Large-scale linear instability of the generalized Kolmogorov flow - hyperviscous version with a general parallel flow

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U of
any shape,
periodic within the interval $[0, 2\pi]$
with the condition that $\int_0^{2\pi} U[y]\, dy = 0$
Streamfunction defined as u=$\partial_y \psi$, $v = -\partial_x \psi$

Settings

- **Last use**

```
DateList[]

{2009, 1, 22, 12, 49, 31.310386}
```

- **General**

```
Off[General::spell1]
```

- **Laplacian**

```
Δ = - ( ϵ^2 ∂_{x,x} # + ∂_{y,y} #) &
```

```
-(ϵ^2 ∂_{x,x} #1 + ∂_{y,y} #1) &
```

Beware of the sign in the definition

- **Definition of the dissipation**

Hyperviscosity

```
DS = - ν Δ[Δ[Δ[Δ[#]]]] &
```

```
-ν Δ[Δ[Δ[Δ[#1]]]] &
```

Definition of the modified Kolmogorov Flow

Velocity and streamfunction

```
Φ[y_] := Integrate[U[s], {s, π/2, y}]
```

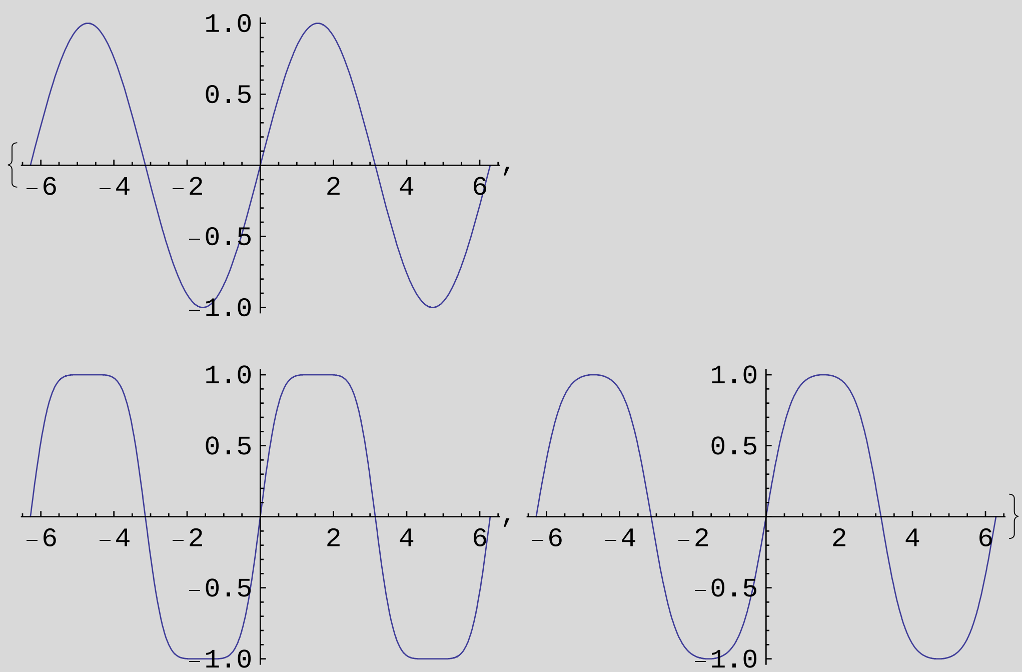
Exemples of flows

Standard Kolmogorov flow in Flow1

```
Flow2 = U → Evaluate[Sin[# + 1/2 Sin[2 #]] &];
Flow1 = U → Evaluate[Sin[#] &];
Flow3 = U → Evaluate[Sin[# + 1/4 Sin[2 #]] &];
```

