Large-scale linear instability of the generalized Kolmogorov flow with modified viscosity

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Smagorinsky-Clark mixed model + friction
U = A Sin[y] and μ =1
Streamfunction defined as u=$\partial_y \psi$, $v = -\partial_x \psi$

Settings

- **Last use**

```
DateList[]

{2009, 1, 14, 18, 21, 49.648320}
```

- **General**

```
Off[General::spell1]
```

- **Laplacian**

```
Δ = - (ϵ^2 ∂x,x # + ∂y,y #) &

- (ϵ^2 ∂x,x #1 + ∂y,y #1) &
```

$$\Delta = -\left(\epsilon^2 \partial_{x,x} \# + \partial_{y,y} \#\right) \&$$

$$-\left(\epsilon^2 \partial_{x,x} \#1 + \partial_{y,y} \#1\right) \&$$

- **Definition of the turbulent viscosity, Smagorinsky-Clark mixed model**

Standard version

```
S̄ = √(4 (ϵ ∂x,y # )^2 + (ϵ^2 ∂x,x # − ∂y,y #)^2 + (ϵ ∂x # )^2 + (∂y # )^2) &
```

$$\sqrt{4\,(\epsilon\,\partial_{x,y}\#1)^2 + \left(\epsilon^2\,\partial_{x,x}\#1 - \partial_{y,y}\#1\right)^2 + (\epsilon\,\partial_x\#1)^2 + (\partial_y\#1)^2}\ \&$$

Dissipation: C in front of the Smagorinsky component, D in front of the Clark component

In standard notations $C=\Gamma^2/12$ and $D = C_S \Gamma^2$

Note that K=-DS in the paper

```
DS =
 C (∂y,y (S̄[#] (−∂y,y # + ϵ^2 ∂x,x #)) − ϵ^2 ∂x,x (S̄[#] ( − ∂y,y # + ϵ^2 ∂x,x #)) −
      4 ϵ^2 ∂x,y (S̄[#] ∂x,y #)) +
    D ( ϵ ∂x,y # ( ϵ^4 ∂x,x,x,x # − ∂y,y,y,y #) +
       ( ϵ ∂x,y,y,y # + ϵ^3 ∂x,x,x,y #) ( ∂y,y # − ϵ^2 ∂x,x #) ) &
```

$$C\left(\partial_{y,y}\left(\bar{S}[\#1]\left(-\partial_{y,y}\#1 + \epsilon^2\,\partial_{x,x}\#1\right)\right) - \epsilon^2\,\partial_{x,x}\left(\bar{S}[\#1]\left(-\partial_{y,y}\#1 + \epsilon^2\,\partial_{x,x}\#1\right)\right) - 4\,\epsilon^2\,\partial_{x,y}\left(\bar{S}[\#1]\,\partial_{x,y}\#1\right)\right) + D\left(\epsilon\,\partial_{x,y}\#1\left(\epsilon^4\,\partial_{x,x,x,x}\#1 - \partial_{y,y,y,y}\#1\right) + \left(\epsilon\,\partial_{x,y,y,y}\#1 + \epsilon^3\,\partial_{x,x,x,y}\#1\right)\left(\partial_{y,y}\#1 - \epsilon^2\,\partial_{x,x}\#1\right)\right)\ \&$$

Definition of the modified Kolmogorov Flow

Velocity and streamfunction

```
U[y_] := A Sin[y]
```

```
Φ[y_] := Integrate[U[s], {s, 0, y}]
```

Plots of the flow

```
Plot[U[y] /. {A → 1}, {y, -2 π, 2 π}];
```

Forcing (or dissipation)

```
Forcing = Simplify[Assuming[A > 0 , Simplify[ DS[ Φ[y]]]]]
```

```
A^2 C Cos[y]
```

Definition of the pertubation problem

2D Navier-Stokes with scaling

We have taken u = $\partial_y\ \psi$ and $v\ =\ -\partial_x\ \psi$ (mechanical definition)

```
NS = ϵ^2 ∂_T Δ[ψ[T, X, y]] - ϵ ∂_X ψ[T, X, y] ∂_y Δ[ψ[T, X, y]] +
   ϵ ∂_y ψ[T, X, y] ∂_X Δ[ψ[T, X, y]] - DS[ψ[T, X, y]] + DS[Φ[y]];
```

Expansion

```
ψ[T_, X_, y_] := Φ[y] + ψ0[T, X, y] +
  ϵ ψ1[T, X, y] + ϵ^2 ψ2[T, X, y] + ϵ^3 ψ3[T, X, y] + ϵ^4 ψ4[T, X, y]
```

```
NS2 = Collect[Normal[Series[NS, {ϵ, 0, 4}]], ϵ];
```

