Large-scale linear instability of the generalized Kolmogorov flow with modified viscosity

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Smagorinsky-Clark mixed model + friction

U = A Sin[θ [y]] where θ [y] = $\Sigma\,\hat{\theta}_p$Sin[2 p y]

Streamfunction defined as u=$\partial_y\,\psi$, $v = -\partial_x\,\psi$

Calculation are done in terms of U, not expanding in θ [y

Settings

Last use

In[13]:=
```
DateList[]
```

Out[13]=
```
{2009, 1, 14, 16, 32, 21.058743}
```

General

In[2]:=
```
Off[General::spell1]
```

Laplacian

In[3]:=

Δ = - ($\epsilon^2\,\partial_{x,x}$ # + $\partial_{y,y}$ #) &

Out[3]=

- ($\epsilon^2\,\partial_{x,x}$ #1 + $\partial_{y,y}$ #1) &

Definition of the turbulent viscosity, Smagorinsky-Clark mixed model

Standard version with a heuristic component accounting for the vertical shear

In[4]:=
```
S̄ = √ 4 (ϵ ∂x,y # )^2 + (ϵ^2 ∂x,x # - ∂y,y #)^2 + μ (ϵ ∂x # )^2 + μ (∂y # )^2 &
```

Out[4]=
$$\sqrt{4\,(\epsilon\,\partial_{x,y}\#1)^2+\left(\epsilon^2\,\partial_{x,x}\#1-\partial_{y,y}\#1\right)^2+\mu\,(\epsilon\,\partial_{x}\#1)^2+\mu\,(\partial_{y}\#1)^2}\;\&$$

Dissipation: C in front of the Smagorinsky component, D in front of the Clark component

In standard notations D=Γ^2/12 and C = $C_S\Gamma^2$

Note that K=-DS in the paper

In[5]:=
```
DS = C (∂y,y (S̄[#] (-∂y,y # + ϵ^2 ∂x,x #)) -
      ϵ^2 ∂x,x (S̄[#] ( - ∂y,y # + ϵ^2 ∂x,x #)) - 4 ϵ^2 ∂x,y (S̄[#] ∂x,y #)) +
   D ( ϵ ∂x,y # ( ϵ^4 ∂x,x,x,x # - ∂y,y,y,y #) + ( ϵ ∂x,y,y,y # + ϵ^3 ∂x,x,x,y #) (∂y,y # - ϵ^2 ∂x,x #) ) &
```

Out[5]=
$$C\left(\partial_{y,y}\left(\bar{S}[\#1]\left(-\partial_{y,y}\#1+\epsilon^2\,\partial_{x,x}\#1\right)\right)-\epsilon^2\,\partial_{x,x}\left(\bar{S}[\#1]\left(-\partial_{y,y}\#1+\epsilon^2\,\partial_{x,x}\#1\right)\right)-4\,\epsilon^2\,\partial_{x,y}\left(\bar{S}[\#1]\,\partial_{x,y}\#1\right)\right)+D\left(\epsilon\,\partial_{x,y}\#1\left(\epsilon^4\,\partial_{x,x,x,x}\#1-\partial_{y,y,y,y}\#1\right)+\left(\epsilon\,\partial_{x,y,y,y}\#1+\epsilon^3\,\partial_{x,x,x,y}\#1\right)\left(\partial_{y,y}\#1-\epsilon^2\,\partial_{x,x}\#1\right)\right)\;\&$$

Reference
Pope, S.B. Turbulent flows, C.U.P., 2000

Definition of the generalized Kolmogorov Flow

Velocity and streamfunction

U[y_] := A Sin[θ [y]]

In[6]:=
```
Φ[y_] := Integrate[U[s], {s, 0., y}]
```

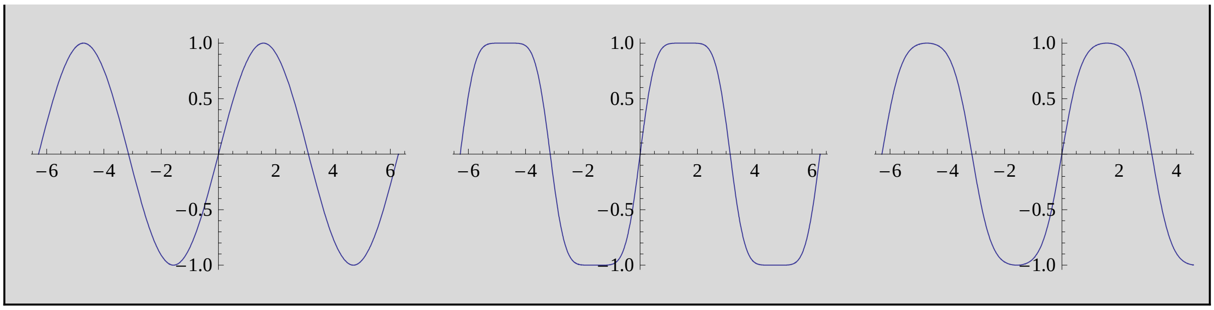
Exemples of flows

Standard Kolmogorov flow in Flow1, "flat" flow in Flow2, intermediate case in Flow3

In[140]:=
```
Flow2 = U → Evaluate[Sin[# + 1/2 Sin[ 2 #]] &];
Flow1 = U → Evaluate[Sin[# ] &];
Flow3 = U → Evaluate[Sin[# + 1/4 Sin[ 2 #]] &];
```

Plots of the flow

```
p1 = Plot[U[y] /. {Flow1, A → 1}, {y, -2 π, 2 π}, DisplayFunction → Identity];
p2 = Plot[U[y] /. {Flow2, A → 1}, {y, -2 π, 2 π}, DisplayFunction → Identity];
p3 = Plot[U[y] /. {Flow3, A → 1}, {y, -2 π, 2 π}, DisplayFunction → Identity];
Show[GraphicsRow[{p1, p2, p3}]]
```



Plots of the streamfunction

Plots of the forcing (or dissipation)