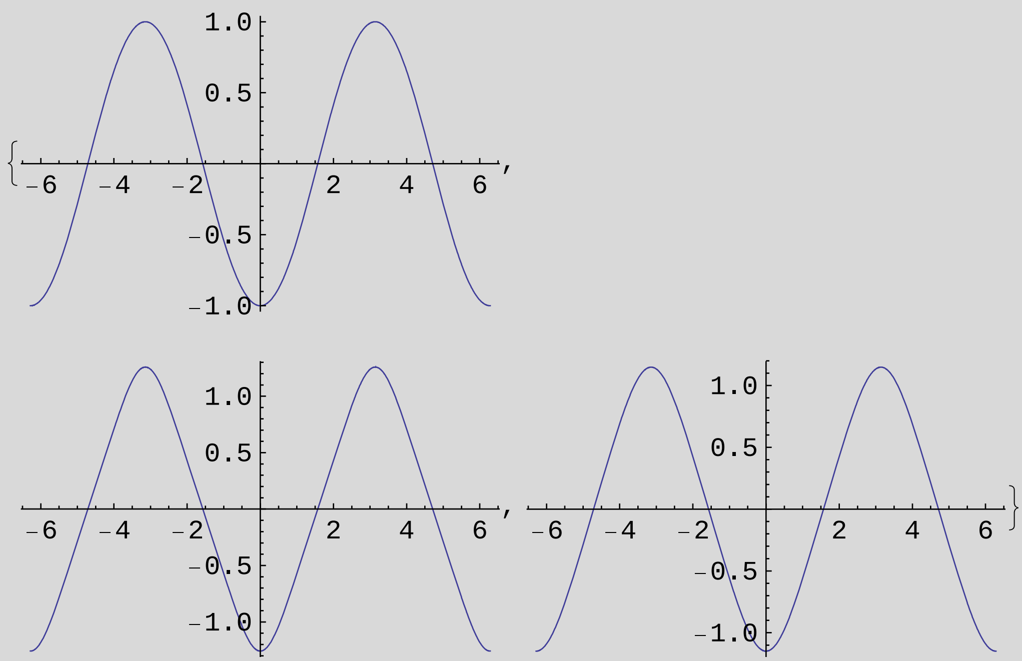
■ Plots of the flow

```
{Plot[U[y] /. {Flow1, A → 1}, {y, -2 π, 2 π}],
 Plot[U[y] /. {Flow2, A → 1}, {y, -2 π, 2 π}],
 Plot[U[y] /. {Flow3, A → 1}, {y, -2 π, 2 π}]}
```



Plots of the streamfunction

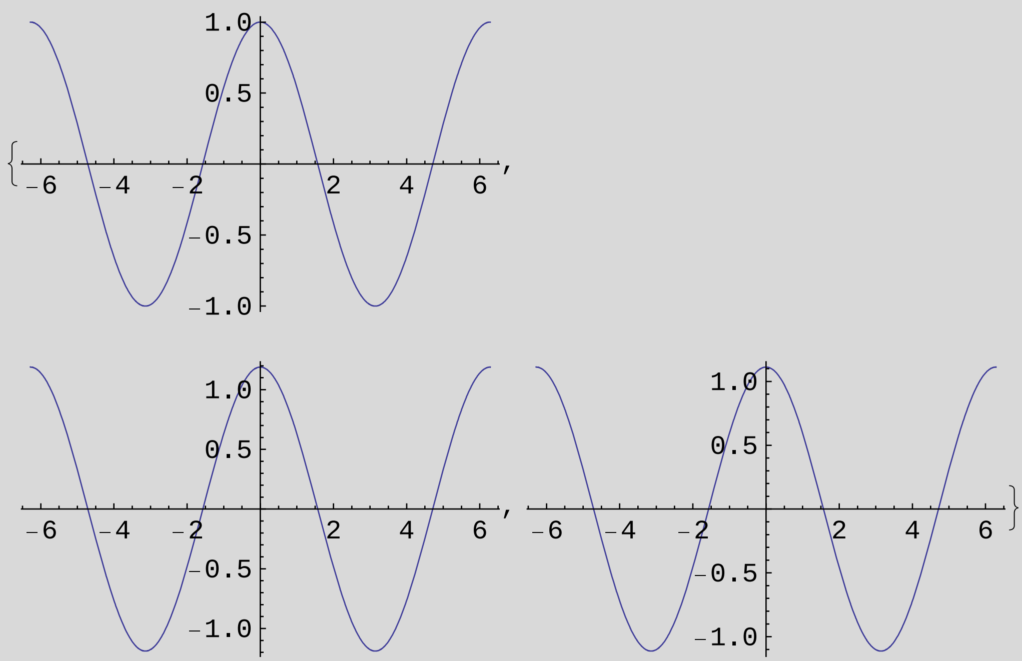
```
Φ1 = NIntegrate[Evaluate[U[y] /. {Flow1, A → 1}], {y, π/2, #}] &;
Φ2 = NIntegrate[Evaluate[U[y] /. {Flow2, A → 1}], {y, π/2, #}] &;
Φ3 = NIntegrate[Evaluate[U[y] /. {Flow3, A → 1}], {y, π/2, #}] &;
{Plot[Φ1[u], {u, -2 π, 2 π}],
 Plot[Φ2[u], {u, -2 π, 2 π}],
 Plot[Φ3[u], {u, -2 π, 2 π}]}
```