Order 0

```
a0 = Coefficient[NS2, ϵ, 0];
```

Symbolic solution

```
Rule0 = ψ0 → (φ0[#1, #2] &)
```

```
ψ0 → (φ0[#1, #2] &)
```

Solvability condition

```
Assuming[{A > 0 }, FullSimplify[a0 /. {Rule0}] ]
```

```
0
```

Solvability condition

```
Solva[expr_] :=
 Simplify[((expr /. y → 2 π) - (expr /. y → 0)) /. g___[2 π] → g[0]]
```

Useful rules for simplifications

```
Rulea = (A^2 Sin[y]^2 + A^2 Cos[y]^2)^n_ → A^(2 n)
```

$(A^2 \mathrm{Cos}[y]^2 + A^2 \mathrm{Sin}[y]^2)^{n\_} \to A^{2n}$

```
Improve = Simplify[Together[#1], A > 0, Trig → True] &
```

```
Simplify[Together[#1], A > 0, Trig → True] &
```

```
Treat[expr_] :=
 Simplify[Simplify[expr /. μ → 1, Trig → True] /. θ → (# &)]
```

Order 1

```
a1 =
 Assuming[A > 0 , Simplify[Expand[Coefficient[NS2, ϵ, 1]] /. Rule0 ]]
```

$$-\frac{1}{2} A \left(2\,\mathrm{Sin}[y]\,\varphi 0^{(0,1)}[T, X] + C\left(4\,\mathrm{Sin}[2y]\,\psi 1^{(0,0,1)}[T, X, y] + 3\,\mathrm{Sin}[2y]\,\psi 1^{(0,0,3)}[T, X, y] - (3 + \mathrm{Cos}[2y])\,\psi 1^{(0,0,4)}[T, X, y]\right)\right)$$

Extraction of the linear operator and verification

```
a1F1 = FullSimplify[Coefficient[a1, ψ1^(0,0,1)[T, X, y]] , Trig → True]
```

$-2\,A\,C\,\mathrm{Sin}[2y]$

```
a1F2 = FullSimplify[Coefficient[a1, ψ1^(0,0,2)[T, X, y]] , Trig → True]
```

0

```
a1F3 = Factor[Simplify[Coefficient[a1, ψ1^(0,0,3)[T, X, y]] ] ]
```

$-3\,A\,C\,\mathrm{Cos}[y]\,\mathrm{Sin}[y]$

```
a1F4 = Factor[Simplify[Coefficient[a1, ψ1^(0,0,4)[T, X, y]]]]
```

$\frac{1}{2}\,A\,C\,(3+\mathrm{Cos}[2\,y])$

$$\mathcal{L}[f\_] := \frac{1}{2}\,A\,C\,(\partial_{y,y}\,(\,(3+\mathrm{Cos}[2\,y])\,\partial_{y,y}f\,)\,+\partial_{y,y}\,(\mathrm{Sin}[2\,y]\,\partial_y f))$$

$$\mathrm{Simplify}[\mathcal{L}[f[y]] - \mathrm{a1F1}\ f^{(1)}[y] - \mathrm{a1F2}\ f^{(2)}[y] - \mathrm{a1F3}\ f^{(3)}[y] - \mathrm{a1F4}\,f^{(4)}[y],\ \mathrm{Trig} \to \mathrm{True}]$$

0

As a reference : complete operator for the full problem

$$\mathcal{L}[f\_] := A\,C\,\partial_{y,y}\left(\frac{\left(\mu\,\mathrm{Sin}[\theta[y]]^2+2\,\mathrm{Cos}[\theta[y]]^2\,\theta'[y]^2\right)}{\sqrt{\mu\,\mathrm{Sin}[\theta[y]]^2+\mathrm{Cos}[\theta[y]]^2\,\theta'[y]^2}}\,\partial_{y,y}f\right)+$$
$$A\,C\;\mu\,\partial_{y,y}\left(\frac{\mathrm{Sin}[2\,\theta[y]]\,\theta'[y]}{2\,\sqrt{\mu\,\mathrm{Sin}[\theta[y]]^2+\mathrm{Cos}[\theta[y]]^2\,\theta'[y]^2}}\,\partial_y f\right)$$

Operator to solve problem after two integrations

$$\mathcal{L}2[f\_] := \frac{1}{2}\,A\,C\;(\,(3+\mathrm{Cos}[2\,y])\,\partial_{y,y}f\;+\mathrm{Sin}[2\,y]\,\partial_y f)$$
$$\mathcal{L}22[h\_] := \frac{1}{2}\,A\,C\;(\,(3+\mathrm{Cos}[2\,y])\,\partial_y h\;+\mathrm{Sin}[2\,y]\,h)$$

