trig → True]
```

Out[22]=
$$\frac{1}{\left(\mu\, U[y]^2 + U'[y]^2\right)^{5/2}} C \left(3\mu^3 U[y]^5 U''[y] - 3\mu^2 U[y]^3 U'[y]^2 U''[y] + 5\mu U[y]^2 U'[y]^3 \left(\mu U'[y] + U^{(3)}[y]\right) + 2 U'[y]^5 \left(\mu U'[y] + U^{(3)}[y]\right) + 3\mu^2 U[y]^4 \left(U''[y]^2 + U'[y] U^{(3)}[y]\right)\right)$$

Third derivative of ψ 1in y

In[23]:=
```
a1F3 = Factor[Simplify[Coefficient[a1, ψ1^(0,0,3)[T, X, y]] ] ]
```

Out[23]=
$$\frac{C\, U'[y] \left(3\mu^2 U[y]^3 + \mu U[y] U'[y]^2 + 6\mu U[y]^2 U''[y] + 4 U'[y]^2 U''[y]\right)}{\left(\mu\, U[y]^2 + U'[y]^2\right)^{3/2}}$$

Fourth derivative of ψ 1in y

In[24]:=
```
a1F4 = Factor[Simplify[Coefficient[a1, ψ1^(0,0,4)[T, X, y]]]]
```

Out[24]=
$$\frac{C\left(\mu\, U[y]^2 + 2 U'[y]^2\right)}{\sqrt{\mu\, U[y]^2 + U'[y]^2}}$$

Complete form of ℒ and verification against the terms extracted from a1

In[25]:=
$$\mathcal{L}[f\_] := C\, \partial_{y,y}\left(\frac{\left(\mu\, U[y]^2 + 2 U'[y]^2\right)}{\sqrt{\mu\, U[y]^2 + U'[y]^2}}\partial_{y,y} f\right) + C\, \mu\, \partial_{y,y}\left(\frac{U[y]\, U'[y]}{\sqrt{\mu\, U[y]^2 + U'[y]^2}}\partial_{y} f\right)$$

In[26]:=
```
Simplify[ℒ[f[y]] - a1F1 f^(1)[y] - a1F2 f^(2)[y] - a1F3 f^(3)[y] - a1F4 f^(4)[y], Trig -> True]
```

Out[26]=
```
0
```

Extraction of second member: a1a

The second member contains only the first derivative in X of the Oth order solution

In[27]:=
```
a1a = -Simplify[Coefficient[a1, φ0^(0,1)[T, X]]]
```

Out[27]=
```
-U''[y]
```

Reconstruction of the first order equation

We check here that the first order equation is $\mathcal{L}[\psi 1[T, X, y]] = -U''[y]\, \varphi 0^{(0,1)}[T, X]$

In[28]:=
```
a1N = ℒ[ψ1[T, X, y]] - a1a φ0^(0,1)[T, X];
```

In[29]:=
```
Simplify[a1N - a1]
```

Out[29]=
```
0
```

Symbolic solution

This solution is meant to replace ψ 1in further formal steps of the perturbative expansion

In[33]:=
```
Rule1 = ψ1 → (f1[#3] φ0^(0,1)[#1, #2] &);
```

Notice that f1 has SinI symmetry

Numerical solution [TO BE ADAPTED]

Order 2

Extract order 2 terms from the general expansion, use formal solution to order 0 and 1 equations and perform algebraic simplication.