Plots of the third primitive of the velocity

```
ξ1sol = NDSolve[{D[ξ1[y], {y, 3}] == U[y] /. {Flow1, A → 1},
    ξ1[π/2] == 0, ξ1''[π/2] == 0, ξ1'[0] == 0}, ξ1, {y, -2 π, 2 π}];
ξ2sol = NDSolve[{D[ξ2[y], {y, 3}] == U[y] /. {Flow2, A → 1},
    ξ2[π/2] == 0, ξ2''[π/2] == 0, ξ2'[0] == 0}, ξ2, {y, -2 π, 2 π}];
ξ3sol = NDSolve[{D[ξ3[y], {y, 3}] == U[y] /. {Flow3, A → 1},
    ξ3[π/2] == 0, ξ3''[π/2] == 0, ξ3'[0] == 0}, ξ3, {y, -2 π, 2 π}];
```

```
{Plot[ξ1[u] /. ξ1sol, {u, -2π, 2π}],
 Plot[ξ2[u] /. ξ2sol, {u, -2π, 2π}],
 Plot[ξ3[u] /. ξ3sol, {u, -2π, 2π}]}
```

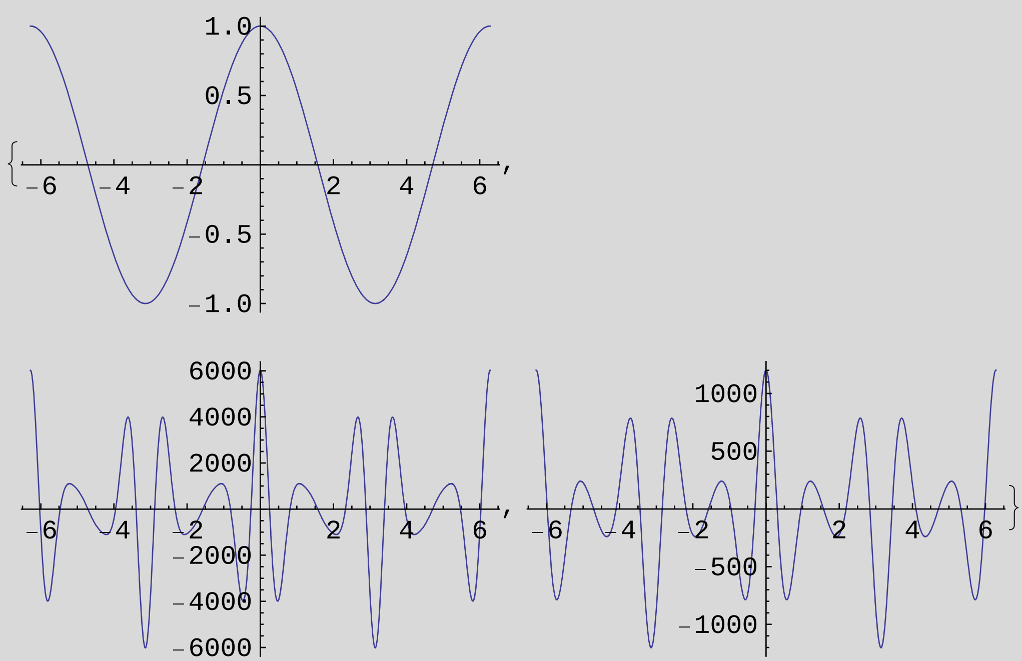


■ Plots of the forcing

```
Forcing = Simplify[DS[Φ[y]]]
```

$-\nu\, U^{(7)}[y]$

```
{Plot[ Forcing //. {Flow1, A → 1, ν → 1},
   {y, -2 π, 2 π}, PlotRange → All],
 Plot[ Forcing //. {Flow2, A → 1, ν → 1},
   {y, -2 π, 2 π}, PlotRange → All],
 Plot[ Forcing //. {Flow3, A → 1, ν → 1},
   {y, -2 π, 2 π}, PlotRange → All]}
```



Definition of the pertubation problem

■ 2D Navier-Stokes with scaling

We have taken u = $\partial_y \psi$ and $v = -\partial_x \psi$ (mechanical definition)

a linear friction term has been included

```
NS = ϵ² ∂T Δ[ψ[T, X, y]] - ϵ ∂X ψ[T, X, y] ∂y Δ[ψ[T, X, y]] +
   ϵ ∂y ψ[T, X, y] ∂X Δ[ψ[T, X, y]] + ϵ² r Δ[ψ[T, X, y]] -
   DS[ψ[T, X, y]] + DS[Φ[y]] - ϵ² r Δ[Φ[y]];
```

■ Expansion

```
ψ[T_, X_, y_] := Φ[y] + ψ0[T, X, y] +
  ϵ ψ1[T, X, y] + ϵ² ψ2[T, X, y] + ϵ³ ψ3[T, X, y] + ϵ⁴ ψ4[T, X, y]
```

```
NS2 = Collect[Normal[Series[NS, {ϵ, 0, 4}]], ϵ];
```

Order 0

```
a0 = Simplify[Coefficient[NS2, ϵ, 0] ]
```

$\nu\, \psi 0^{(0,0,8)}[T, X, y]$

Symbolic solution

```
Rule0 = ψ0 → (φ0[#1, #2] &);
```

Check solution

```
Simplify[a0 /. {Rule0}]
```

0

Solvability condition

```
Solva[expr_, Rules___] :=
 Integrate[expr //. Flatten[ {Rules}], {y, 0, 2 π}] /. g___[2 π] → g[0]
```