Extraction of second member

```
a1a = -Simplify[Coefficient[a1, φ0^(0,1)[T, X]]]
```

```
A Sin[y]
```

Reconstruction of the first order equation

```
a1N = ℒ[ψ1[T, X, y]] - a1a φ0^(0,1)[T, X];
```

```
Simplify[a1N - a1]
```

```
0
```

Solvability condition

The solvability condition is obviously satisfied

Symbolic solution

```
Rule1 = ψ1 → (f1[#3] φ0^(0,1)[#1, #2] &)
```

```
ψ1 → (f1[#3] φ0^(0,1)[#1, #2] &)
```

f1 has SinI symmetry

Solution of the problem

```
solf1 = f1 → (1/C Sin[#] &)
```

```
f1 → (Sin[#1]/C &)
```

```
Simplify[ℒ[f1[y]] - a1a /. solf1]
```

```
0
```

Order 2

```
a2 = Assuming[A > 0 ,
  Simplify[Coefficient[NS2, ϵ, 2]  //. {Rule0, Rule1} , Trig → True] ]
```

Extraction of second member and solvability

- **term in factor of $\varphi 0^{(0,1)}[T, X]^2$: a2a**

```
a2a = Collect[-Simplify[Coefficient[a2 , φ0^(0,1)[T, X], 2] ] , {A, D, C}]
```

$$-f1^{(3)}[y] + \frac{1}{8} C \Big(4\,Cos[y] + 3\,(Cos[y] + 3\,Cos[3\,y])\, f1'[y]^2 + (-9\,Cos[y] + Cos[3\,y])\, f1''[y]^2 + 2\,(-9\,Cos[y] + Cos[3\,y])\, f1^{(3)}[y]^2 + 2\, f1''[y]\,\big(12\,Sin[y]^3\, f1^{(3)}[y] + (-9\,Cos[y] + Cos[3\,y])\, f1^{(4)}[y]\big) + 2\, f1'[y]\,\big(12\,Sin[y]^3\, f1''[y] + (-9\,Cos[y] + 5\,Cos[3\,y])\, f1^{(3)}[y] - 4\,Sin[y]^3\, f1^{(4)}[y]\big)\Big)$$

```
a2aF = FullSimplify[a2a /. solf1]
```

$$\frac{(3 + C^2)\, Cos[y]}{2\,C}$$

Solution of the perturbation problem
(- sign because the problem is already integrated twice)

```
ss2a = Flatten[DSolve[
    {ℒ2[f2a[y]] == -a2aF, f2a[π/2] == 0, f2a'[0] == 0}, f2a[y], y]];
solf2a = f2a → Function[y, Evaluate[f2a[y] /. ss2a]]
```

$$f2a \to Function\Big[y, \frac{(3 + C^2)\, Cos[y]}{4\,A\,C^2}\Big]$$

- **term in factor of $\varphi 0^{(0,2)}$[T, X] : a2b**

```
a2b = -Collect[Simplify[Coefficient[a2 , φ0^(0,2)[T, X], 1]] , {A, D, C }]
```

$$-A\left(2\,C\,\mathrm{Cos}[2\,y]-\mathrm{f1}[y]\,\mathrm{Sin}[y]-\mathrm{Sin}[y]\,\mathrm{f1}''[y]+D\left(-\mathrm{Cos}[y]\,\mathrm{f1}'[y]-\mathrm{Cos}[y]\,\mathrm{f1}^{(3)}[y]\right)\right)$$

```
a2bF = a2b /. solf1
a2bFI2 = Integrate[Integrate[a2bF, y], y]
```

$$-2\,A\,C\,\mathrm{Cos}[2\,y]$$

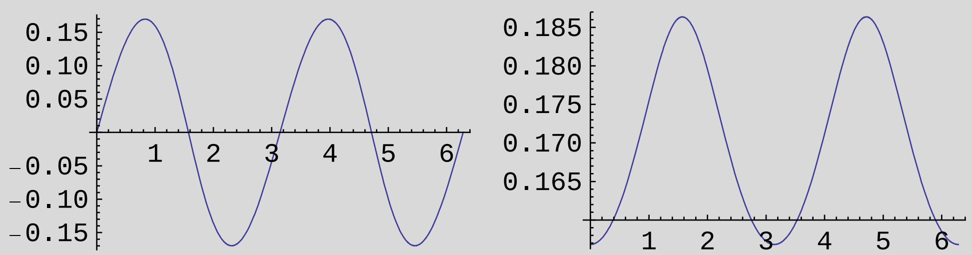
$$\frac{1}{2}\,A\,C\,\mathrm{Cos}[2\,y]$$