Need to be put under the form ℒ [ψ2[T, X, y]] = second member

In[34]:=
```
a2 = Simplify[Coefficient[NS2, ϵ, 2]  //. {Rule0, Rule1} ];
```

Extraction of second member and solvability

Here we extract the terms of the second member, factor per factor, and we test the solvability condition

term in factor of $\varphi 0^{(0,1)}[T, X]^2$: a2a

In[35]:=
```
a2a = Collect[-Simplify[Coefficient[a2 , φ0^(0,1)[T, X], 2] ] , {A, D, C}];
```

Check that this simplification can be somewhat improved for the term which is not in factor of C

In[36]:=
```
Simplify[a2a /. C → 0]
```

Out[36]=
```
-f1^(3)[y]
```

This term integrates twice in y and satisfies the solvability condition

In[37]:=
```
a2a1 = Coefficient[a2a, C];
```

a2a1I = Integrate[a2a1, y];
a2a1I2 = Integrate[a2aI, y]

In[38]:=
```
a2a1I2 = -1/2 ((2 μ^2 U[y]^3 f1′[y] f1″[y] + U′[y]^3 (μ + μ f1′[y]^2 + 2 f1″[y]^2) +
      μ U[y]^2 U′[y] (μ + 3 f1″[y]^2)) / (μ U[y]^2 + U′[y]^2)^(3/2));
Simplify[D[a2a1I2, {y, 2}] - a2a1]
```

Out[39]= 0

term in factor of $\varphi 0^{(0,2)}$[T, X] : a2b

In[40]:=
```
a2b = -Collect[Simplify[Coefficient[a2 , φ0^(0,2)[T, X], 1]] , {A, D, C }]
```

Split it in several part for integration. First isolate the part that does not depend on C and D and integrate it once.

In[41]:=
```
a2b0 = a2b /. {D → 0 , C → 0};
a2b1I = Integrate[a2b0, y]
```

Out[42]= U[y] f1′[y] - f1[y] U′[y]

Then isolate the factor of C and checks it integrates twice

In[43]:=
```
a2b1 = Coefficient[a2b, C];
```

a2b2I2 = Integrate[Integrate[a2b1, y], y]

In[44]:=
```
a2b2I2 = (μ U[y]^2 + 2 U′[y]^2) / Sqrt[μ U[y]^2 + U′[y]^2];
Simplify[D[a2b2I2, {y, 2}] - a2b1]
```

Out[45]= 0

Isolate the factor of D and integrate it once

In[46]:=
```
a2b2 = Coefficient[a2b , D];
a2b0I = Integrate[a2b2, y]
```

Out[47]= U′[y] f1″[y] - f1′[y] U″[y]

Complete the solvability proof for a2b by check that nothing has been forgotten in a2b

In[48]:=
```
Simplify[a2b - a2b0 - C a2b1 - D a2b2]
```

Out[48]= 0

term in factor of $\varphi0^{(1,1)}$[T, X] : a2d

In[56]:=
```
a2d = -Simplify[Coefficient[a2 , φ0^(1,1)[T, X], 1]]
```

Out[56]=
```
f1''[y]
```

Reconstruction of the second order equation

Check that nothing has been forgotten in the extraction of the components of a2

In[70]:=
```
a2N = ℒ[ψ2[T, X, y]] - a2b φ0^(0,2)[T, X] - a2a φ0^(0,1)[T, X]^2 - a2d φ0^(1,1)[T, X];
Simplify[a2N - a2 ]
```

Out[71]=
```
0
```

Symbolic solution

In[59]:=
```
Rule2 =  ψ2  → (f2a[#3] φ0^(0,1)[#1, #2]^2 + f2b[#3] φ0^(0,2)[#1, #2] + f2d φ0^(1,1)[T, X] &)
```

Out[59]=
```
ψ2 → (f2a[#3] φ0^(0,1)[#1, #2]^2 + f2b[#3] φ0^(0,2)[#1, #2] + f2d φ0^(1,1)[T, X] &)
```

f2a has CosI symmetry
f2b has CosP symmetry
f2d has SinI symmetry

Order 3

Extract order 3 terms from the general expansion, use formal solution to order 0, 1 and 2 equations and perform algebraic simplication.