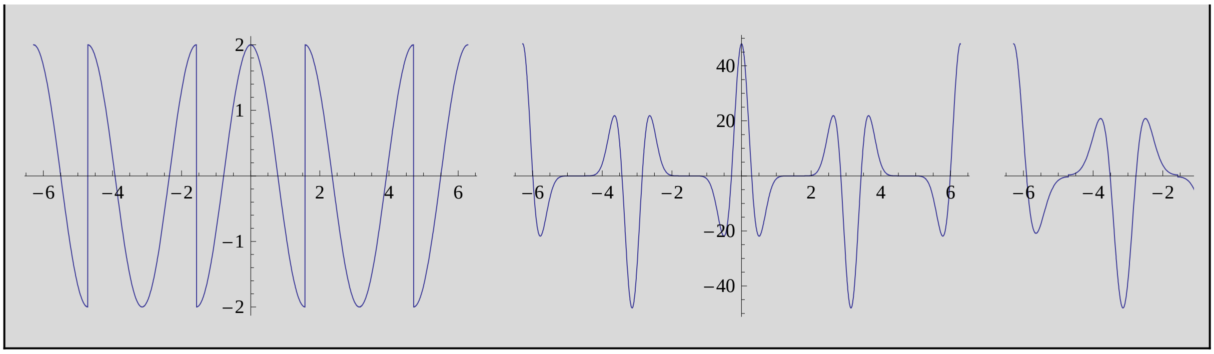
Forcing is calculated to equilibrate the dissipation since there is no advection for a flow that depends only on y
Some arbitrary values are assumed for the parameters

```
Forcing = Assuming[A > 0 && θ'[y] > 0, Simplify[ DS[ Φ[y]]]]
```

$$-\frac{1}{\left(\mu\, U[y]^2 + U'[y]^2\right)^{3/2}} C \left(\mu\, U'[y]^5 + 3\,\mu\, U[y]^2\, U'[y]\, U''[y]\, (\mu\, U[y] + U''[y]) + U'[y]^3\, U''[y]\, (\mu\, U[y] + 2\, U''[y]) + \mu^2\, U[y]^4\, U^{(3)}[y] + 3\,\mu\, U[y]^2\, U'[y]^2\, U^{(3)}[y] + 2\, U'[y]^4\, U^{(3)}[y]\right)$$

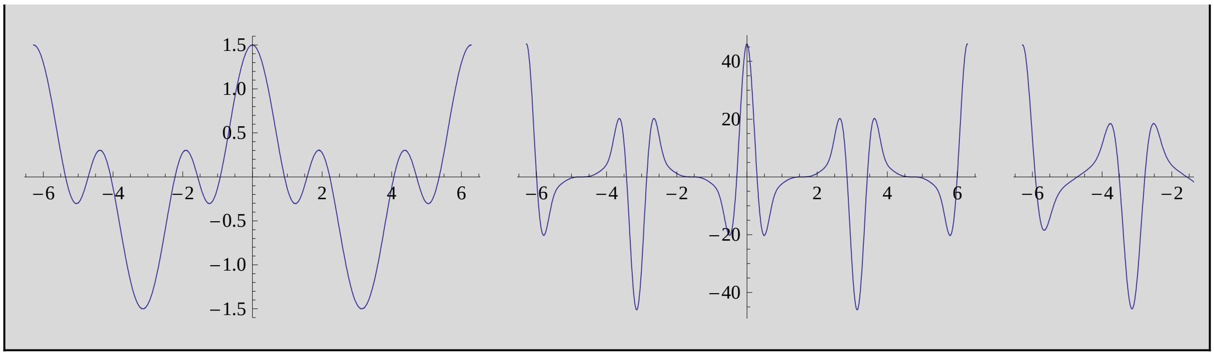
Setting μ -> 0. shows that Flow1 requires a forcing with discontinuities that disappear in Flow2 and Flow3.

```
p1 = Plot[ Forcing //. {Flow1, A → 1, C → 1, μ → 0.},
   {y, -2 π, 2 π}, PlotRange → All, DisplayFunction → Identity];
p2 = Plot[ Forcing //. {Flow2, A → 1, C → 1, μ → 0.}, {y, -2 π, 2 π},
   PlotRange → All, DisplayFunction → Identity];
p3 = Plot[ Forcing //. {Flow3, A → 1, C → 1, μ → 0.}, {y, -2 π, 2 π},
   PlotRange → All, DisplayFunction → Identity];
Show[GraphicsRow[{p1, p2, p3}]]
```



Setting μ -> 0.5 shows that the discontinuities disappear in Flow1, while the forcings in Flow2 and Flow3 are hardly modified

```
p1 = Plot[ Forcing //. {Flow1, A → 1, C → 1, μ → 0.5},
   {y, -2 π, 2 π}, PlotRange → All, DisplayFunction → Identity];
p2 = Plot[ Forcing //. {Flow2, A → 1, C → 1, μ → 0.5}, {y, -2 π, 2 π},
   PlotRange → All, DisplayFunction → Identity];
p3 = Plot[ Forcing //. {Flow3, A → 1, C → 1, μ → 0.5}, {y, -2 π, 2 π},
   PlotRange → All, DisplayFunction → Identity];
Show[GraphicsRow[{p1, p2, p3}]]
```



Definition of the pertubation problem

2D Navier-Stokes with scaling

We have taken u = $\partial_y \psi$ and $v = -\partial_x \psi$ (definition used in fluid mechanics, opposite to the convention in GFD)

In[7]:=
```
NS = ϵ^2 ∂_T Δ[ψ[T, X, y]] - ϵ ∂_X ψ[T, X, y] ∂_y Δ[ψ[T, X, y]] +
   ϵ ∂_y ψ[T, X, y] ∂_X Δ[ψ[T, X, y]] - DS[ψ[T, X, y]] + DS[Φ[y]];
```

Expansion

The streamfunction is expanded up to the fourth order in ϵ

In[8]:=
```
ψ[T_, X_, y_] :=
 Φ[y] + ψ0[T, X, y] + ϵ ψ1[T, X, y] + ϵ^2 ψ2[T, X, y] + ϵ^3 ψ3[T, X, y] + ϵ^4 ψ4[T, X, y]
```

In[9]:=
```
NS2 = Collect[Normal[Series[NS, {ϵ, 0, 4}]], ϵ];
```

Order 0

Extract order 0 terms from the general expansion and perform algebraic simplication.
Fairly lengthy expression anyway.