Solving for h2b = f2b’ introducing an integration constant K2b

```
ss2b = Flatten[DSolve[{ℒ22[h2b[y]] == a2bFI2 + K2b [y],
    K2b'[y] == 0, h2b[0] == 0, h2b[π] == 0}, {h2b[y], K2b[y]}, y]]
```

$$\left\{\mathrm{h2b}[y]\to\frac{1}{4\,\mathrm{EllipticE}\left[\frac{1}{2}\right]}\left(2\sqrt{3+\mathrm{Cos}[2\,y]}\,\mathrm{EllipticE}\left[\frac{1}{2}\right]\mathrm{EllipticF}\left[y,\frac{1}{2}\right]-2\sqrt{3+\mathrm{Cos}[2\,y]}\,\mathrm{EllipticE}\left[y,\frac{1}{2}\right]\mathrm{EllipticK}\left[\frac{1}{2}\right]+\mathrm{EllipticK}\left[\frac{1}{2}\right]\mathrm{Sin}[2\,y]\right),\right.$$
$$\left.\mathrm{K2b}[y]\to\frac{3\,A\,C\,\mathrm{EllipticE}\left[\frac{1}{2}\right]-2\,A\,C\,\mathrm{EllipticK}\left[\frac{1}{2}\right]}{2\,\mathrm{EllipticE}\left[\frac{1}{2}\right]}\right\}$$

```
p21 = Plot[h2b[y] /. ss2b, {y, 0, 2 π}, DisplayFunction → Identity];
p22 = Plot[h2b[y] / Sin[2 y] /. ss2b,
   {y, 0, 2 π}, DisplayFunction → Identity];
Show[GraphicsArray[{p21, p22}]]
```



Reconstruction and check of the second order equation

```
a2N = ℒ[ψ2[T, X, y]] - a2b φ0^(0,2)[T, X] - a2a φ0^(0,1)[T, X]^2;
```

```
Simplify[a2N - a2 ]
```

```
0
```

Symbolic solution

```
Rule2 = ψ2 → (f2a[#3] φ0^(0,1)[#1, #2]^2 + f2b[#3] φ0^(0,2)[#1, #2] &)
```

```
ψ2 → (f2a[#3] φ0^(0,1)[#1, #2]^2 + f2b[#3] φ0^(0,2)[#1, #2] &)
```

f2a has CosI symmetry
f2b has CosP symmetry
f2c has SinI symmetry

```
solf2 = Flatten[{solf2a}]
```

$$\left\{\text{f2a} \to \text{Function}\left[y, \frac{(3 + C^2)\,\text{Cos}[y]}{4\,A\,C^2}\right]\right\}$$

Order 3

```
a3 = Coefficient[NS2, ϵ, 3]  //. {Rule0, Rule1, Rule2};
```

Extraction of second member and solvability

```
a31 = Collect[Expand[a3],
   {φ0^(1,1)[T, X], φ0^(0,1)[T, X], φ0^(0,2)[T, X], φ0^(0,3)[T, X]}];
```

- Term in $\varphi0^{(0,1)}[T, X]^3$: a3a

```
a3a = Collect[Coefficient[-a31 , φ0^(0,1)[T, X]^3] , {C, A}] ;
a3aF = Assuming[A > 0, Simplify[a3a /. Flatten[{solf1, solf2}]]]
```

$$-\frac{\left(1+C^2\right)\,\mathrm{Sin}[y]}{2\,A\,C^2}$$

```
ss3a = Flatten[DSolve[
    {ℒ2[f3a[y]] == -a3aF, f3a'[π/2] == 0, f3a[0] == 0}, f3a[y], y]];
solf3a = f3a → Function[y, Evaluate[f3a[y] /. ss3a]]
```

$$\mathrm{f3a}\to\mathrm{Function}\left[y,\,-\frac{\left(1+C^2\right)\,\mathrm{Sin}[y]}{2\,A^2\,C^3}\right]$$

- Terms in factor of $\varphi0^{(0,1)}[T, X]$

```
a33 = Coefficient[Expand[a31], φ0^(0,1)[T, X]];
```

- Term in $\varphi0^{(0,1)}[T, X]\,\varphi0^{(0,2)}[T, X]$: a3b

```
a3b = Assuming[A > 0,
   Collect[Coefficient[-a33 , φ0^(0,2)[T, X]] , {D, C, A }]];
```

Solvability is satisfied since all contributions can be integrated twice

```
a3bF = Assuming[A > 0, Simplify[a3b /. solf1 /. solf2]]
a3bI2 = -1/2 (2 Cos[2 y] f2b'[y] + Sin[2 y] (3 - 2 f2b''[y])) - f2b'[y];
Simplify[D[a3bI2, {y, 2}] - a3bF]
```

$$4\,\mathrm{Cos}[2\,y]\,\mathrm{f2b}'[y]+(-1+3\,\mathrm{Cos}[2\,y])\,\mathrm{f2b}^{(3)}[y]+\mathrm{Sin}[2\,y]\left(6+\mathrm{f2b}^{(4)}[y]\right)$$

```
0
```