Need to be put under the form ℒ[ψ3[T, X, y]] = second member

No attemps to simplify here

In[60]:=
```
a3 =  Coefficient[NS2, ϵ, 3]  //. {Rule0, Rule1, Rule2} ;
```

Extraction of second member and solvability

First reorganize a3 in order to make further extraction easier

In[61]:=
```
a31 = Collect[Expand[a3], {φ0^(1,1)[T, X], φ0^(0,1)[T, X],  φ0^(0,2)[T, X], φ0^(0,3)[T, X]}];
```

Term in $\varphi0^{(0,1)}$[T, X]^3 : a3a

Terms in factor of $\varphi0^{(0,1)}$[T, X]

Term in $\varphi0^{(0,1)}$[T, X] $\varphi0^{(0,2)}$[T, X] : a3b

Term in $\varphi 0^{(0,3)}$[T, X] : a3d [S]

In[81]:=
```
a3d = Collect[Coefficient[a31 , φ0^(0,3)[T, X]] , {D, C }];
```

Extract the contribution that does not contain C

In[82]:=
```
a3d0 = a3d /. C → 0
```

Out[82]= $-U[y] - U[y]\, f2b''[y] + f2b[y]\, U''[y] + D\left(-U'[y]\, f2b^{(3)}[y] + f2b'[y]\, U^{(3)}[y]\right)$

In[83]:=
```
Integrate[a3d0 + U[y], y]
```

Out[83]= $-U[y]\, f2b'[y] + U'[y]\left(f2b[y] - D\, f2b''[y]\right) + D\, f2b'[y]\, U''[y]$

Coefficient in front of C in a3d

In[84]:=
```
a3d2 = Improve[ Coefficient[a3d, C] ];
```

Mathematica does not integrate this expression but after a few manipulations we get an expression for a3d2 as the sum of two derivatives plus a residual term.
The residual term does not vanish in general in the absence of symmetry but we notice that it
is in factor of μ and hence disappears in the $\mu \to 0$ limit

In[85]:=

$$a3d2P = D\left[\frac{2\,\mu\left(\mu\, U[y]^3 + U[y]\, U'[y]^2\right)^2}{\left(\mu\, U[y]^2 + U'[y]^2\right)^{5/2}}\, f1'[y],\, y\right] - D\left[\frac{\mu\, U[y]^2 + 2\, U'[y]^2}{\sqrt{\mu\, U[y]^2 + U'[y]^2}},\, \{y, 2\}\right] f1[y] - \frac{\mu\, U[y]\, f1'[y]\, U'[y]}{\sqrt{\mu\, U[y]^2 + U'[y]^2}};$$

```
Simplify[a3d2P - a3d2]
```

Out[86]= 0

Term in $\varphi 0^{(1,1)}$[T, X] $\varphi 0^{(0,1)}$[T, X] : a3g

In[89]:=
```
a3g = Coefficient[a33, φ0^(1,1)[T, X]]
```

Out[89]= $-2\, f2a''[y]$

Term in $\varphi 0^{(1,2)}$[T, X] : a3h

In[92]:=
```
a3h = Coefficient[a31, φ0^(1,2)[T, X]]
```

Out[92]= $-1 - f2b''[y]$

This term generates a contribution to the amplitude equation

Amplitude equation

The amplitude equation at this stage is

$$\partial_T \varphi 0 + < \frac{\mu\, U[y]\; f1'[y]\; U'[y]}{\sqrt{\mu\, U[y]^2 + U'[y]^2}} > \partial_X \varphi 0 = 0$$

It is purely propagative and hence cannot produce an instability per se. It is still necessary to go to higer order. Also we see that the propagtive effect is proportional to μ that we want to be small.

Verification

Assemble all the contribution found at order 3 plus the linear term in ψ 3,and check the sum against a3

In[95]:=
```
a3N = a3g φ0^(0,1)[T, X] φ0^(1,1)[T, X] + a3d φ0^(0,3)[T, X] +
   a3b φ0^(0,1)[T, X] φ0^(0,2)[T, X] + a3a φ0^(0,1)[T, X]^3 + a3h φ0^(1,2)[T, X] + ℒ[ψ3[T, X, y]];
Simplify[Together[Expand[a3N - a3 ]] ]
```

Out[96]=
```
0
```

Symbolic solution

In[97]:=
```
Rule3 = ψ3 → (f3g[#3] φ0^(1,1)[T, X] + f3d[#3] φ0^(0,3)[T, X] +
      f3b[#3] φ0^(0,1)[T, X] φ0^(0,2)[T, X] + f3a[#3] φ0^(0,1)[T, X]^3 + f3h φ0^(1,2)[T, X] &);
```

f3a has SinI symmetry
f3b has SinP symmetry
f3d has SinI symmetry
f3g has SinI symmetry
f3h has CosP symmetry

Order 4

Extract order 4 terms from the general expansion, use formal solution to order 0, 1, 2 and 3 equations and perform algebraic simplication.