Order 1

```
a1 = Simplify[Expand[Coefficient[NS2, ϵ, 1]] /. Rule0 ]
```

$U''[y]\, \varphi 0^{(0,1)}[T, X] + \nu\, \psi 1^{(0,0,8)}[T, X, y]$

Define linear operator

```
ℒ[f_] := + ∂y,y ( ∂y,y ( ∂y,y (ν ∂y,y f )))
```

Extraction and integration of second member

```
a1a = -Simplify[Coefficient[a1, φ0^(0,1)[T, X]]]
```

$-U''[y]$

```
a1aI2 = Integrate[Integrate[a1a, y], y]
```

$-U[y]$

Reconstruction and check of the first order equation

```
a1N = ℒ[ψ1[T, X, y]] - a1a φ0^(0,1)[T, X];
Simplify[a1N - a1]
```

0

Solvability condition

The solvability condition is obviously satisfied with this second member since it has a primitive (even two primitives)

```
Solva[a1a] /. g___[2 π] → g[0]
```

0

Symbolic solution

```
Rule1 = ψ1 → (f1[#3] φ0^(0,1)[#1, #2] &);
```

Order 2

```
a2 = Simplify[Coefficient[NS2, ϵ, 2]  //. {Rule0, Rule1}, Trig → True]
```

$f1^{(3)}[y]\,\varphi0^{(0,1)}[T, X]^2 - U[y]\,f1''[y]\,\varphi0^{(0,2)}[T, X] + f1[y]\,U''[y]\,\varphi0^{(0,2)}[T, X] + \nu\,\psi2^{(0,0,8)}[T, X, y]$

Extraction of second member and solvability

- **term in factor of $\varphi 0^{(0,1)}[T, X]^2$**

```
a2a = -Factor[Simplify[Coefficient[a2 , φ0^(0,1)[T, X], 2] ]]
Solva[a2a]
```

$-f1^{(3)}[y]$

0

- **term in factor of $\varphi 0^{(0,2)}[T, X]$**

```
a2b = -Simplify[Coefficient[a2 , φ0^(0,2)[T, X], 1] ]
Integrate[a2b, y]
Solva[a2b]
```

$U[y]\, f1''[y] - f1[y]\, U''[y]$

$U[y]\, f1'[y] - f1[y]\, U'[y]$

0

Reconstruction and check of the second order equation

```
a2N = ℒ[ψ2[T, X, y]] -
   a2c φ1^(0,1)[T, X] - a2b φ0^(0,2)[T, X] - a2a φ0^(0,1)[T, X]^2;
Simplify[a2N - a2]
```

$-a2c\, \varphi 1^{(0,1)}[T, X]$

Symbolic solution

```
Rule2 = ψ2 → (f2a[#3] φ0^(0,1)[#1, #2]^2 + f2b[#3] φ0^(0,2)[#1, #2] &);
```

Order 3

```
a3 = FullSimplify[
  Coefficient[NS2, ϵ, 3]  //. {Rule0, Rule1, Rule2} , Trig → True]
```

$$f2a^{(3)}[y]\,\varphi 0^{(0,1)}[T,X]^3 + 2\,f2a[y]\,U''[y]\,\varphi 0^{(0,1)}[T,X]\,\varphi 0^{(0,2)}[T,X] + f1[y]\,f1^{(3)}[y]\,\varphi 0^{(0,1)}[T,X]\,\varphi 0^{(0,2)}[T,X] + f2b^{(3)}[y]\,\varphi 0^{(0,1)}[T,X]\,\varphi 0^{(0,2)}[T,X] + f2b[y]\,U''[y]\,\varphi 0^{(0,3)}[T,X] + 4\,\nu\,f1^{(6)}[y]\,\varphi 0^{(0,3)}[T,X] - U[y]\left(2\,f2a''[y]\,\varphi 0^{(0,1)}[T,X]\,\varphi 0^{(0,2)}[T,X] + \left(1 + f2b''[y]\right)\varphi 0^{(0,3)}[T,X]\right) - f1''[y]\left(\varphi 0^{(0,1)}[T,X]\left(r + f1'[y]\,\varphi 0^{(0,2)}[T,X]\right) + \varphi 0^{(1,1)}[T,X]\right) + \nu\,\psi 3^{(0,0,8)}[T,X,y]$$