In[14]:=
```
a0 = Simplify[Coefficient[NS2, ϵ, 0] ]
```

Symbolic solution

The 0 th order solution satisfies a0 = 0 as soon as it does not depend on the fast variable y

In[16]:=
```
Rule0 = ψ0 → (φ0[#1, #2] &)
```
Out[16]=
```
ψ0 → (φ0[#1, #2] &)
```

Verification

Check that φ 0[X,T] satisfies the 0 th order solution

In[17]:=
```
FullSimplify[a0 /. {Rule0}]
```
Out[17]=
```
0
```

Useful rules for simplifications

In[10]:=
```
Rulea = (A^2 μ Sin[θ[y]]^2 + A^2 Cos[θ[y]]^2 θ'[y]^2)^n_ → A^(2 n) (μ Sin[θ[y]]^2 + Cos[θ[y]]^2 θ'[y]^2)^n
```
Out[10]=
```
(A^2 μ Sin[θ[y]]^2 + A^2 Cos[θ[y]]^2 θ'[y]^2)^n_ → A^(2 n) (μ Sin[θ[y]]^2 + Cos[θ[y]]^2 θ'[y]^2)^n
```

In[11]:=
```
Improve = Simplify[Together[#1]] &
```
Out[11]=
```
Simplify[Together[#1]] &
```

Periodicity in space

In[12]:=
```
Solva[expr_] := ((expr /. y → 2 π) - (expr /. y → 0)) /. g___[2 π] → g[0]
```

Order 1

Extract order 1 terms from the general expansion, use formal solution to order 0 equation and perform algebraic simplication.
Need to be put under the form $\mathcal{L}$[ψ1[T, X, y]] = second member

In[18]:=
```
a1 = Simplify[Expand[Coefficient[NS2, ϵ, 1]] /. Rule0 ]
```

Extraction of the linear operator and verification

First derivative of ψ 1in y

In[19]:=
```
a1F1 = FullSimplify[Coefficient[a1, ψ1^(0,0,1)[T, X, y]] , Trig → True]
```

Out[19]=
$$\frac{1}{\left(\mu\, U[y]^2 + U'[y]^2\right)^{5/2}} C\,\mu\,\left(-3\,\mu\, U[y]\, U'[y]^5 + 7\,\mu\, U[y]^2\, U'[y]^3\, U''[y] + U'[y]^5\, U''[y] + \mu^2\, U[y]^5\, U^{(3)}[y] + \mu\, U[y]^3\, U'[y]\,\left(-3\, U''[y]^2 + U'[y]\, U^{(3)}[y]\right)\right)$$

Second derivative of ψ 1in y

In[20]:=
```
a1F2 = FullSimplify[Coefficient[a1, ψ1^(0,0,2)[T, X, y]] , Trig → True]
```

Out[20]=
$$\frac{1}{\left(\mu\, U[y]^2 + U'[y]^2\right)^{5/2}} C\,\left(3\,\mu^3\, U[y]^5\, U''[y] - 3\,\mu^2\, U[y]^3\, U'[y]^2\, U''[y] + 5\,\mu\, U[y]^2\, U'[y]^3\,\left(\mu\, U'[y] + U^{(3)}[y]\right) + 2\, U'[y]^5\,\left(\mu\, U'[y] + U^{(3)}[y]\right) + 3\,\mu^2\, U[y]^4\,\left(U''[y]^2 + U'[y]\, U^{(3)}[y]\right)\right)$$

Third derivative of ψ 1in y

In[21]:=
```
a1F3 = Factor[Simplify[Coefficient[a1, ψ1^(0,0,3)[T, X, y]] ] ]
```

Out[21]=
$$\frac{C\, U'[y]\,\left(3\,\mu^2\, U[y]^3 + \mu\, U[y]\, U'[y]^2 + 6\,\mu\, U[y]^2\, U''[y] + 4\, U'[y]^2\, U''[y]\right)}{\left(\mu\, U[y]^2 + U'[y]^2\right)^{3/2}}$$

Fourth derivative of ψ 1in y

In[22]:=
```
a1F4 = Factor[Simplify[Coefficient[a1, ψ1^(0,0,4)[T, X, y]]]]
```

Out[22]=
$$\frac{C\,\left(\mu\, U[y]^2 + 2\, U'[y]^2\right)}{\sqrt{\mu\, U[y]^2 + U'[y]^2}}$$

Complete form of $\mathcal{L}$ and verification against the terms extracted from a1

In[23]:=
$$\mathcal{L}[f\_] := C\,\partial_{y,y}\left(\frac{\left(\mu\, U[y]^2 + 2\, U'[y]^2\right)}{\sqrt{\mu\, U[y]^2 + U'[y]^2}}\,\partial_{y,y} f\right) + C\,\mu\,\partial_{y,y}\left(\frac{U[y]\, U'[y]}{\sqrt{\mu\, U[y]^2 + U'[y]^2}}\,\partial_y f\right)$$

In[24]:=
```
Simplify[ℒ[f[y]] - a1F1 f^(1)[y] - a1F2 f^(2)[y] - a1F3 f^(3)[y] - a1F4 f^(4)[y], Trig -> True]
```

Out[24]=
```
0
```

Extraction of second member: a1a

The second member contains only the first derivative in X of the Oth order solution

In[25]:=
```
a1a = -Simplify[Coefficient[a1, φ0^(0,1)[T, X]]]
```
Out[25]=
```
-U''[y]
```

Reconstruction of the first order equation

We check here that the first order equation is $\mathcal{L}[\psi 1[T, X, y]] = -U''[y]\, \varphi 0^{(0,1)}[T, X]$

In[26]:=
```
a1N = ℒ[ψ1[T, X, y]] - a1a φ0^(0,1)[T, X];
```
In[27]:=
```
Simplify[a1N - a1]
```
Out[27]=
```
0
```

Solvability condition

The solvability condition is that the integral in y from 0 to 2π of the second member vanishes. It is obviously satisfied with the second member of the first order equation which is proportional to the second derivative of Sin[θ [y]], as checked here

In[28]:=
```
Simplify[a1a + A D[Sin[θ[y]], {y, 2}]]
```
Out[28]=
```
-A Sin[θ[y]] θ'[y]^2 - U''[y] + A Cos[θ[y]] θ''[y]
```

Symbolic solution

This solution is meant to replace ψ 1in further formal steps of the perturbative expansion

In[29]:=
```
Rule1 = ψ1 → (f1[#3] φ0^(0,1)[#1, #2] &)
```
Out[29]=
```
ψ1 → (f1[#3] φ0^(0,1)[#1, #2] &)
```

Notice that f1 has SinI symmetry

Order 2

Extract order 2 terms from the general expansion, use formal solution to order 0 and 1 equations and perform algebraic simplication.
Need to be put under the form $\mathcal{L}[\psi 2[T, X, y]]$ = second member

In[30]:=
```
a2 = Simplify[Coefficient[NS2, ϵ, 2] //. {Rule0, Rule1} ];
```

Extraction of second member and solvability

Here we extract the terms of the second member, factor per factor, and we test the solvability condition

term in factor of $\varphi 0^{(0,1)}[T, X]^2$: a2a

In[31]:=
```
a2a = Collect[-Simplify[Coefficient[a2 , φ0^(0,1)[T, X], 2]] , {A, D, C}];
```