Need to be put under the form $\mathcal{L}[\psi 4[T, X, y]]$ = second member

No attemps to simplify here

In[98]:=
```
a4 =
  Assuming[A > 0 && Θ'[y] > 0, Coefficient[NS2, ϵ, 4]  //. {Rule0, Rule1, Rule2, Rule3} ];
```

Rearrange a4 to prepair extraction

In[99]:=
```
a41 = Collect[Expand[a4], {φ0^(1,1)[T, X], φ0^(1,2)[T, X],
    φ0^(0,1)[T, X], φ0^(0,2)[T, X], φ0^(0,3)[T, X], , φ0^(0,4)[T, X], φ1^(0,1)[T, X]}];
```

Term in $\varphi 0^{(0,1)}[T, X]^4$: a4a

Terms in $\varphi 0^{(0,1)}[T, X]^2$

Term in $\varphi 0^{(0,1)}[T, X]^2\ \varphi 0^{(0,2)}[T, X]$: a4c

Terms in $\varphi 0^{(0,1)}[T, X]$

Term in $\varphi 0^{(0,1)}[T, X]\ \varphi 0^{(0,3)}[T, X]$: a4g [S]

In[173]:=
```
a4g = Collect[Coefficient[a43, φ0^(0,3)[T, X]] , {D, C}];
```

Contribution that does not contain C.
Can be integrated and satisfies the solvability condition.

In[174]:=
```
a4g0 = a4g /. C → 0;
```

In[175]:=
```
a4gOI = Integrate[a4g0, y]
Solva[a4gOI]
```

Out[175]=
```
f2b[y] f1''[y] - D f1''[y] f2b''[y] + f3d''[y] + f2b'[y] (-f1'[y] + D f1^(3)[y])
```

Out[176]=
```
0
```

Get the rest of a4g

In[185]:=
```
a4g1 = Coefficient[a4g, C];
a4gl = CoefficientList[ Coefficient[a4g, C], A];
Simplify[a4g - a4g0 - C a4g1]
```

Out[187]=
```
0
```

This contribution cannot be integrated but, after some manipulations, it is possible to extract an integrable part on which the solvability condition can be tested and a non integrable part for which the solvability condition is obtained due to the symmetries

In[188]:=
```
a4g1 = Improve[a4g1];
```

In[189]:=

$$a4g1Pa = -\frac{\mu\, U'[y]}{\sqrt{\mu\, U[y]^2 + U'[y]^2}} - \frac{U'[y]\left(3\,\mu\, U[y]^2 + 2\, U'[y]^2\right) f1''[y]^2}{\left(\mu\, U[y]^2 + U'[y]^2\right)^{3/2}} - \frac{\mu\, f1'[y]^2\, U'[y]}{\left(\mu\, U[y]^2 + U'[y]^2\right)^{5/2}}\left(-2\,\mu\, U[y]^2\, U'[y]^2 + U'[y]^4 + 3\,\mu\, U[y]^3\, U''[y]\right) + \frac{(2\, U'[y]\,(3\,\mu\, U[y] + 4\, U''[y]))}{\sqrt{\mu\, U[y]^2 + U'[y]^2}}\, f2a'[y] + \frac{6\,\mu\, U[y]^2 + 4\, U'[y]^2}{\sqrt{\mu\, U[y]^2 + U'[y]^2}}\, f2a''[y];$$

$$a4g1Pb = \frac{\mu\, U[y]\, f1'[y]^2\left(3\,\mu\, U[y]^2 + 6\, U'[y]^2 - 2\, U[y]\, U''[y]\right)}{\left(\mu\, U[y]^2 + U'[y]^2\right)^{3/2}};$$