- **Term in $\varphi 0^{(0,3)}$[T, X] : a3d**

```
a3d = Collect[Coefficient[-a31 , φ0^(0,3)[T, X]] , {D, A, C }];
```

```
a3dF = Assuming[A > 0, Simplify[a3d /. solf1 /. solf2]]
```

$$\frac{1}{4}\,A\,\big(11\,\mathrm{Sin}[y]+4\,\mathrm{f2b}[y]\,\mathrm{Sin}[y]-9\,\mathrm{Sin}[3\,y]+4\,D\,\mathrm{Cos}[y]\,\mathrm{f2b}'[y]+4\,\mathrm{Sin}[y]\,\mathrm{f2b}''[y]+4\,D\,\mathrm{Cos}[y]\,\mathrm{f2b}^{(3)}[y]\big)$$

Check the first integral

```
a3dI = -11/4 A Cos[y] + 3/4 A Cos[3 y] - A Cos[y] f2b[y] +
   A Sin[y] f2b'[y] + A D (Sin[y] f2b'[y] + Cos[y] f2b''[y]);
Simplify[D[a3dI, y] - a3dF]
```

```
0
```

- **Term in $\varphi 0^{(1,1)}$[T, X] : a3g**

```
a3g = Coefficient[-a31, φ0^(1,1)[T, X]]
```

$$\mathrm{f1}''[y]$$

```
a3gF = a3g /. solf1
```

$$-\frac{\mathrm{Sin}[y]}{C}$$

```
ss3g = Flatten[DSolve[
    {ℒ2[f3g[y]] == -a3gF, f3g'[π/2] == 0, f3g[0] == 0}, f3g[y], y]];
solf3g = f3g → Function[y, Evaluate[f3g[y] /. ss3g]]

f3g → Function[y, -Sin[y]/(A C^2)]
```

- **Verification**

```
a3N = -a3g φ0^(1,1)[T, X] - a3d φ0^(0,3)[T, X] -
   a3b φ0^(0,1)[T, X] φ0^(0,2)[T, X] - a3a φ0^(0,1)[T, X]^3 + ℒ[ψ3[T, X, y]];
Assuming[A > 0, Simplify[Together[Expand[a3N - a3 ]] ]]

0
```

Solution

```
Rule3 = ψ3 → (f3g[#3] φ0^(1,1)[T, X] + f3d[#3] φ0^(0,3)[T, X] +
      f3b[#3] φ0^(0,1)[T, X] φ0^(0,2)[T, X] + f3a[#3] φ0^(0,1)[T, X]^3 &);
```

f3a has SinI symmetry
f3b has SinP symmetry
f3c has CosI symmetry
f3d has SinI symmetry
f3e has CosP symmetry
f3f has SinI symmetry
f3g has SinI symmetry
f3h has SinI symmetry

```
solf3 = Flatten[{solf3a, solf3g}]

{f3a → Function[y, -((1 + C^2) Sin[y])/(2 A^2 C^3)], f3g → Function[y, -Sin[y]/(A C^2)]}
```

Order 4

```
a4 = Assuming[A > 0 ,
   Coefficient[NS2, ϵ, 4]  //. {Rule0, Rule1, Rule2, Rule3} ];
a41 = Collect[Expand[a4], {φ0^(1,1)[T, X], φ0^(1,2)[T, X],
    φ0^(0,1)[T, X], φ0^(0,2)[T, X], φ0^(0,3)[T, X], , φ0^(0,4)[T, X]}];
```

- **Term in $\varphi 0^{(0,1)}[T, X]^4$: a4a**

```
a4a = Collect[Coefficient[-a41, φ0^(0,1)[T, X]^4] , {A, D, C}];
Assuming[A > 0, Simplify[a4a /. solf1 /. solf2 /. solf3]]
```

$$-\frac{\left(9+14\,C^2+C^4\right)\,\mathrm{Cos}[y]}{16\,A^2\,C^3}$$

Solvability is satisfied for this term

- **Terms in $\varphi 0^{(0,1)}[T, X]^2$**

```
a42 = Coefficient[a41, φ0^(0,1)[T, X]^2];
```