■ Extraction of second member and solvability

```
a31 = Collect[Expand[a3],
  {φ0^(1,1)[T, X], φ0^(0,1)[T, X], φ0^(0,2)[T, X], φ0^(0,3)[T, X]}]
```

$$f2a^{(3)}[y]\,\varphi 0^{(0,1)}[T,X]^3 + \varphi 0^{(0,1)}[T,X]\left(-r\,f1''[y] + \left(-f1'[y]\,f1''[y] - 2\,U[y]\,f2a''[y] + 2\,f2a[y]\,U''[y] + f1[y]\,f1^{(3)}[y] + f2b^{(3)}[y]\right)\varphi 0^{(0,2)}[T,X]\right) + \left(-U[y] - U[y]\,f2b''[y] + f2b[y]\,U''[y] + 4\,\nu\,f1^{(6)}[y]\right)\varphi 0^{(0,3)}[T,X] - f1''[y]\,\varphi 0^{(1,1)}[T,X] + \nu\,\psi 3^{(0,0,8)}[T,X,y]$$

- **Term in $\varphi 0^{(0,1)}[T,X]^3$: a3a**

```
a3a = -Coefficient[a31, φ0^(0,1)[T, X]^3]
Solva[a3a]
```

$$-f2a^{(3)}[y]$$

```
0
```

- **Term in $\varphi0^{(0,1)}$[T, X] $\varphi0^{(0,2)}$[T, X] : a3b**

```
a3b =
 -Coefficient[Coefficient[Expand[a31], φ0^(0,1)[T, X], 1], φ0^(0,2)[T, X]]
Integrate[a3b, y]
Solva[a3b]
```

$f1'[y]\, f1''[y] + 2\, U[y]\, f2a''[y] - 2\, f2a[y]\, U''[y] - f1[y]\, f1^{(3)}[y] - f2b^{(3)}[y]$

$f1'[y]^2 + 2\, U[y]\, f2a'[y] - 2\, f2a[y]\, U'[y] - f1[y]\, f1''[y] - f2b''[y]$

0

- **Term in $\varphi0^{(0,3)}$[T, X] : a3d**

```
a3d = -Coefficient[a31, φ0^(0,3)[T, X]]
Integrate[a3d - U[y], y]
Solva[a3d - U[y]]
```

$U[y] + U[y]\, f2b''[y] - f2b[y]\, U''[y] - 4\, \nu\, f1^{(6)}[y]$

$U[y]\, f2b'[y] - f2b[y]\, U'[y] - 4\, \nu\, f1^{(5)}[y]$

0

The solvability condition is satisfied due to the hypothesis that $\int_0^{2\pi} U[y]\, dy = 0$,

no other contribution

- **Term in $\varphi 0^{(1,1)}$[T, X] : a3g**

```
a3g = -Coefficient[a31, φ0^(1,1)[T, X]]
Solva[a3g]
```

```
f1''[y]
```

```
0
```

- **Terms due to friction**

```
a3r = -Coefficient[a31, r φ0^(0,1)[T, X]]
Solva[a3r]
```

```
f1''[y]
```

```
0
```

- **Reconstruction and check of the third order equation**

```
a3N =
  -a3g φ0^(1,1)[T, X] - a3d φ0^(0,3)[T, X] - φ0^(0,1)[T, X] (a3b φ0^(0,2)[T, X] ) -
   a3a φ0^(0,1)[T, X]^3 - r a3r φ0^(0,1)[T, X] + ℒ[ψ3[T, X, y]];
Simplify[a3N - a3 ]
```

```
0
```

Symbolic solution

```
Rule3 = ψ3 -> (f3g[#3] φ0^(1,1)[#1, #2] + f3d[#3] φ0^(0,3)[#1, #2] +
      φ0^(0,1)[#1, #2] (f3b[#3] φ0^(0,2)[#1, #2] ) +
      f3a[#3] φ0^(0,1)[#1, #2]^3 + r f3r[#3] φ0^(0,1)[#1, #2]  &);
```

Order 4

```
a4 = Simplify[Coefficient[NS2, ϵ, 4]  //.
   {Rule0, Rule1, Rule2, Rule3} , Trig → True]
```