Check that this simplification can be somewhat improved for the term which is not in factor of C

In[32]:=
```
Simplify[a2a /. C → 0]
```

Out[32]= $-f1^{(3)}[y]$

This term integrates twice in y and satisfies the solvability condition

In[33]:=
```
a2a1 = Coefficient[a2a, C];
```

a2a1I = Integrate[a2a1, y];
a2a1I2 = Integrate[a2aI, y]

In[34]:=

$$a2a1I2 = -\frac{1}{2}\left(\left(2\,\mu^2\,U[y]^3\,f1'[y]\,f1''[y] + U'[y]^3\left(\mu + \mu\,f1'[y]^2 + 2\,f1''[y]^2\right) + \mu\,U[y]^2\,U'[y]\left(\mu + 3\,f1''[y]^2\right)\right) \Big/ \left(\mu\,U[y]^2 + U'[y]^2\right)^{3/2}\right);$$

```
Simplify[D[a2a1I2, {y, 2}] - a2a1]
```

Out[35]= 0

term in factor of $\varphi 0^{(0,2)}[T, X]$: a2b

In[38]:=
```
a2b = -Collect[Simplify[Coefficient[a2 , φ0^(0,2)[T, X], 1]] , {A, D, C }];
```

Split it in several part for integration. First isolate the part that does not depend on C and D and integrate it once.

In[39]:=
```
a2b0 = a2b /. {D → 0 , C → 0};
a2b1I = Integrate[a2b0, y]
```

Out[40]= $U[y]\,f1'[y] - f1[y]\,U'[y]$

Then isolate the factor of C and checks it integrates twice

In[41]:=
```
a2b1 = Coefficient[a2b, C];
```

a2b2I2 = Integrate[Integrate[a2b1, y], y]

In[42]:=

$$a2b2I2 = \frac{\mu\,U[y]^2 + 2\,U'[y]^2}{\sqrt{\mu\,U[y]^2 + U'[y]^2}};$$

```
Simplify[D[a2b2I2, {y, 2}] - a2b1]
```

Out[43]= 0

Isolate the factor of D and integrate it once

In[44]:=
```
a2b2 = Coefficient[a2b , D];
a2b0I = Integrate[a2b2, y]
```

Out[45]=
```
U′[y] f1″[y] - f1′[y] U″[y]
```

Complete the solvability proof for a2b by check that nothing has been forgotten in a2b

In[46]:=
```
Simplify[a2b - a2b0 - C a2b1 - D a2b2]
```

Out[46]=
```
0
```

Reconstruction and verification of the second order equation

Check that nothing has been forgotten in the extraction of the components of a2

In[47]:=
```
a2N = ℒ[ψ2[T, X, y]] - a2b φ0^(0,2)[T, X] - a2a φ0^(0,1)[T, X]^2;
Simplify[a2N - a2 ]
```

Out[48]=
```
0
```

Symbolic solution

In[49]:=
```
Rule2 =  ψ2  → (f2a[#3] φ0^(0,1)[#1, #2]^2 + f2b[#3] φ0^(0,2)[#1, #2] &)
```

Out[49]=
```
ψ2 → (f2a[#3] φ0^(0,1)[#1, #2]^2 + f2b[#3] φ0^(0,2)[#1, #2] &)
```

f2a has CosI symmetry
f2b has CosP symmetry

Order 3

Extract order 3 terms from the general expansion, use formal solution to order 0, 1 and 2 equations and perform algebraic simplication.
Need to be put under the form ℒ[ψ3[T, X, y]] = second member

No attemps to simplify here
NOTICE THAT, FOR HISTORICAL REASONS, THE axx TERMS AT THIS ORDER AND THE NEXT ONE IN THIS NOTEBOOK ARE DEFINED AS THE NEGATIVE OF THOSE USED IN THE PAPER AND OTHER NOTEBOOK (SEE THE VERIFICATION STEP)
The terms marked as [S] require the symmetry assumption to satisfy the solvability condition. Only term 3d is involved at this order.
Pre-defined integrals at this order and the next one in this notebook have been derived from the calculation of primitives obtained with *Mathematica* with U=Sin[θ [y]]For some unknown reasons, *Mathematica* is more able to find the primitive under this form than using U alone.

In[50]:=
```
a3 = Coefficient[NS2, ϵ, 3]  //. {Rule0, Rule1, Rule2} ;
```

Extraction of second member and solvability

First reorganize a3 in order to make further extraction easier

In[51]:=
```
a31 = Collect[Expand[a3], {φ0^(1,1)[T, X], φ0^(0,1)[T, X], φ0^(0,2)[T, X], φ0^(0,3)[T, X]}];
```

Term in $\varphi 0^{(0,1)}[T, X]^3$: a3a

In[52]:=
```
a3a = Collect[Coefficient[a31 /. Rulea , φ0^(0,1)[T, X]^3] , C] ;
```

Term excluding factor C, which can be integrated in y

In[54]:=
```
a3a /. C → 0.;
```

Get term in factor of C and check that the combination, adding previous contribution, reconstructs a3a

In[55]:=
```
a3a1 = Coefficient[a3a, C];
a3a - C a3a1 - (a3a /. C → 0)
```

Out[56]=
```
0
```

a3a1I = Integrate[a3a1, y];
a3a1I2 = Integrate[a3a1I, y];

In[57]:=
```
a3a1I2 = (μ^2 U[y] (-f1′[y] U′[y] + U[y] f1″[y])
      ((1 + f1′[y]^2) U′[y]^2 - 2 U[y] f1′[y] U′[y] f1″[y] + U[y]^2 (μ + f1″[y]^2))) /
    (2 (μ U[y]^2 + U′[y]^2)^(5/2)) + (μ f1′[y] ( f2a′[y] U′[y]^3 + μ U[y]^3 f2a″[y]) +
      f1″[y] (μ^2 U[y]^3 f2a′[y] + U′[y] (3 μ U[y]^2 + 2 U′[y]^2) f2a″[y])) / (μ U[y]^2 + U′[y]^2)^(3/2);
Simplify[D[a3a1I2, {y, 2}] - a3a1]
```

Out[58]=
```
0
```

Solvability is satisfied since all components of a3a integrate twice in y. Notice that it would be sufficient to check that the solvability condition is satisfied on the first integral but more compact expressions are obtained when a second primitive can be obtained.