- **Term in $\varphi 0^{(0,1)}[T, X]^2\,\varphi 0^{(0,2)}[T, X]$: a4c**

```
a4c = Collect[Coefficient[-a42, φ0^(0,2)[T, X]] /. Rulea, {D, A, C }];
a4cF = Assuming[A > 0, Simplify[a4c /. solf1 /. solf2 /. solf3]]
a4cFFinal = + 1/(2 A C) (19 + 5 C^2) Cos[2 y] - (3 + C^2)/(8 A C)
    ((12 f2b′[y] + 9 f2b^(3)[y]) Sin[2 y] - (1 + 3 Cos[2 y]) f2b^(4)[y]) +
   4 Cos[2 y] f3b′[y] - (1 - 3 Cos[2 y]) f3b^(3)[y] + Sin[2 y] f3b^(4)[y];
Simplify[a4cF - a4cFFinal /. {solf2a, solf1}]
```

$$\frac{1}{8\,A\,C}\Big(76\,\mathrm{Cos}[2\,y]+20\,C^2\,\mathrm{Cos}[2\,y]-12\,\left(3+C^2\right)\,\mathrm{Sin}[2\,y]\,\mathrm{f2b}'[y]+32\,A\,C\,\mathrm{Cos}[2\,y]\,\mathrm{f3b}'[y]-27\,\mathrm{Sin}[2\,y]\,\mathrm{f2b}^{(3)}[y]-9\,C^2\,\mathrm{Sin}[2\,y]\,\mathrm{f2b}^{(3)}[y]-8\,A\,C\,\mathrm{f3b}^{(3)}[y]+24\,A\,C\,\mathrm{Cos}[2\,y]\,\mathrm{f3b}^{(3)}[y]+3\,\mathrm{f2b}^{(4)}[y]+C^2\,\mathrm{f2b}^{(4)}[y]+9\,\mathrm{Cos}[2\,y]\,\mathrm{f2b}^{(4)}[y]+3\,C^2\,\mathrm{Cos}[2\,y]\,\mathrm{f2b}^{(4)}[y]+8\,A\,C\,\mathrm{Sin}[2\,y]\,\mathrm{f3b}^{(4)}[y]\Big)$$

```
0
```

Check that the contribution can be integrated twice

```
a4cI2 = - 1/(8 A C) (19 + 5 C^2) Cos[2 y] -
   2 Cos[y] (Cos[y] f3b′[y] - Sin[y ] f3b″[y]) +
   (3 + C^2)/(8 A C) (3 Sin[2 y] f2b′[y] + (3 Cos[2 y] + 1) f2b″[y]);
Simplify[D[a4cI2, {y, 2}] - a4c /. {solf2a, solf1} ]

0
```

No contribution from this part

Hence, there is no term like $\varphi 0^{(0,1)}[T, X]^2\, \varphi 0^{(0,2)}[T, X]$ in the amplitude equation. Finding this term in previous versions of this calculation was a spurious consequence of the singularities induced by vanishing viscosity at the center of the jet.

- **Terms in $\varphi 0^{(0,1)}[T, X]$**

```
a43 = Coefficient[a41, φ0^(0,1)[T, X]];
```

- **Term in $\varphi 0^{(0,1)}[T, X]\, \varphi 0^{(0,3)}[T, X]$: a4g**

```
a4g = Collect[Coefficient[-a43, φ0^(0,3)[T, X]] /. Rulea, {A, D, C}];
a4gF = Simplify[a4g /. solf1 /. solf2 /. solf3 ]
```

$$\frac{1}{8\,C}\Big(15\,\text{Cos}[y] + 9\,C^2\,\text{Cos}[y] - 63\,\text{Cos}[3\,y] - 9\,C^2\,\text{Cos}[3\,y] + 8\,(1 + C^2)\,\text{Cos}[y]\,\text{f2b}[y] + 8\,(2\,C^2 - D)\,\text{Sin}[y]\,\text{f2b}'[y] + 32\,C\,\text{Cos}[2\,y]\,\text{f3d}'[y] + 8\,\text{Cos}[y]\,\text{f2b}''[y] - 8\,C^2\,\text{Cos}[y]\,\text{f2b}''[y] - 8\,D\,\text{Sin}[y]\,\text{f2b}^{(3)}[y] - 8\,C\,\text{f3d}^{(3)}[y] + 24\,C\,\text{Cos}[2\,y]\,\text{f3d}^{(3)}[y] + 8\,C\,\text{Sin}[2\,y]\,\text{f3d}^{(4)}[y]\Big)$$

```
a4gI2 = + 1/(8 C) (-15 Cos[y] - 9 C^2 Cos[y] + 7 Cos[3 y] + C^2 Cos[3 y]) -
    (1 + Cos[2 y]) f3d'[y] + Sin[2 y] f3d''[y]  - C Cos[y] f2b[y];
a4gI1 = - D/C (-Cos[y] f2b'[y] + Sin[y] f2b''[y]) +
    1/C ( f2b[y] Sin[y]  + Cos[y] f2b'[y]);
Simplify[a4gF - D[a4gI2, {y, 2}] - D[a4gI1, y]]
```