$$
\begin{aligned}
&-r\,\varphi 0^{(0,2)}[T,X]-\\
&f1''[y]\,\varphi 0^{(0,2)}[T,X]\left(f2a'[y]\,\varphi 0^{(0,1)}[T,X]^2+f2b'[y]\,\varphi 0^{(0,2)}[T,X]\right)-\\
&r\left(f2a''[y]\,\varphi 0^{(0,1)}[T,X]^2+f2b''[y]\,\varphi 0^{(0,2)}[T,X]\right)+\\
&f1[y]\,\varphi 0^{(0,2)}[T,X]\left(f2a^{(3)}[y]\,\varphi 0^{(0,1)}[T,X]^2+f2b^{(3)}[y]\,\varphi 0^{(0,2)}[T,X]\right)+\\
&f1'[y]\,\varphi 0^{(0,1)}[T,X]\,\varphi 0^{(0,3)}[T,X]+f1^{(3)}[y]\,\varphi 0^{(0,1)}[T,X]\\
&\quad\left(2\,f2a[y]\,\varphi 0^{(0,1)}[T,X]\,\varphi 0^{(0,2)}[T,X]+f2b[y]\,\varphi 0^{(0,3)}[T,X]\right)-f1'[y]\,\varphi 0^{(0,1)}[\\
&\quad T,X]\left(2\,f2a''[y]\,\varphi 0^{(0,1)}[T,X]\,\varphi 0^{(0,2)}[T,X]+\left(1+f2b''[y]\right)\varphi 0^{(0,3)}[T,X]\right)+\\
&4\,\nu\left(2\,f2a^{(6)}[y]\left(\varphi 0^{(0,2)}[T,X]^2+\varphi 0^{(0,1)}[T,X]\,\varphi 0^{(0,3)}[T,X]\right)+\right.\\
&\qquad\left.f2b^{(6)}[y]\,\varphi 0^{(0,4)}[T,X]\right)-2\,f2a''[y]\,\varphi 0^{(0,1)}[T,X]\,\varphi 0^{(1,1)}[T,X]+\\
&\varphi 0^{(0,1)}[T,X]\left(r\,f3r^{(3)}[y]\,\varphi 0^{(0,1)}[T,X]+f3a^{(3)}[y]\,\varphi 0^{(0,1)}[T,X]^3+\right.\\
&\qquad f3b^{(3)}[y]\,\varphi 0^{(0,1)}[T,X]\,\varphi 0^{(0,2)}[T,X]+f3d^{(3)}[y]\,\varphi 0^{(0,3)}[T,X]+\\
&\qquad\left.f3g^{(3)}[y]\,\varphi 0^{(1,1)}[T,X]\right)-\varphi 0^{(1,2)}[T,X]-f2b''[y]\,\varphi 0^{(1,2)}[T,X]+\\
&U''[y]\left(r\,f3r[y]\,\varphi 0^{(0,2)}[T,X]+3\,f3a[y]\,\varphi 0^{(0,1)}[T,X]^2\,\varphi 0^{(0,2)}[T,X]+\right.\\
&\qquad f3b[y]\,\varphi 0^{(0,2)}[T,X]^2+f3b[y]\,\varphi 0^{(0,1)}[T,X]\,\varphi 0^{(0,3)}[T,X]+\\
&\qquad\left.f3d[y]\,\varphi 0^{(0,4)}[T,X]+f3g[y]\,\varphi 0^{(1,2)}[T,X]\right)-\\
&U[y]\left(r\,f3r''[y]\,\varphi 0^{(0,2)}[T,X]+3\,f3a''[y]\,\varphi 0^{(0,1)}[T,X]^2\,\varphi 0^{(0,2)}[T,X]+f3b''[y]\right.\\
&\qquad\varphi 0^{(0,2)}[T,X]^2+f3b''[y]\,\varphi 0^{(0,1)}[T,X]\,\varphi 0^{(0,3)}[T,X]+f1[y]\,\varphi 0^{(0,4)}[T,X]+\\
&\qquad\left.f3d''[y]\,\varphi 0^{(0,4)}[T,X]+f3g''[y]\,\varphi 0^{(1,2)}[T,X]\right)+\nu\,\psi 4^{(0,0,8)}[T,X,y]
\end{aligned}
$$

Terms containing $\varphi 0^{(0,1)}[T, X]$

Terms in $\varphi 0^{(0,1)}[T, X]^4$: a4a

```
a4a = -Coefficient[Expand[a4], φ0^(0,1)[T, X]^4]
Solva[a4a]
```

$-f3a^{(3)}[y]$

0

Terms in $\varphi 0^{(0,1)}[T, X]^3$

```
Coefficient[Expand[a4], φ0^(0,1)[T, X]^3]
```

0

Terms in $\varphi 0^{(0,1)}[T, X]^2$

```
a42 = Coefficient[Expand[a4], φ0^(0,1)[T, X]^2]
```

$$-r\, f2a''[y] + r\, f3r^{(3)}[y] - f2a'[y]\, f1''[y]\, \varphi 0^{(0,2)}[T, X] - 2\, f1'[y]\, f2a''[y]\, \varphi 0^{(0,2)}[T, X] - 3\, U[y]\, f3a''[y]\, \varphi 0^{(0,2)}[T, X] + 3\, f3a[y]\, U''[y]\, \varphi 0^{(0,2)}[T, X] + 2\, f2a[y]\, f1^{(3)}[y]\, \varphi 0^{(0,2)}[T, X] + f1[y]\, f2a^{(3)}[y]\, \varphi 0^{(0,2)}[T, X] + f3b^{(3)}[y]\, \varphi 0^{(0,2)}[T, X]$$