$$a4g1Pc = \Big(\mu\, f2b[y]\, U'[y]\left(\mu\, U[y]^2 + U'[y]^2\right) + 2\, f1'[y]^2\, U'[y]\left(\mu\, U[y]^2 + U'[y]^2\right) - f1[y]\Big(\mu^2\, U[y]^3\, f1'[y] + U'[y]\left(3\,\mu\, U[y]^2 + 2\, U'[y]^2\right) f1''[y]\Big) + \mu\, f1'[y]\left(f3d'[y]\, U'[y]^3 + \mu\, U[y]^3\, f3d''[y]\right) + f1''[y]\Big(\mu^2\, U[y]^3\, f3d'[y] + U'[y]\left(3\,\mu\, U[y]^2 + 2\, U'[y]^2\right) f3d''[y]\Big)\Big) \Big/ \left(\mu\, U[y]^2 + U'[y]^2\right)^{3/2} - \frac{2\left(\mu\, U[y]^2 + 2\, U'[y]^2\right)}{\sqrt{\mu\, U[y]^2 + U'[y]^2}}\, f2a[y];$$

```
a4g1P = a4g1Pa + D[a4g1Pb, y] + D[a4g1Pc, {y, 2}];
Simplify[a4g1 - a4g1P]
```

Out[193]= 0

In general and in the absence of symmetries,
the term a4g1Pa does not integrate to zero over $[0, 2\pi]$,
hence it contributes to the amplitude equation. In the limit $\mu \to 0$, this term is

$$-\frac{2\, U'[y]^3\, f1''[y]^2}{\left(U'[y]^2\right)^{3/2}} + 4\sqrt{U'[y]^2}\; f2a''[y] + \frac{8\, f2a'[y]\, U'[y]\, U''[y]}{\sqrt{U'[y]^2}}$$

Term in $\varphi0^{(0,1)}[T, X]\,\varphi0^{(1,1)}[T, X]$: a4h

Term in $\varphi0^{(0,4)}[T, X]$: a4l (contributes to the amplitude equation)

In[127]:=

$$a4l = Collect\left[Coefficient\left[a41,\ \varphi0^{(0,4)}[T, X]\right], \{D, C\}\right];$$

Contribution independent of C.
The two terms of a4l0 are of symmetry CosP, they have no primitive and provide a contribution to the amplitude equation.

In[128]:=

```
a4l0 = a4l /. C → 0
```

Out[128]= $-f1[y]\, U[y] - D\, f1'[y]\, U'[y]$

Now get the coefficient of C. It has no primitive but can be described as a sum of derivatives and residual terms.

In[129]:=

```
a4l1 = Improve[Coefficient[a4l, C]];
```

In[130]:=

$$a4l1Pa = \frac{\mu\,U[y]^2 + 2\,U'[y]^2}{\sqrt{\mu\,U[y]^2 + U'[y]^2}};$$

$$a4l1Pb = \frac{2\,\mu\,U[y]^2}{\sqrt{\mu\,U[y]^2 + U'[y]^2}}\,f2b[y];$$

$$a4l1Pc = \frac{\mu\,U[y]\,U'[y]\,\left(3\,\mu\,U[y]^2 + 5\,U'[y]^2 - 2\,U[y]\,U''[y]\right)}{\left(\mu\,U[y]^2 + U'[y]^2\right)^{3/2}}\,f2b[y];$$

$$a4l1Pd = \frac{-\mu\,U[y]\,U'[y]^3 + 3\,\mu\,U[y]^2\,U'[y]\,U''[y] + 2\,U'[y]^3\,U''[y]}{\left(\mu\,U[y]^2 + U'[y]^2\right)^{3/2}};$$

```
a4l1P = a4l1Pa + D[a4l1Pb, {y, 2}] - D[a4l1Pc, y] - f2b[y] D[a4l1Pd, y];
Simplify[a4l1 - a4l1P]
```

Out[135]= 0

The first and the fourth term contribute as they are product of trig functions with same symmetry.