0

- **Term in $\varphi 0^{(0,1)}$[T, X] $\varphi 0^{(1,1)}$[T, X] : a4h**

```
a4h = Collect[Coefficient[-a43, φ0^(1,1)[T, X]] /. Rulea, {A, D, C}];
Simplify[a4h /. solf1 /. solf2 /. solf3 ]
```

$$-\frac{\left(7+C^2\right) \text{Cos}[y]}{2\,A\,C^2}$$

- **Term in $\varphi 0^{(0,4)}$[T, X] : a4l**

```
a4l = Collect[Coefficient[-a41,  φ0^(0,4)[T, X]] /. Rulea, {A, D, C }];
a4lF = Simplify[a4l /. solf1 /. solf2 /. solf3]
```

$$\frac{1}{C} A \left(D\,\text{Cos}[y]^2+\text{Sin}[y]^2-\frac{1}{2} C^2 \left(3+\text{Cos}[2\,y]+4\,\text{Cos}[2\,y]\,\text{f2b}[y]+3\,\text{Sin}[2\,y]\,\text{f2b}'[y]+2\,\text{f2b}''[y]-2\,\text{Cos}[2\,y]\,\text{f2b}''[y]\right)\right)$$

Eliminate the terms that can be integrated and keep only the contributing terms

```
Expand[Simplify[Expand[a4lF +  D[(A C)/4 Sin[2 y]  + A C  f2b'[y], y]]]]
```

$$-\frac{3\,A\,C}{2}+\frac{A\,D\,\text{Cos}[y]^2}{C}-2\,A\,C\,\text{Cos}[2\,y]\,\text{f2b}[y]+\frac{A\,\text{Sin}[y]^2}{C}-3\,A\,C\,\text{Cos}[y]\,\text{Sin}[y]\,\text{f2b}'[y]+A\,C\,\text{Cos}[2\,y]\,\text{f2b}''[y]$$

- **Term in $\varphi 1^{(1,1)}[T, X]$: a4m**

```
a4m = Coefficient[-a41,  φ1^(1,1)[T, X]]
```

0

```
a4m /. solf2
```

0

- **Term in $\varphi 0^{(1,2)}[T, X]$: a4n**

```
a4n = Collect[Coefficient[-a41,  φ0^(1,2)[T, X]], {A, D}]
```

$1 + \text{f2b}''[y]$

The firxt term contributes, the second one does not

- **Term in $\varphi 0^{(0,2)}[T, X]^2$: a4o**

```
a4o = Collect[Coefficient[-a41,  φ0^(0,2)[T, X]^2] /. Rulea, {A, D, C }];
a4oF = Simplify[a4o /. solf1 /. solf2 /. solf3]
```

$$\frac{1}{8\,C}\Big(3\,C^2\,(\text{Cos}[y] + 3\,\text{Cos}[3\,y])\,\text{f2b}'[y]^2 + C^2\,(-9\,\text{Cos}[y] + \text{Cos}[3\,y])\,\text{f2b}''[y]^2 + 4\,\text{f2b}''[y]\,\Big(6\,C^2\,\text{Sin}[y]^3\,\text{f2b}^{(3)}[y] + \text{Cos}[y]\,\Big(-3\,C^2 - 2\,D + 3\,C^2\,\text{Cos}[2\,y] + C^2\,(-5 + \text{Cos}[2\,y])\,\text{f2b}^{(4)}[y]\Big)\Big) + 2\,\Big(2\,\Big(-2 - 5\,C^2 + 5\,C^2\,\text{Cos}[2\,y]\Big)\,\text{Sin}[y]\,\text{f2b}^{(3)}[y] + C^2\,(-9\,\text{Cos}[y] + \text{Cos}[3\,y])\,\text{f2b}^{(3)}[y]^2 - 2\,\text{Cos}[y]\,\Big(-6 - 9\,C^2 - 2\,D + 9\,C^2\,\text{Cos}[2\,y] + \Big(-5\,C^2 + 2\,D + C^2\,\text{Cos}[2\,y]\Big)\,\text{f2b}^{(4)}[y]\Big)\Big) + 2\,\text{f2b}'[y]\,\Big(12\,C^2\,\text{Sin}[y]^3\,\text{f2b}''[y] + C^2\,(-9\,\text{Cos}[y] + 5\,\text{Cos}[3\,y])\,\text{f2b}^{(3)}[y] + 2\,\text{Sin}[y]\,\Big(-2 + 3\,C^2 + 9\,C^2\,\text{Cos}[2\,y] - 2\,C^2\,\text{Sin}[y]^2\,\text{f2b}^{(4)}[y]\Big)\Big)\Big)$$