Terms in r $\varphi 0^{(0,1)}[T, X]^2$: a4u

```
a4u = -Coefficient[Expand[a42], r]
Solva[a4u]
```

$f2a''[y] - f3r^{(3)}[y]$

0

- **Terms in $\varphi0^{(0,1)}[T, X]^2\, \varphi0^{(0,2)}[T, X]$: a4c**

```
a4c = -Coefficient[Expand[a42], φ0^(0,2)[T, X]]
Integrate[a4c, y]
Solva[a4c]
```

$f2a'[y]\, f1''[y] + 2\, f1'[y]\, f2a''[y] + 3\, U[y]\, f3a''[y] - 3\, f3a[y]\, U''[y] - 2\, f2a[y]\, f1^{(3)}[y] - f1[y]\, f2a^{(3)}[y] - f3b^{(3)}[y]$

$3\, f1'[y]\, f2a'[y] + 3\, U[y]\, f3a'[y] - 3\, f3a[y]\, U'[y] - 2\, f2a[y]\, f1''[y] - f1[y]\, f2a''[y] - f3b''[y]$

0

- **Terms in $\varphi0^{(0,1)}[T, X]$**

```
a43 = Coefficient[ Expand[a4], φ0^(0,1)[T, X]]
```

$-f1'[y]\, f2b''[y]\, \varphi0^{(0,3)}[T, X] - U[y]\, f3b''[y]\, \varphi0^{(0,3)}[T, X] + f3b[y]\, U''[y]\, \varphi0^{(0,3)}[T, X] + f2b[y]\, f1^{(3)}[y]\, \varphi0^{(0,3)}[T, X] + f3d^{(3)}[y]\, \varphi0^{(0,3)}[T, X] + 8\, \nu\, f2a^{(6)}[y]\, \varphi0^{(0,3)}[T, X] - 2\, f2a''[y]\, \varphi0^{(1,1)}[T, X] + f3g^{(3)}[y]\, \varphi0^{(1,1)}[T, X]$

- **Term in $\varphi0^{(0,1)}[T, X]\, \varphi0^{(0,3)}[T, X]$: a4g**

```
a4g = -Coefficient[ Expand[a43], φ0^(0,3)[T, X]]
Integrate[a4g, y]
Solva[a4g]
```

$f1'[y]\, f2b''[y] + U[y]\, f3b''[y] - f3b[y]\, U''[y] - f2b[y]\, f1^{(3)}[y] - f3d^{(3)}[y] - 8\, \nu\, f2a^{(6)}[y]$

$f1'[y]\, f2b'[y] + U[y]\, f3b'[y] - f3b[y]\, U'[y] - f2b[y]\, f1''[y] - f3d''[y] - 8\, \nu\, f2a^{(5)}[y]$

0

- **Term in $\varphi0^{(0,1)}[T, X]\, \varphi0^{(1,1)}[T, X]$: a4h**

```
a4h = -Coefficient[Expand[a43], φ0^(1,1)[T, X]]
Solva[a4h]
```

$2\, f2a''[y] - f3g^{(3)}[y]$

0

Select terms that do not contain $\varphi0^{(0,1)}[T, X]$

```
a45 = a4 /. φ0^(0,1)[T, X] → 0
```

$$-r\, \varphi0^{(0,2)}[T, X] - r\, f2b''[y]\, \varphi0^{(0,2)}[T, X] - f2b'[y]\, f1''[y]\, \varphi0^{(0,2)}[T, X]^2 + f1[y]\, f2b^{(3)}[y]\, \varphi0^{(0,2)}[T, X]^2 + 4\, \nu \left(2\, f2a^{(6)}[y]\, \varphi0^{(0,2)}[T, X]^2 + f2b^{(6)}[y]\, \varphi0^{(0,4)}[T, X]\right) - \varphi0^{(1,2)}[T, X] - f2b''[y]\, \varphi0^{(1,2)}[T, X] + U''[y] \left(r\, f3r[y]\, \varphi0^{(0,2)}[T, X] + f3b[y]\, \varphi0^{(0,2)}[T, X]^2 + f3d[y]\, \varphi0^{(0,4)}[T, X] + f3g[y]\, \varphi0^{(1,2)}[T, X]\right) - U[y] \left(r\, f3r''[y]\, \varphi0^{(0,2)}[T, X] + f3b''[y]\, \varphi0^{(0,2)}[T, X]^2 + f1[y]\, \varphi0^{(0,4)}[T, X] + f3d''[y]\, \varphi0^{(0,4)}[T, X] + f3g''[y]\, \varphi0^{(1,2)}[T, X]\right) + \nu\, \psi4^{(0,0,8)}[T, X, y]$$