Terms in factor of $\varphi 0^{(0,1)}[T, X]$

Group of all terms containing $\varphi 0^{(0,1)}[T, X]$ in factor

In[59]:=
```
a33 = Coefficient[Expand[a31], φ0^(0,1)[T, X]];
```

Term in $\varphi 0^{(0,1)}[T, X]\, \varphi 0^{(0,2)}[T, X]$: a3b

In[60]:=
```
a3b = Collect[Coefficient[a33 /. Rulea, φ0^(0,2)[T, X]] , {D, C }];
```

Term not in factor of C within a3b

In[61]:=
```
a3b0 = a3b /. C → 0
```

Out[61]=
$$-f1'[y]\,f1''[y] - 2\,U[y]\,f2a''[y] + 2\,f2a[y]\,U''[y] + f1[y]\,f1^{(3)}[y] + f2b^{(3)}[y] + D\left(-f1''[y]\,f1^{(3)}[y] - 2\,U'[y]\,f2a^{(3)}[y] + 2\,f2a'[y]\,U^{(3)}[y] + f1'[y]\,f1^{(4)}[y]\right)$$

In[62]:=
```
a3b0I = Integrate[a3b0, y]
```

Out[62]=
$$-f1'[y]^2 - 2\,U[y]\,f2a'[y] + f1[y]\,f1''[y] - D\,f1''[y]^2 + 2\,U'[y]\left(f2a[y] - D\,f2a''[y]\right) + f2b''[y] + 2\,D\,f2a'[y]\,U''[y] + D\,f1'[y]\,f1^{(3)}[y]$$

Term in factor of C within a3b, checking that it combines with a3b0 to reconstruct a3b

In[63]:=
```
a3b2 = Coefficient[a3b, C];
a3b - C a3b2 - a3b0
```

Out[64]=
```
0
```

Integrate twice a3b2, completing the proof of solvability for a3b

a3b2I = Integrate[a3b2, y]
a3b2I2 = Integrate[a3b2I, y]

In[65]:=
$$\texttt{a3b2I2} = \left(\mu\,f1[y]\,U'[y]\left(\mu\,U[y]^2 + U'[y]^2\right) + \mu\,f1'[y]\left(f2b'[y]\,U'[y]^3 + \mu\,U[y]^3\left(-1 + f2b''[y]\right)\right) + f1''[y]\left(\mu^2\,U[y]^3\,f2b'[y] + U'[y]\left(3\,\mu\,U[y]^2 + 2\,U'[y]^2\right)\left(-1 + f2b''[y]\right)\right)\right) \Big/ \left(\mu\,U[y]^2 + U'[y]^2\right)^{3/2};$$

```
Simplify[D[a3b2I2, {y, 2}] - a3b2]
```

Out[66]=
```
0
```

Term in $\varphi 0^{(0,3)}$[T, X] : a3d [S]

In[67]:=
$$\texttt{a3d} = \texttt{Collect}\left[\texttt{Coefficient}\left[\texttt{a31 /. Rulea}, \varphi 0^{(0,3)}[T, X]\right], \{D, C\}\right];$$

Extract the contribution that does not contain C

In[68]:=
```
a3d0 = a3d /. C → 0
```

Out[68]=
$$-U[y] - U[y]\,f2b''[y] + f2b[y]\,U''[y] + D\left(-U'[y]\,f2b^{(3)}[y] + f2b'[y]\,U^{(3)}[y]\right)$$

In[69]:=
```
Integrate[a3d0 + U[y], y]
```

Out[69]=
$$-U[y]\,f2b'[y] + U'[y]\left(f2b[y] - D\,f2b''[y]\right) + D\,f2b'[y]\,U''[y]$$

Coefficient in front of C in a3d

In[70]:=
```
a3d2 = Improve[ Coefficient[a3d, C] ];
```

Mathematica does not integrate this expression but after a few manipulations we get an expression for a3d2 as the sum of two derivatives plus a residual term.
In the residual term, f1'[y] has CosI symmetry. Its product with U[y]U'[y] has SinI symmetry with zero mean.
Hence the solvability condition is satisfied for all the terms contributing to a3d

In[71]:=

$$\text{a3d2P} = D\left[\frac{2\,\mu\,\left(\mu\,U[y]^3 + U[y]\,U'[y]^2\right)^2}{\left(\mu\,U[y]^2 + U'[y]^2\right)^{5/2}}\,\text{f1}'[y],\,y\right] - D\left[\frac{\mu\,U[y]^2 + 2\,U'[y]^2}{\sqrt{\mu\,U[y]^2 + U'[y]^2}},\,\{y,\,2\}\right]\text{f1}[y] - \frac{\mu\,U[y]\,\text{f1}'[y]\,U'[y]}{\sqrt{\mu\,U[y]^2 + U'[y]^2}};$$

```
Simplify[a3d2P - a3d2]
```

Out[72]= 0

Term in $\varphi 0^{(1,1)}[T, X]$: a3g

In[73]:=

$$\text{a3g} = \text{Coefficient}\left[\text{a31},\,\varphi 0^{(1,1)}[T,\,X]\right]$$

Out[73]= $-\text{f1}''[y]$

Verification

Assemble all the contribution found at order 3 plus the linear term in ψ 3,and check the sum against a3

In[74]:=

$$\text{a3N} = \text{a3g}\,\varphi 0^{(1,1)}[T,\,X] + \text{a3d}\,\varphi 0^{(0,3)}[T,\,X] + \varphi 0^{(0,1)}[T,\,X]\,\left(\text{a3b}\,\varphi 0^{(0,2)}[T,\,X]\right) + \text{a3a}\,\varphi 0^{(0,1)}[T,\,X]^3 + \mathcal{L}[\psi 3[T,\,X,\,y]];$$

```
Simplify[Together[Expand[a3N - a3 /. Rulea]] ]
```

Out[75]= 0

Symbolic solution

In[76]:=

$$\text{Rule3} = \psi 3 \to \left(\text{f3g}[\#3]\,\varphi 0^{(1,1)}[T,\,X] + \text{f3d}[\#3]\,\varphi 0^{(0,3)}[T,\,X] + \varphi 0^{(0,1)}[T,\,X]\,\left(\text{f3b}[\#3]\,\varphi 0^{(0,2)}[T,\,X] + \text{f3h}\;\tilde{r}\right) + \text{f3a}[\#3]\,\varphi 0^{(0,1)}[T,\,X]^3\;\&\right);$$

f3a has SinI symmetry
f3b has SinP symmetry
f3d has SinI symmetry
f3g has SinI symmetry
f3h has SinI symmetry

Order 4

Extract order 4 terms from the general expansion, use formal solution to order 0, 1, 2 and 3 equations and perform algebraic simplication.
Need to be put under the form $\mathcal{L}[\psi 4[T, X, y]]$ = second member

No attemps to simplify here.
The terms a4g, a4à and a4r require the symmetry assumption to satisfy the solvability condition.
The terms a4l and a4l provide the amplitude equation.