Primitive for the whole expression

$$\text{a4oI} = \frac{3\,\text{Sin}[y]}{C} + \frac{9}{4} C\,\text{Sin}[y] - \frac{3}{4} C\,\text{Sin}[3\,y] - \frac{1}{C}\Big(-\text{Cos}[y]\,\text{f2b}'[y] + (1+D)\,\text{Sin}[y]\,\text{f2b}''[y] + D\,\big(-\text{Sin}[y] + \text{Cos}[y]\,\text{f2b}^{(3)}[y]\big)\Big) - \frac{1}{4} C\,\Big(-6\,\text{Cos}[y]^2\,\text{Sin}[y]\,\text{f2b}'[y]^2 - 2\,\text{Sin}[y]^3\,\text{f2b}''[y]^2 + (-9\,\text{Cos}[y] + \text{Cos}[3\,y])\,\text{f2b}^{(3)}[y] + \text{f2b}''[y]\,\big(8\,\text{Sin}[y]^3 - (-9\,\text{Cos}[y] + \text{Cos}[3\,y])\,\text{f2b}^{(3)}[y]\big) - 2\,\text{f2b}'[y]\,\big((-3\,\text{Cos}[y] + \text{Cos}[3\,y])\,\text{f2b}''[y] - 2\,\text{Sin}[y]^2\,(-3\,\text{Cos}[y] + \text{Sin}[y]\,\text{f2b}^{(3)}[y])\big)\Big);$$

```
Simplify[D[a4oI, y] - a4oF]
```

```
0
```

Better expression

$$\text{a4oIFinal} = 3\,\text{Sin}[y]\,\left(\frac{1}{C}\left(1+\frac{D}{3}\right) + C\,\text{Sin}[y]^2\right) - \frac{1}{C}\big(\text{Sin}[y]\,\text{f2b}''[y] - \text{Cos}[y]\,\text{f2b}'[y]\big) - \frac{D}{C}\big(\text{Sin}[y]\,\text{f2b}''[y] + \text{Cos}[y]\,\text{f2b}^{(3)}[y]\big) - \frac{1}{4} C\,D\Big[2\,\text{Cos}[y]^3\,\text{f2b}'[y]^2 + 4\,\text{Sin}[y]^3\,\text{f2b}'[y]\,\big(-1 + \text{f2b}''[y]\big) - \frac{1}{2}(-9\,\text{Cos}[y] + \text{Cos}[3\,y])\,\big(-2 + \text{f2b}''[y]\big)\,\text{f2b}''[y],\ y\Big];$$

```
Simplify[a4oIFinal - a4oI]
```

```
0
```

- **Verification**

$$\text{a4NN} = -\text{a4o}\,\varphi0^{(0,2)}[T,X]^2 - \text{a4n}\,\varphi0^{(1,2)}[T,X] - \text{a4l}\,\varphi0^{(0,4)}[T,X] - \varphi0^{(0,1)}[T,X]\,\big(\text{a4g}\,\varphi0^{(0,3)}[T,X] + \text{a4h}\,\varphi0^{(1,1)}[T,X]\big) - \text{a4c}\,\varphi0^{(0,1)}[T,X]^2\,\varphi0^{(0,2)}[T,X] - \text{a4a}\,\varphi0^{(0,1)}[T,X]^4 + \mathcal{L}[\psi4[T,X,y]];$$

```
Simplify[a4NN - a4 /. Rulea /. solf1 /. solf2a]
```

```
0
```

Dump

```
DumpSave["CH-eff1-nu-9b_compact.m6.mx", "Global`"]
```

```
{Global`}
```

```
<< CH-eff1-nu-9b_compact.m6.mx
```

Amplitude equation

Calculation of $-\frac{3}{2}\frac{1}{2\pi}\int_0^{2\pi}\partial_y$ f2b $\sin 2y\, dy$

$$II = -\frac{3}{2}\frac{1}{2\pi}\,\texttt{Integrate[Evaluate[Sin[2 y] h2b[y] /. ss2b], \{y, 0, 2}\,\pi\texttt{\}]}$$

```
N[II]
```

$$-\frac{3}{4}\left(2-\frac{4\,\text{EllipticK}\left[\frac{1}{2}\right]}{3\,\text{EllipticE}\left[\frac{1}{2}\right]}\right)$$

```
-0.127266
```

Stability threshold

$$DD = \frac{1}{144} + 4\,Cs^2\,(3 + 2\,II)\ \texttt{/.}\ Cs \to 0.13$$

$$\sqrt{\frac{1}{2\,(3 + 2\,II)\,Cs^2}\left(\frac{1}{12} + \sqrt{DD}\right)}\ \texttt{/.}\ Cs \to 0.13$$

```
0.192538
```

```
2.37203
```