- **Term in $\varphi 0^{(0,4)}$[T, X] : a4l : contribution to the amplitude equation**

```
a4l = -Coefficient[a45, φ0^(0,4)[T, X]]
Integrate[a4l - f1[y] U[y], y]
Solva[a4l - f1[y] U[y]]
Integrate[f1[y] U[y], {y, 0, 2 π}]
```

$$f1[y]\,U[y] + U[y]\,f3d''[y] - f3d[y]\,U''[y] - 4\,\nu\,f2b^{(6)}[y]$$

$$U[y]\,f3d'[y] - f3d[y]\,U'[y] - 4\,\nu\,f2b^{(5)}[y]$$

$$0$$

$$\int_0^{2\pi} f1[y]\,U[y]\,dy$$

The last line is the contribution to the amplitude equation

- **Term in $\varphi 0^{(1,2)}$[T, X] : a4n : Contributes to the amplitude equation**

```
a4n = -Coefficient[Expand[a45], φ0^(1,2)[T, X]]
Integrate[a4n, y]
Solva[a4n]
```

$$1 + f2b''[y] + U[y]\,f3g''[y] - f3g[y]\,U''[y]$$

$$y + f2b'[y] + U[y]\,f3g'[y] - f3g[y]\,U'[y]$$

$$2\,\pi$$

- **Term in $\varphi 0^{(0,2)}[T, X]^2$: a4o**

```
a4o = -Coefficient[Expand[a45], φ0^(0,2)[T, X]^2]
Integrate[a4o, y]
Solva[a4o]
```

$f2b'[y]\, f1''[y] + U[y]\, f3b''[y] - f3b[y]\, U''[y] - f1[y]\, f2b^{(3)}[y] - 8\,\nu\, f2a^{(6)}[y]$

$f1'[y]\, f2b'[y] + U[y]\, f3b'[y] - f3b[y]\, U'[y] - f1[y]\, f2b''[y] - 8\,\nu\, f2a^{(5)}[y]$

0

- **Term in $\varphi 0^{(0,2)}[T, X]$: a4p : Contributes to the amplitude equation (if $r \neq 0$)**

```
a4p = -Coefficient[Expand[a45] /. φ1^(0,1)[T, X] → 0, φ0^(0,2)[T, X]]
Integrate[a4p, y]
Solva[a4p]
```

$r + r\, f2b''[y] + r\, U[y]\, f3r''[y] - r\, f3r[y]\, U''[y]$

$r\, y + r\, f2b'[y] + r\, U[y]\, f3r'[y] - r\, f3r[y]\, U'[y]$

$2\,\pi\, r$

- **Verification**

```
a4NN = -a4o φ0^(0,2)[T, X]^2 - a4n φ0^(1,2)[T, X] - a4l φ0^(0,4)[T, X] -
    φ0^(0,1)[T, X] (a4g φ0^(0,3)[T, X] + a4h φ0^(1,1)[T, X]) -
    φ0^(0,1)[T, X]^2 (a4c φ0^(0,2)[T, X]) - a4a φ0^(0,1)[T, X]^4 -
    a4u r φ0^(0,1)[T, X]^2 - a4p φ0^(0,2)[T, X] + ℒ[ψ4[T, X, y]];
Simplify[a4 - a4NN]
```

0

Discussion

The amplitude equation is

$$\varphi 0^{(1,2)}[T, X] + r\,\varphi 0^{(0,2)}[T, X] + \left(\frac{1}{2\pi}\int_0^{2\pi} f1[y]\, U[y]\, dy\right)\varphi 0^{(0,4)}[T, X] == 0$$

After integration by part, using the fact that $\nu\, f1^{(6)}$=-U, the integral is $\frac{1}{2\pi\nu}\int_0^{2\pi}\Psi^2\, dy$

Hence we have again a negative viscosity effect with a threshold which requires to calculate f1[y].

Friction appears in this equation because it has been assumed of order ϵ^2.

The amplitude aquation has been obtained independently of any symmetry hypothesis

The only hypothesis is that $\int_0^{2\pi} U[y]\, dy = 0$.

Instability coefficient (with no friction)

```
{NIntegrate[1/(2 π) ξ1[s]^2 /. ξ1sol , {s, 0, 2 π}],

 NIntegrate[1/(2 π) ξ2[s]^2 /. ξ2sol , {s, 0, 2 π}],
 NIntegrate[1/(2 π) ξ3[s]^2 /. ξ3sol, {s, 0, 2 π}]}

{{0.5}, {0.697102}, {0.614353}}
```

We recover values which are close to the instability criterion in the viscous case. Not surprising since the streamfunction is in all cases dominated by the leading mode and that higher modes are further damped when ξ is calculated.

Dump the current state of this workspace and reload it

```
DumpSave["CH-lin2-nu-12.m6.mx", "Global`"]

{Global`}
```

```
<< CH-lin2-nu-12.m6.mx
```