In[77]:=
```
a4 =
  Assuming[A > 0 && θ′[y] > 0, Coefficient[NS2, ϵ, 4]  //. {Rule0, Rule1, Rule2, Rule3} ];
```

Rearrange a4 to prepair extraction

In[78]:=
```
a41 = Collect[Expand[a4],
   {φ0^(1,1)[T, X], φ0^(1,2)[T, X], φ0^(0,1)[T, X], φ0^(0,2)[T, X], φ0^(0,3)[T, X], φ0^(0,4)[T, X]}];
```

Term in $\varphi 0^{(0,1)}[T, X]^4$: a4a

In[79]:=
```
a4a = Collect[Coefficient[a41, φ0^(0,1)[T, X]^4] , {D, C}];
```

Contribution that does not contain C.
Can be integrated and satisfies the solvability condition.

In[80]:=
```
a4a0 = a4a  /. C → 0
```

Out[80]=
```
f3a^(3)[y]
```

Get the rest of a4a

In[81]:=
```
a4a1 = Coefficient[a4a, C];
Simplify[a4a - C a4a1 - a4a0]
```

Out[82]=
```
0
```

Integrate and check

a4a1I = Integrate[a4a1, y];
a4a1I2 = Integrate[a4a1I, y]

In[83]:=
```
a4a1I2a = -(μ^2 ((1 + f1'[y]^2) U'[y]^2 - 2 U[y] f1'[y] U'[y] f1''[y] + U[y]^2 (μ + f1''[y]^2))
    ((1 + f1'[y]^2) U'[y]^3 + 4 μ U[y]^3 f1'[y] f1''[y] - 6 U[y] f1'[y] U'[y]^2 f1''[y] +
      U[y]^2 U'[y] (μ - 4 μ f1'[y]^2 + 5 f1''[y]^2))) / (8 (μ U[y]^2 + U'[y]^2)^(7/2));
a4a1I2b = (μ^2 U[y] ((1 + 3 f1'[y]^2) U'[y]^2 - 6 U[y] f1'[y] U'[y] f1''[y] + U[y]^2 (μ + 3 f1''[y]^2))
    (-f2a'[y] U'[y] + U[y] f2a''[y])) / (2 (μ U[y]^2 + U'[y]^2)^(5/2));
a4a1I2c = (μ f2a'[y]^2 U'[y]^3 + 2 μ^2 U[y]^3 f3a'[y] f1''[y] + 2 μ^2 U[y]^3 f2a'[y] f2a''[y] +
    3 μ U[y]^2 U'[y] f2a''[y]^2 + 2 U'[y]^3 f2a''[y]^2 +
    6 μ U[y]^2 U'[y] f1''[y] f3a''[y] + 4 U'[y]^3 f1''[y] f3a''[y] +
    2 μ f1'[y] (f3a'[y] U'[y]^3 + μ U[y]^3 f3a''[y])) / (2 (μ U[y]^2 + U'[y]^2)^(3/2));
Simplify[Expand[D[a4a1I2a + a4a1I2b + a4a1I2c, {y, 2}] - a4a1]]
```

Out[86]=
```
0
```

Terms in $\varphi 0^{(0,1)}[T, X]^2$

Group of all terms containing a factor $\varphi 0^{(0,1)}[T, X]^2$

In[87]:=
```
a42 = Coefficient[a41, φ0^(0,1)[T, X]^2];
```

Term in $\varphi 0^{(0,1)}[T, X]^2 \varphi 0^{(0,2)}[T, X]$: a4c

In[88]:=
```
a4c = Collect[Coefficient[a42, φ0^(0,2)[T, X]] , {D, C }];
```

Contribution that does not contain C.
Can be integrated and satisfies the solvability condition.

In[89]:=
```
a4c0 = a4c  /. C → 0;
a4c0I = Integrate[a4c0, y]
Solva[a4c0I]
```

Out[90]=
```
2 f2a[y] f1''[y] + (f1[y] - 3 D f1''[y]) f2a''[y] +
  f3b''[y] + f2a'[y] (-3 f1'[y] + 2 D f1^(3)[y]) + D f1'[y] f2a^(3)[y]
```

Out[91]=
```
0
```

Get the rest of a4c in powers of A

In[92]:=
```
a4c1 = Coefficient[a4c, C];
Simplify[a4c - a4c0 - C a4c1]
```

Out[93]=
```
0
```

Integrate and check

a4c1I = Improve[Integrate[a4c1, y]];

a4c1I2 = Integrate[a4c1I, y]

In[94]:=
```
a4c1I2a = (μ^2 U[y] (-(2 f1[y] f1′[y] + (1 + 3 f1′[y]^2) f2b′[y]) U′[y]^3 -
      U[y]^2 U′[y] (2 μ f1[y] f1′[y] + f2b′[y] (μ + 3 f1″[y]^2) + 6 f1′[y] f1″[y] (-1 + f2b″[y])) +
      U[y]^3 (2 μ f1[y] f1″[y] + (μ + 3 f1″[y]^2) (-1 + f2b″[y])) +
      U[y] U′[y]^2 (-1 + 2 f1[y] f1″[y] + 6 f1′[y] f2b′[y] f1″[y] +
         3 f1′[y]^2 (-1 + f2b″[y]) + f2b″[y]))) / (2 (μ U[y]^2 + U′[y]^2)^(5/2));
a4c1I2b = (μ^2 U[y]^3 (f3b′[y] f1″[y] + f2b′[y] f2a″[y] + f2a′[y] (-1 + f2b″[y]) +
       f1′[y] f3b″[y]) + U′[y]^3 (2 μ f2a[y] + μ f2a′[y] f2b′[y] + μ f1′[y] f3b′[y] -
       2 f2a″[y] + 2 f2a″[y] f2b″[y] + 2 f1″[y] f3b″[y]) + μ U[y]^2 U′[y]
     (2 μ f2a[y] + 3 (f2a″[y] (-1 + f2b″[y]) + f1″[y] f3b″[y]))) / ((μ U[y]^2 + U′[y]^2)^(3/2));
Simplify[D[a4c1I2a + a4c1I2b, {y, 2}] - a4c1]
```

Out[96]=
```
0
```

Terms in $\varphi 0^{(0,1)}$[T, X]

Group of all terms containing a factor $\varphi 0^{(0,1)}$[T, X]

In[97]:=
```
a43 = Coefficient[a41, φ0^(0,1)[T, X]];
```

Term in $\varphi 0^{(0,1)}$[T, X] $\varphi 0^{(0,3)}$[T, X] : a4g [S]

In[98]:=
```
a4g = Collect[Coefficient[a43, φ0^(0,3)[T, X]] , {D, C}];
```

Contribution that does not contain C.
Can be integrated and satisfies the solvability condition.