Check that nothing has been forgotten

In[136]:=

```
Simplify[a4l - a4l0 - C a4l1 ]
```

Out[136]= 0

Term in $\varphi 0^{(1,2)}[T, X]$: a4n

In[137]:=

$$a4n = Collect\left[Coefficient\left[a41,\ \varphi 0^{(1,2)}[T, X]\right], \{D\}\right]$$

Out[137]= 0

No contribution of that sort

Term in $\varphi 0^{(0,2)}[T, X]^2$: a4o [S]

In[138]:=

$$a4o = Collect\left[Coefficient\left[a41,\ \varphi 0^{(0,2)}[T, X]^2\right], \{D, C\}\right];$$

The contribution independent of C has a primitive and satisfies the solvability condition

In[139]:=

```
a4o0 = a4o /. C → 0;
a4o0I = Integrate[a4o0, y]
Solva[a4oI0]
```

Out[140]=

$$-f1'[y]\,f2b'[y] + D\,f1''[y] + \left(f1[y] - D\,f1''[y]\right)\,f2b''[y] + D\,f1'[y]\,f2b^{(3)}[y]$$

Out[141]= 0

Expand the rest of a4o in power of A

In[142]:=
```
a4o1 = Coefficient[a4o, C];
Simplify[a4o - a4o0 - C a4o1]
```

Out[143]= 0

The term in C cannot be integrated but it can be sorted in several parts

In[144]:=

$$a4o1aP = D\Big[\Big(\frac{\mu}{2}\,f1[y]^2 + 4\,f1'[y]^2\Big)\,\frac{U'[y]}{\sqrt{\mu\,U[y]^2 + U'[y]^2}} - \frac{2\,(\mu\,U[y]^2 + 2\,U'[y]^2)}{\sqrt{\mu\,U[y]^2 + U'[y]^2}}\,f2a[y] + \Big(\mu\,f2b'[y]^2\,U'[y]^3 + 2\,\mu^2\,U[y]^3\,f2b'[y]\,(-1 + f2b''[y]) + U'[y]\,(3\,\mu\,U[y]^2 + 2\,U'[y]^2)\,(-2 + f2b''[y])\,f2b''[y]\Big) \Big/ \Big(2\,(\mu\,U[y]^2 + U'[y]^2)^{3/2}\Big),\ \{y, 2\}\Big] + D\Big[\mu\,U[y]\,(6\,U'[y]^2 + U[y]\,(3\,\mu\,U[y] - 2\,U''[y])) \Big/ (\mu\,U[y]^2 + U'[y]^2)^{3/2}\,f1'[y]^2,\ y\Big];$$

$$a4o1bP = -\Big(\mu\,U'[y]\,(-2\,\mu\,U[y]^2\,U'[y]^2 + U'[y]^4 + 3\,\mu\,U[y]^3\,U''[y])\Big) \Big/ (\mu\,U[y]^2 + U'[y]^2)^{5/2}\,f1'[y]^2 - U'[y]\,(3\,\mu^2\,U[y]^4 + 5\,\mu\,U[y]^2\,U'[y]^2 + 2\,U'[y]^4) \Big/ \Big((\mu\,U[y]^2 + U'[y]^2)^{5/2}\Big)\,f1''[y]^2 + \Big(4\,D\big[U'[y]^2\,f2a'[y],\ y\big] + 6\,\mu\,U[y]\,D[U[y]\,f2a'[y],\ y]\Big) \Big/ \Big(\sqrt{\mu\,U[y]^2 + U'[y]^2}\Big);$$

It is easy to check that all the terms which are not under the derivative in a4o1bP are with CosI symmetry and integrate to zero

In[146]:=

$$a4o1c = \Big(-2\,\mu\,U'[y]^7 + 3\,\mu^2\,U[y]^2\,U'[y]\,\big(-5\,U'[y]^4 + 7\,U[y]\,U'[y]^2\,U''[y] - 5\,U[y]^2\,U''[y]^2 + U[y]^2\,U'[y]\,U^{(3)}[y]\big) + 3\,U[y]^5\,\mu^3\,(-3\,U'[y]\,U''[y] + U[y]\,U^{(3)}[y]) - 2\,\mu^4\,U[y]^6\,U'[y]\Big) \Big/ \Big(2\,(\mu\,U[y]^2 + U'[y]^2)^{7/2}\Big);$$

The numerator is a sum of terms with CosI symmetry, which is preserved by the division by the denominator in CosP, integrating a4o1c to zero. Let us check it numerically for some examples.