In[99]:=
```
a4g0 = a4g /. C → 0;
```

In[100]:=
```
a4gOI = Integrate[a4g0, y]
Solva[a4gOI]
```

Out[100]=
```
f2b[y] f1″[y] - D f1″[y] f2b″[y] + f3d″[y] + f2b′[y] (-f1′[y] + D f1^(3)[y])
```

Out[101]=
```
0
```

Get the rest of a4g

In[102]:=
```
a4g1 = Coefficient[a4g, C];
a4gl = CoefficientList[ Coefficient[a4g, C], A];
Simplify[a4g - a4g0 - C a4g1]
```

Out[104]=
```
0
```

This contribution cannot be integrated but, after some manipulations, it is possible to extract an integrable part on which the solvability condition can be tested and a non integrable part for which the solvability condition is obtained due to the symmetries

In[105]:=
```
a4g1 = Improve[a4gl];
```

In[106]:=

$$a4g1Pa = -\frac{\mu\, U'[y]}{\sqrt{\mu\, U[y]^2 + U'[y]^2}} - \frac{U'[y]\left(3\,\mu\, U[y]^2 + 2\, U'[y]^2\right) f1''[y]^2}{\left(\mu\, U[y]^2 + U'[y]^2\right)^{3/2}} - \frac{\mu\, f1'[y]^2\, U'[y]}{\left(\mu\, U[y]^2 + U'[y]^2\right)^{5/2}}\left(-2\,\mu\, U[y]^2\, U'[y]^2 + U'[y]^4 + 3\,\mu\, U[y]^3\, U''[y]\right) + \frac{(2\, U'[y]\,(3\,\mu\, U[y] + 4\, U''[y]))}{\sqrt{\mu\, U[y]^2 + U'[y]^2}}\, f2a'[y] + \frac{6\,\mu\, U[y]^2 + 4\, U'[y]^2}{\sqrt{\mu\, U[y]^2 + U'[y]^2}}\, f2a''[y];$$

$$a4g1Pb = \frac{\mu\, U[y]\, f1'[y]^2\left(3\,\mu\, U[y]^2 + 6\, U'[y]^2 - 2\, U[y]\, U''[y]\right)}{\left(\mu\, U[y]^2 + U'[y]^2\right)^{3/2}};$$

$$a4g1Pc = \left(\mu\, f2b[y]\, U'[y]\left(\mu\, U[y]^2 + U'[y]^2\right) + 2\, f1'[y]^2\, U'[y]\left(\mu\, U[y]^2 + U'[y]^2\right) - f1[y]\left(\mu^2\, U[y]^3\, f1'[y] + U'[y]\left(3\,\mu\, U[y]^2 + 2\, U'[y]^2\right) f1''[y]\right) + \mu\, f1'[y]\left(f3d'[y]\, U'[y]^3 + \mu\, U[y]^3\, f3d''[y]\right) + f1''[y]\left(\mu^2\, U[y]^3\, f3d'[y] + U'[y]\left(3\,\mu\, U[y]^2 + 2\, U'[y]^2\right) f3d''[y]\right)\right) \Big/ \left(\mu\, U[y]^2 + U'[y]^2\right)^{3/2} - \frac{2\left(\mu\, U[y]^2 + 2\, U'[y]^2\right)}{\sqrt{\mu\, U[y]^2 + U'[y]^2}}\, f2a[y];$$

```
a4g1P = a4g1Pa + D[a4g1Pb, y] + D[a4g1Pc, {y, 2}];
Simplify[a4g1 - a4g1P]
```

Out[110]= {0}

All the non integrated terms in a4g1Pa have CosI symmetry and hence sum to zero.

f2a''[y] has symmetry CosI, its product with a CosP term is CosI and integrates to zero.
f2a'[y] has SinI symmetry, and U'[y] U[y] has SinP symmetry,
hence there are two products CosI SinI SinI and CosI
CosI SinP SinI which have CosI symmetry and integrate to zero.

Term in $\varphi 0^{(0,1)}[T, X]\, \varphi 0^{(1,1)}[T, X]$: a4h

In[111]:=
```
a4h = Collect[Coefficient[a43, φ0^(1,1)[T, X]] , {D, C}];
```

Contribution that does not contain C.
Can be integrated and satisfies the solvability condition.

In[112]:=
```
a4h0 = a4h /. C → 0
```

Out[112]= $-2\, f2a''[y] + f3g^{(3)}[y]$

Get contribution in C, integrate it and check solvability

In[113]:=
```
a4h1 = Improve[Coefficient[a4h, C]];
Simplify[a4h - a4h0 - C a4h1] 3
```
Out[114]=
```
0
```

a4h1I = Improve[Integrate[a4h1, y]];
a4h1I2 = Integrate[a4h1I, y]

Integrate and check

In[115]:=
```
a4h1I2 = (μ f1′[y] (f3g′[y] U′[y]^3 + μ U[y]^3 f3g″[y]) +
    f1″[y] (μ^2 U[y]^3 f3g′[y] + U′[y] (3 μ U[y]^2 + 2 U′[y]^2) f3g″[y])) / (μ U[y]^2 + U′[y]^2)^(3/2);
Simplify[D[a4h1I2, {y, 2}] - a4h1]
```
Out[116]=
```
0
```

Term in $\varphi 0^{(0,4)}$[T, X] : a4l (contributes to the amplitude equation)

In[117]:=
```
a4l = Collect[Coefficient[a41,  φ0^(0,4)[T, X]] , {D, C }];
```

Contribution independent of C.
The two terms of a4l0 are of symmetry CosP, they have no primitive and provide a contribution to the amplitude equation.

In[118]:=
```
a4l0 = a4l /. C → 0
```
Out[118]=
```
-f1[y] U[y] - D f1′[y] U′[y]
```

Now get the coefficient of C. It has no primitive but can be described as a sum of derivatives and residual terms.

In[119]:=
```
a4l1 = Improve[Coefficient[a4l, C]];
```

In[120]:=

$$\texttt{a4l1Pa} = \frac{\mu\, U[y]^2 + 2\, U'[y]^2}{\sqrt{\mu\, U[y]^2 + U'[y]^2}};$$

$$\texttt{a4l1Pb} = \frac{2\,\mu\, U[y]^2}{\sqrt{\mu\, U[y]^2 + U'[y]^2}}\, \texttt{f2b}[y];$$

$$\texttt{a4l1Pc} = \frac{\mu\, U[y]\, U'[y]\, \left(3\,\mu\, U[y]^2 + 5\, U'[y]^2 - 2\, U[y]\, U''[y]\right)}{\left(\mu\, U[y]^2 + U'[y]^2\right)^{3/2}}\, \texttt{f2b}[y];$$

$$\texttt{a4l1Pd} = \frac{-\mu\, U[y]\, U'[y]^3 + 3\,\mu\, U[y]^2\, U'[y]\, U''[y] + 2\, U'[y]^3\, U''[y]}{\left(\mu\, U[y]^2 + U'[y]^2\right)^{3/2}};$$

```
a4l1P = a4l1Pa + D[a4l1Pb, {y, 2}] - D[a4l1Pc, y] - f2b[y] D[a4l1Pd, y];
Simplify[a4l1 - a4l1P]
```
Out[125]=
```
0
```

The first and the fourth term contribute as they are product of trig functions with same symmetry.

Check that nothing has been forgotten

In[126]:=
```
Simplify[a4l - a4l0 - C a4l1 ]
```

Out[126]=
```
0
```

Term in $\varphi 0^{(1,2)}$[T, X] : a4n (contributes to the amplitude equation)

In[127]:=
```
a4n = Collect[Coefficient[a41,  φ0^(1,2)[T, X]] , {D}]
```

Out[127]=
$-1 - f2b''[y]$

The first term makes a contribution to the amplitude equation and the secondone satisfies the solvability condition.

Term in $\varphi 0^{(0,2)}[T, X]^2$: a4o [S]

In[128]:=
```
a4o = Collect[Coefficient[a41,  φ0^(0,2)[T, X]^2 ] , {D, C }];
```

The contribution independent of C has a primitive and satisfies the solvability condition

In[129]:=
```
a4o0 = a4o /. C → 0;
a4o0I = Integrate[a4o0, y]
Solva[a4oI0]
```

Out[130]=
$-f1'[y]\, f2b'[y] + D\, f1''[y] + (f1[y] - D\, f1''[y])\, f2b''[y] + D\, f1'[y]\, f2b^{(3)}[y]$

Out[131]=
```
0
```

Expand the rest of a4o in power of A

In[132]:=
```
a4o1 = Coefficient[a4o, C];
Simplify[a4o - a4o0 - C a4o1]
```

Out[133]=
```
0
```

The term in C cannot be integrated but it can be sorted in several parts

In[134]:=
```
a4o1aP = D[((μ/2) f1[y]^2 + 4 f1′[y]^2) U′[y]/Sqrt[μ U[y]^2 + U′[y]^2] - 2 (μ U[y]^2 + 2 U′[y]^2)/Sqrt[μ U[y]^2 + U′[y]^2] f2a[y] +
    (μ f2b′[y]^2 U′[y]^3 + 2 μ^2 U[y]^3 f2b′[y] (-1 + f2b′′[y]) + U′[y] (3 μ U[y]^2 + 2 U′[y]^2)
        (-2 + f2b′′[y]) f2b′′[y]) / (2 (μ U[y]^2 + U′[y]^2)^(3/2)), {y, 2}] +
   D[μ U[y] (6 U′[y]^2 + U[y] (3 μ U[y] - 2 U′′[y])) / (μ U[y]^2 + U′[y]^2)^(3/2) f1′[y]^2, y];
a4o1bP = -(μ U′[y] (-2 μ U[y]^2 U′[y]^2 + U′[y]^4 + 3 μ U[y]^3 U′′[y])) / (μ U[y]^2 + U′[y]^2)^(5/2)
    f1′[y]^2 - U′[y] (3 μ^2 U[y]^4 + 5 μ U[y]^2 U′[y]^2 + 2 U′[y]^4) / ((μ U[y]^2 + U′[y]^2)^(5/2)) f1′′[y]^2 +
   (4 D[U′[y]^2 f2a′[y], y] + 6 μ U[y] D[U[y] f2a′[y], y]) / (Sqrt[μ U[y]^2 + U′[y]^2]);
```

It is easy to check that all the terms which are not under the derivative in a4o1bP are with CosI symmetry and integrate to zero

In[136]:=
```
a4o1c = (-2 μ U′[y]^7 +
     3 μ^2 U[y]^2 U′[y] (-5 U′[y]^4 + 7 U[y] U′[y]^2 U′′[y] - 5 U[y]^2 U′′[y]^2 + U[y]^2 U′[y] U^(3)[y]) +
     3 U[y]^5 μ^3 (-3 U′[y] U′′[y] + U[y] U^(3)[y]) - 2 μ^4 U[y]^6 U′[y]) / (2 (μ U[y]^2 + U′[y]^2)^(7/2));
```

The numerator is a sum of terms with CosI symmetry, which is preserved by the division by the denominator in CosP, integrating a4o1c to zero. Let us check it numerically for some examples.

In[143]:=
```
NIntegrate[Evaluate[a4o1c /. {Flow1, μ → 0.5}], {y, 0, 2 π}]
NIntegrate[Evaluate[a4o1c /. {Flow2, μ → 0.5}], {y, 0, 2 π}]
NIntegrate[Evaluate[a4o1c /. {Flow3, μ → 0.5}], {y, 0, 2 π}]
```

Out[143]= $0. \times 10^{-13}$

Out[144]= $0. \times 10^{-15}$

Out[145]= $0. \times 10^{-15}$

Chek that a4o1 is reconstructed

In[146]:=
```
Simplify[a4o1 - a4o1aP - a4o1bP - a4o1c]
```

Out[146]= 0

Verification

In[147]:=
```
Simplify[Expand[a4] - a41]
```

Out[147]= 0

In[153]:=

```
a4NN = a4o φ0^(0,2)[T, X]^2 + a4n φ0^(1,2)[T, X] +
   a4l φ0^(0,4)[T, X] + φ0^(0,1)[T, X] (a4g φ0^(0,3)[T, X] + a4h φ0^(1,1)[T, X]) +
   a4c φ0^(0,1)[T, X]^2 φ0^(0,2)[T, X] + a4a φ0^(0,1)[T, X]^4 + ℒ[ψ4[T, X, y]];
Simplify[Expand[a4NN - a41] /. Rulea]
```

Out[154]=

```
0
```

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