In[153]:=
```
NIntegrate[Evaluate[a4o1c  /. {Flow1, μ → 0.5}], {y, 0, 2π}]
NIntegrate[Evaluate[a4o1c  /. {Flow2, μ → 0.5}], {y, 0, 2π}]
NIntegrate[Evaluate[a4o1c  /. {Flow3, μ → 0.5}], {y, 0, 2π}]
```

Out[153]= $0. \times 10^{-13}$

Out[154]= $0. \times 10^{-15}$

Out[155]= $0. \times 10^{-15}$

Chek that a4o1 is reconstructed

```
Simplify[a4o1 - a4o1aP - a4o1bP - a4o1c]
```

```
0
```

Term in $\varphi 0^{(1,3)}[T, X]$: a4p

In[182]:=
```
a4p = Coefficient[a41, φ0^(1,3)[T, X]]
```

Out[182]=
```
-f1[y]
```

Verification

$$a4NN = a4p\,\varphi 0^{(1,3)}[T,X] + a4o\,\varphi 0^{(0,2)}[T,X]^2 + a4l\,\varphi 0^{(0,4)}[T,X] + \varphi 0^{(0,1)}[T,X]\left(a4g\,\varphi 0^{(0,3)}[T,X] + a4h\,\varphi 0^{(1,1)}[T,X]\right) + \varphi 0^{(0,1)}[T,X]^2\left(a4c\,\varphi 0^{(0,2)}[T,X]\right) + a4a\,\varphi 0^{(0,1)}[T,X]^4 + \mathcal{L}[\psi 4[T,X,y]];$$

```
Simplify[Expand[a4NN - a41] ]
```

Out[206]=
```
0
```

Analysis of the contributions to the amplitude equation

The contribution to the amplitude equation which have been identified at order 4 are

$C < a4o1c + a4o1bP > \varphi 0^{(0,2)}[T, X]^2 + C < a4g1Pa > \varphi 0^{(0,1)}[T, X]\,\varphi 0^{(0,3)}[T, X] + < -f1[y]\,U[y] - D\,f1'[y]\,U'[y] + (a4l1Pa - f2b[y]\,D[a4l1Pd, y]) > \varphi 0^{(0,4)}[T, X]$

It is interesting to see how the first terms combine. Actually we find :

In[220]:=
$$Simplify\left[-D\left[\frac{U'[y]\left(3\,\mu\,U[y]^2 + 2\,U'[y]^2\right)}{2\left(\mu\,U[y]^2 + U'[y]^2\right)^{3/2}}, \{y, 2\}\right] - a4g1Pa + a4o1c + a4o1bP\right]$$

Out[220]=
```
0
```

This means that $< a4o1c + a4o1bP > = < a4g1Pa >$
and hence that the nonlinear terms are of the form Constant $\times \left((\partial_X \varphi 0)^2\right)$ ''

The ful amplitude equation is then (after 2 integrations in y)

$$\partial_T\,\varphi 0 + P\,\partial_X\,\varphi 0 + \epsilon\left(Q\,(\partial_X\,\varphi 0)^2 - \nu T\,\partial_{X^2}\,\varphi 0\right) = 0$$

where $P = < \dfrac{\mu\,U[y]\,f1'[y]\,U'[y]}{\sqrt{\mu\,U[y]^2 + U'[y]^2}} >$, $Q = < a4g1Pa >$,

$\nu T = - < -f1[y]\,U[y] - D\,f1'[y]\,U'[y] + (a4l1Pa - f2b[y]\,D[a4l1Pd, y]) >$

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